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ON WEIGHTED DENSITIES

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Abstract. The continuity of densities given by the weight functions n^α, $\alpha \in [-1,\infty[$, with respect to the parameter α is investigated.

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1. Introduction

Let $f\colon \mathbb{N} \to [0,\infty[$ be a nonzero function, let $A \subset \mathbb{N}$ and $n \in \mathbb{N}$. We denote

$$A_f(n) = \sum_{a\in A, a\leqslant n} f(a)$$

and define

$$\underline{d}_f(A) = \liminf_{n\to\infty} \frac{A_f(n)}{\mathbb{N}_f(n)} \quad \text{and} \quad \bar{d}_f(A) = \limsup_{n\to\infty} \frac{A_f(n)}{\mathbb{N}_f(n)},$$

i.e. the lower and the upper f-densities of the set A, respectively. Put moreover

$$D_f(A) = (\bar{d}_f(A), \underline{d}_f(A)) \in \{(x,y);\ x\in[0,1],\ y\in[0,x]\}.$$

We call $D_f(A)$ the f*-density point* of the set A. Two important cases of densities are those of asymptotic densities (denoted by $\underline{d}$, $\bar{d}$) with $f(n)=1$, $n\in\mathbb{N}$, and logarithmic

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densities (denoted by $\underline{\delta}, \bar{\delta}$) with $f(n) = 1/n$, $n \in \mathbb{N}$. There is a well known relation between these four values (see, for instance, [2], p. 241–242)

$$\text{(I)} \qquad 0 \leqslant \underline{d}(A) \leqslant \underline{\delta}(A) \leqslant \bar{\delta}(A) \leqslant \bar{d}(A) \leqslant 1$$

which holds for every $A \subset \mathbb{N}$. Also, examples of sets are known for which the values of the asymptotic densities differ from the corresponding ones of the logarithmic densities. There exist even sets with arbitrary prescribed values of all four densities respecting the relation (I) (see [4]). In this paper we will deal with the class of densities determined by weight functions $f_\alpha(n) = n^\alpha$, $\alpha \in [-1, \infty[$. Notice that the asymptotic densities correspond to $\alpha = 0$ and the logarithmic densities correspond to $\alpha = -1$.

In the sequel we shall write A_α in place of A_{f_α} and $\underline{d}_\alpha$, $\bar{d}_\alpha$, D_α in place of $\underline{d}_{f_\alpha}$, $\bar{d}_{f_\alpha}$ and D_{f_α}, respectively, and moreover we shall use the term α-density point instead of the f_α-density point.

In [6] it is proved that both the upper and lower α-densities vary monotonously with respect to the parameter α. This provides an extension of inequalities (I): Let $-1 \leqslant \alpha < \beta < \infty$. Then the inequalities

$$\text{(R)} \qquad \underline{d}_\beta(A) \leqslant \underline{d}_\alpha(A) \quad \text{and} \quad \bar{d}_\alpha(A) \leqslant \bar{d}_\beta(A)$$

hold for every $A \subset \mathbb{N}$.

A natural question arises whether the coordinates of the α-density point of a set A depend on the parameter α continuously. The aim of the present paper is to discuss this problem. As there are no well known examples of sets with different α-density points for $\alpha \in [-1, \infty[$, we will start with the following example. It shows that there are sets $A \subset \mathbb{N}$ for which both the functions $\alpha \mapsto \underline{d}_\alpha(A)$ and $\alpha \mapsto \bar{d}_\alpha(A)$ are injective on $[-1, \infty[$.

Example 1. Let $a > 1$ be a real number. Denote $A = \bigcup_{k=0}^{\infty}][a^{2k}], [a^{2k+1}]] \cap \mathbb{N}$, where $[r]$ means the integer part of the real number r, i.e. the largest integer less than or equal to r. Then for every $\alpha \in [-1, \infty[$

$$\underline{d}_\alpha(A) = \frac{1}{a^{\alpha+1} + 1} \quad \text{and} \quad \bar{d}_\alpha(A) = \frac{a^{\alpha+1}}{a^{\alpha+1} + 1}.$$

First, let $\alpha > -1$. Then both densities can be calculated using the technique in [5], integrating the function x^α in the corresponding intervals and cancelling the constant multipliers $1/(\alpha+1)$:

$$\bar{d}_\alpha(A) = \limsup_{n\to\infty} \frac{\sum\limits_{k=0}^{n} \sum\limits_{i=[a^{2k}]+1}^{[a^{2k+1}]} i^\alpha}{\sum\limits_{j=1}^{a^{2n+1}} j^\alpha} = \limsup_{n\to\infty} \frac{\sum\limits_{k=0}^{n} (a^{2k+1})^{\alpha+1} - (a^{2k})^{\alpha+1}}{(a^{2n+1})^{\alpha+1}}$$

$$= \lim_{n\to\infty} (a^{\alpha+1}-1) \frac{\sum\limits_{k=0}^{n} (a^{2\alpha+2})^k}{(a^{2n+1})^{\alpha+1}} = (a^{\alpha+1}-1) \lim_{n\to\infty} \frac{\frac{(a^{2\alpha+2})^{n+1}-1}{a^{2\alpha+2}-1}}{(a^{2n+1})^{\alpha+1}}$$

$$= \frac{1}{a^{\alpha+1}+1} \lim_{n\to\infty} \frac{a^{2\alpha n+2n+2\alpha+2}}{a^{2\alpha n+2n+\alpha+1}} = \frac{a^{\alpha+1}}{a^{\alpha+1}+1}$$

and, similarly, or using the fact that in this case $\underline{d}_\alpha(A) = \bar{d}_\alpha(A)/a^{\alpha+1}$, we get

$$\underline{d}_\alpha(A) = \frac{1}{a^{\alpha+1}+1}.$$

Calculation of $\underline{d}_{-1}(A)$ and $\bar{d}_{-1}(A)$ can be performed using the same technique to get

$$\underline{d}_{-1}(A) = \frac{1}{2} = \bar{d}_{-1}(A).$$

Notice that the same result can be obtained using Theorem 2 below on continuity at $\alpha = -1$, as the set A fulfils its assumptions and

$$\lim_{\alpha\to-1^+} \frac{a^{\alpha+1}}{a^{\alpha+1}+1} = \lim_{\alpha\to-1^+} \frac{1}{a^{\alpha+1}+1} = \frac{1}{2}.$$

2. Continuity on $]-1,\infty[$

Now we are going to answer the question about the continuity of the dependence of α-density points on the parameter α. First we will consider the case $\alpha \in\ ]-1,\infty[$.

Theorem 1. *Let $\alpha \in\ ]-1,\infty[$ and $\delta > 0$. Then for every set $A \subset \mathbb{N}$*

$$|\underline{d}_\alpha(A) - \underline{d}_{\alpha+\delta}(A)| \leqslant \frac{2\delta}{\alpha+1} \quad \textit{and} \quad |\bar{d}_\alpha(A) - \bar{d}_{\alpha+\delta}(A)| \leqslant \frac{2\delta}{\alpha+1}.$$

Proof. We calculate

$$\Delta_n(A) = \left|\frac{A_\alpha(n)}{\mathbb{N}_\alpha(n)} - \frac{A_{\alpha+\delta}(n)}{\mathbb{N}_{\alpha+\delta}(n)}\right| = \left|\sum_{a\in A, a\leqslant n} \frac{a^\alpha}{\mathbb{N}_\alpha(n)} - \sum_{a\in A, a\leqslant n} \frac{a^{\alpha+\delta}}{\mathbb{N}_{\alpha+\delta}(n)}\right|$$

$$= \left|(\alpha+1) \sum_{a\in A, a\leqslant n} \frac{a^\alpha}{n^{\alpha+1}} \frac{\frac{n^{\alpha+1}}{\alpha+1}}{\mathbb{N}_\alpha(n)} - (\alpha+\delta+1) \sum_{a\in A, a\leqslant n} \frac{a^{\alpha+\delta}}{n^{\alpha+\delta+1}} \frac{\frac{n^{\alpha+\delta+1}}{\alpha+\delta+1}}{\mathbb{N}_{\alpha+\delta}(n)}\right|.$$

Denote

$$\frac{\frac{n^{\alpha+1}}{\alpha+1}}{\mathbb{N}_\alpha(n)} = 1 + \varepsilon_1(n) \quad \text{and, similarly,} \quad \frac{\frac{n^{\alpha+\delta+1}}{\alpha+\delta+1}}{\mathbb{N}_{\alpha+\delta}(n)} = 1 + \varepsilon_2(n)$$

and notice that both

$$\varepsilon_1(n) \to 0, \qquad \varepsilon_2(n) \to 0 \quad \text{as} \quad n \to \infty.$$

Let us continue the calculation:

$$\begin{aligned}
\Delta_n(A) = \Bigg| &(\alpha+1) \sum_{a\in A, a\leqslant n} \frac{a^\alpha}{n^{\alpha+1}} + \varepsilon_1(n)(\alpha+1) \sum_{a\in A, a\leqslant n} \frac{a^\alpha}{n^{\alpha+1}} \\
&- (\alpha+\delta+1) \sum_{a\in A, a\leqslant n} \frac{a^{\alpha+\delta}}{n^{\alpha+\delta+1}} - \varepsilon_2(n)(\alpha+\delta+1) \sum_{a\in A, a\leqslant n} \frac{a^{\alpha+\delta}}{n^{\alpha+\delta+1}} \Bigg| \\
= \Bigg| &\frac{\alpha+1}{n^{\alpha+1}} \sum_{a\in A, a\leqslant n} a^\alpha \Big(1 - \Big(\frac{a}{n}\Big)^\delta\Big) - \frac{\delta}{n^{\alpha+\delta+1}} \sum_{a\in A, a\leqslant n} a^{\alpha+\delta} \\
&+ \varepsilon_1(n) \frac{\alpha+1}{n^{\alpha+1}} \sum_{a\in A, a\leqslant n} a^\alpha - \varepsilon_2(n) \frac{\alpha+\delta+1}{n^{\alpha+\delta+1}} \sum_{a\in A, a\leqslant n} a^{\alpha+\delta} \Bigg| \\
\leqslant\ &S_1(n) + S_2(n) + |\varepsilon_1(n)| S_3(n) + |\varepsilon_2(n)| S_4(n)
\end{aligned}$$

where

$$\begin{aligned}
S_1(n) &= \frac{\alpha+1}{n^{\alpha+1}} \sum_{a\leqslant n} a^\alpha \Big(1 - \Big(\frac{a}{n}\Big)^\delta\Big), \\
S_2(n) &= \frac{\delta}{n^{\alpha+\delta+1}} \sum_{a\leqslant n} a^{\alpha+\delta}, \\
S_3(n) &= \frac{\alpha+1}{n^{\alpha+1}} \sum_{a\leqslant n} a^\alpha
\end{aligned}$$

and

$$S_4(n) = \frac{\alpha+\delta+1}{n^{\alpha+\delta+1}} \sum_{a\leqslant n} a^{\alpha+\delta}.$$

It is clear that

$$\lim_{n\to\infty} S_2(n) = \frac{\delta}{\alpha+\delta+1} \leqslant \frac{\delta}{\alpha+1}$$

and also

$$\lim_{n\to\infty} S_3(n) = \lim_{n\to\infty} S_4(n) = 1.$$

We have

$$S_1(n) = (\alpha+1) \sum_{k=1}^{n} \Big(\frac{k}{n}\Big)^\alpha \Big(1 - \Big(\frac{k}{n}\Big)^\delta\Big) \frac{1}{n}.$$

The last sum is an integral sum of the (integrable) function $\varphi(x) = x^\alpha(1-x^\delta)$ in the interval $[0,1]$; hence

$$\begin{aligned}\lim_{n\to\infty}\sum_{k=1}^{n}\Big(\frac{k}{n}\Big)^\alpha\Big(1-\Big(\frac{k}{n}\Big)^\delta\Big)\frac{1}{n} &= \int_0^1 x^\alpha(1-x^\delta)\,\mathrm{d}x\\ &= \frac{1}{\alpha+1}-\frac{1}{\alpha+\delta+1}=\frac{\delta}{(\alpha+1)(\alpha+\delta+1)}\leqslant\frac{\delta}{(\alpha+1)^2}.\end{aligned}$$

Thus we have

$$\begin{aligned}|\underline{d}_\alpha(A)-\underline{d}_{\alpha+\delta}(A)| &\leqslant \limsup_{n\to\infty}\Delta_n(A)\\ &\leqslant \lim_{n\to\infty}(S_1(n)+S_2(n)+|\varepsilon_1(n)|S_3(n)+|\varepsilon_2(n)|S_4(n))\\ &\leqslant \frac{\delta}{\alpha+1}+\frac{\delta}{\alpha+1}+0+0=\frac{2\delta}{\alpha+1}.\end{aligned}$$

The corresponding inequality for the upper densities can be derived by simple observation that

$$\begin{aligned}|\bar{d}_\alpha(A)-\bar{d}_{\alpha+\delta}(A)| &= |(1-\underline{d}_\alpha(\mathbb{N}-A))-(1-\underline{d}_{\alpha+\delta}(\mathbb{N}-A))|\\ &= |\underline{d}_\alpha(\mathbb{N}-A)-\underline{d}_{\alpha+\delta}(\mathbb{N}-A)|\leqslant\frac{2\delta}{\alpha+1}\end{aligned}$$

as the last inequality has already been proved for all subsets of $\mathbb{N}$. □

Remark 1. Since for all $\alpha > -1$ and all δ such that $0 < \delta < \alpha + 1$ we have $\alpha - \delta > -1$, the statement of the theorem can be applied to the pair $a - \delta > -1$ and $\alpha = (\alpha - \delta) + \delta$ to get

$$|\underline{d}_\alpha(A)-\underline{d}_{\alpha-\delta}(A)|\leqslant\frac{2\delta}{\alpha-\delta+1}\quad\text{and}\quad|\bar{d}_\alpha(A)-\bar{d}_{\alpha-\delta}(A)|\leqslant\frac{2\delta}{\alpha-\delta+1}$$

for all $A \subset \mathbb{N}$.

Thus we have direct consequences of the above theorem.

Corollary 1. *Given a set $A \subset \mathbb{N}$, the function $\alpha \mapsto D_\alpha(A)$ is Lipschitzian on each closed half-line $[a_0, \infty[$, with $a_0 > -1$ fixed.*

Corollary 2. *Given a set $A \subset \mathbb{N}$, the function $\alpha \mapsto D_\alpha(A)$ is continuous on* $]-1, \infty[$.

3. The continuity at -1

Let A be a fixed subset of $\mathbb{N}$. In this section we shall study continuity of the α-density points as $\alpha \to -1^+$. We assume that the set $A \subseteq \mathbb{N}$ is neither finite nor cofinite, so that it can be written in the form

$$A = \mathbb{N} \cap \left(\bigcup_{n=1}^{\infty}]a_n, b_n] \right)$$

for two suitable sequences of integers $(a_n)_{n\geqslant 1}$ and $(b_n)_{n\geqslant 1}$ such that $a_n < b_n < a_{n+1}$ for every n. We recall that

$$\mathbb{N}_\alpha(n) = \sum_{k=1}^{n} k^\alpha, \qquad n \in \mathbb{N}.$$

By an application of Theorem 8.2 of [1], we are able to calculate the upper and lower α-densities of A as follows:

Theorem A. *The following relations hold:*

$$\underline{d}_\alpha(A) = \liminf_{n\to\infty} \frac{\sum_{k=1}^{n-1} (\mathbb{N}_\alpha(b_k) - \mathbb{N}_\alpha(a_k))}{\mathbb{N}_\alpha(a_n)}, \tag{1}$$

$$\bar{d}_\alpha(A) = \limsup_{n\to\infty} \frac{\sum_{k=1}^{n} (\mathbb{N}_\alpha(b_k) - \mathbb{N}_\alpha(a_k))}{\mathbb{N}_\alpha(b_n)}.$$

The following result is also easy to prove:

Lemma B. *For $\alpha > -1$ the values $\bar{d}_\alpha(A)$ and $\underline{d}_\alpha(A)$ can be also calculated as*

$$\bar{d}_\alpha(A) = \limsup_{n\to\infty} \frac{\sum_{k=1}^{n} (b_k^{1+\alpha} - a_k^{1+\alpha})}{b_n^{1+\alpha}} \tag{2}$$

and

$$(3) \qquad \underline{d}_\alpha(A) = \liminf_{n\to\infty} \frac{\sum_{k=1}^{n-1} (b_k^{1+\alpha} - a_k^{1+\alpha})}{a_n^{1+\alpha}},$$

while for $\alpha = -1$ *we have*

$$(4) \qquad \bar{d}_{-1}(A) = \limsup_{n\to\infty} \frac{\sum_{k=1}^{n} (\log b_k - \log a_k)}{\log b_n},$$

$$(5) \qquad \underline{d}_{-1}(A) = \liminf_{n\to\infty} \frac{\sum_{k=1}^{n-1} (\log b_k - \log a_k)}{\log a_n}$$

respectively.

Lemma B follows from the equivalence relations, as $n \to \infty$,

$$n^\alpha \sim \begin{cases} \dfrac{1}{1+\alpha}((n+1)^{\alpha+1} - n^{\alpha+1}) & \text{for } \alpha > -1, \\ \log(n+1) - \log n & \text{for } \alpha = -1 \end{cases}$$

using the same arguments as in Theorem 3.2 of [1] or in Lemma 1 of [5].

In the sequel we set, for each n,

$$C_n = \log b_n - \log a_n; \quad B_n = \log b_n - \log b_{n-1}; \quad A_n = \log a_n - \log a_{n-1}.$$

Also we will suppose that the sequence $(B_n)_{n\geqslant 1}$ is bounded (assumption (*H*)). This easily implies that $(A_n)_{n\geqslant 1}$ and $(C_n)_{n\geqslant 1}$ are bounded as well.

We have the following result.

Theorem 2. *In addition to assumption (H), suppose that*

$$(6) \qquad L \doteq \liminf_{n\to\infty} C_n > 0.$$

Then we have

$$(7) \qquad \lim_{\alpha\to -1^+} \bar{d}_\alpha(A) = \bar{d}_{-1}(A),$$

$$(8) \qquad \lim_{\alpha\to -1^+} \underline{d}_\alpha(A) = \underline{d}_{-1}(A).$$

The following example shows that assumption (*H*) cannot be dropped.

Example 2. For the set $A = \mathbb{N} \cap \left(\bigcup_{n=1}^{\infty}]a_n, b_n] \right)$ with

$$a_n = n((n-1)!)^2, \qquad b_n = (n!)^2,$$

assumption (H) is not satisfied.

In fact, we have

$$C_n = \log(n!)^2 - \log n((n-1)!)^2 = \log n,$$

which is not bounded. Now, by means of Theorem A and relations (2), (3), (4) and (5) it is easy to verify that

$$\underline{d}_{-1}(A) = \bar{d}_{-1}(A) = d_{-1}(A) = \frac{1}{2},$$

while for every $\alpha > -1$ we have

$$\underline{d}_{\alpha}(A) = 0; \qquad \bar{d}_{\alpha}(A) = 1,$$

hence neither of the functions $\alpha \mapsto \underline{d}_{\alpha}(A)$, $\alpha \mapsto \bar{d}_{\alpha}(A)$ is continuous at -1.

Theorem 2 covers evidently the rather relevant case of sets such as the set E_r of numbers beginning by a fixed digit r ($r \in \{1, 2, \ldots, 9\}$), i.e.

$$E_r = \mathbb{N} \cap \left(\bigcup_{n=1}^{\infty}]r10^n - 1, (r+1)10^n - 1] \right),$$

but it is not useful for instance for the set of even numbers (or the set of multiples of any other integer, of course). In fact, here we have $a_n = 2n - 1$, $b_n = 2n$ and

$$\liminf_{n \to \infty} (\log b_n - \log a_n) = 0.$$

Observe that in this case the limit

$$\lim_{n \to \infty} \frac{\log b_n - \log a_n}{\log b_n - \log b_{n-1}} = \lim_{n \to \infty} \frac{C_n}{B_n}$$

exists $(= 1/2)$.

In fact, for a general set $A = \mathbb{N} \cap \left(\bigcup_{n=1}^{\infty}]a_n, b_n] \right)$ the following result holds (we keep the notation used for Theorem 2):

Theorem 3. *Let assumption (H) hold and suppose that the limit*

$$\lim_{n\to\infty} \frac{C_n}{B_n}$$

exists and is equal to L. Put $b_0 = 1$. Then

(i) *A possesses logarithmic density $d_{-1}(A) = L$;*

(ii) *there exists α_0 and a positive constant c such that for $-1 < \alpha < \alpha_0$ we have*

$$\limsup_{n\to\infty} \left| \frac{\sum\limits_{k=1}^{n} (b_k^{1+\alpha} - a_k^{1+\alpha})}{\sum\limits_{k=1}^{n} (b_k^{1+\alpha} - b_{k-1}^{1+\alpha})} - \frac{\sum\limits_{k=1}^{n} C_k}{\sum\limits_{k=1}^{n} B_k} \right| \leqslant c(1+\alpha). \tag{9}$$

As a consequence we get

$$\lim_{\alpha\to -1^+} \underline{d}_\alpha(A) = \lim_{\alpha\to -1^+} \bar{d}_\alpha(A) = d_{-1}(A).$$

The rest of this section is devoted to the proofs of Theorems 2 and 3.

P r o o f o f T h e o r e m 2. We shall prove relation (7) only, since (8) has an identical proof (simply replace (b_n) with (a_n) and use the part of Theorem A that concerns lower density, along with relations (3) and (5)). We start with a remark.

Remark 2. Since $C_n \leqslant B_n$ and $C_n \leqslant A_{n+1}$, assumption (6) implies that also $M \doteq \liminf\limits_{n\to\infty} B_n > 0$ and $N \doteq \liminf\limits_{n\to\infty} A_n > 0$.

We need a famous result:

Lemma (Abel) [3]. *Let $(r_n)_n$ and $(s_n)_n$ be any two sequences of real numbers. Then*

$$\sum_{k=1}^{n} r_k s_k = \left(\sum_{k=1}^{n} r_k\right) s_n - \sum_{k=1}^{n-1} \left(\sum_{h=1}^{k} r_h\right)(s_{k+1} - s_k).$$

In (2), the fraction can be replaced by

$$\frac{\sum\limits_{k=1}^{n} (b_k^{1+\alpha} - a_k^{1+\alpha})}{b_n^{1+\alpha} - 1} = \frac{\sum\limits_{k=1}^{n} (b_k^{1+\alpha} - a_k^{1+\alpha})}{\sum\limits_{k=1}^{n} (b_k^{1+\alpha} - b_{k-1}^{1+\alpha})} \qquad (b_0 = 1). \tag{10}$$

Concerning its numerator, Abel's lemma will be applied with

$$r_k = C_k, \qquad s_k = \frac{b_k^{1+\alpha} - a_k^{1+\alpha}}{C_k},$$

so we obtain

$$\sum_{k=1}^{n}(b_k^{1+\alpha}-a_k^{1+\alpha}) = \Big(\sum_{k=1}^{n} C_k\Big)\frac{b_n^{1+\alpha}-a_n^{1+\alpha}}{C_n} - \sum_{k=1}^{n-1}\Big(\sum_{h=1}^{k} C_h\Big)\Big(\frac{b_{k+1}^{1+\alpha}-a_{k+1}^{1+\alpha}}{C_{k+1}} - \frac{b_k^{1+\alpha}-a_k^{1+\alpha}}{C_k}\Big). \tag{11}$$

As to the denominator of (10), another application of Abel's lemma with

$$r_k = B_k, \qquad s_k = \frac{b_k^{1+\alpha}-b_{k-1}^{1+\alpha}}{B_k},$$

gives

$$\sum_{k=1}^{n}(b_k^{1+\alpha}-b_{k-1}^{1+\alpha}) = \Big(\sum_{k=1}^{n} B_k\Big)\frac{b_n^{1+\alpha}-b_{n-1}^{1+\alpha}}{B_n} - \sum_{k=1}^{n-1}\Big(\sum_{h=1}^{k} B_h\Big)\Big(\frac{b_{k+1}^{1+\alpha}-b_k^{1+\alpha}}{B_{k+1}} - \frac{b_k^{1+\alpha}-b_{k-1}^{1+\alpha}}{B_k}\Big). \tag{12}$$

In view of the above formulas (11) and (12), in order to get the statement of Theorem 2 it will be enough to show that

$$\lim_{\alpha\to-1^+}\sup_n\Big|\frac{b_n^{1+\alpha}-a_n^{1+\alpha}}{(1+\alpha)C_n b_n^{1+\alpha}} - 1\Big| = 0, \tag{13}$$

$$\lim_{\alpha\to-1^+}\sup_n \frac{\sum\limits_{k=1}^{n-1}\Big(\sum\limits_{h=1}^{k} C_h\Big)\Big(\frac{b_{k+1}^{1+\alpha}-a_{k+1}^{1+\alpha}}{C_{k+1}} - \frac{b_k^{1+\alpha}-a_k^{1+\alpha}}{C_k}\Big)}{(1+\alpha)b_n^{1+\alpha}\Big(\sum\limits_{k=1}^{n} C_k\Big)} = 0, \tag{14}$$

and two analogous relations concerning (12) (with B_k replacing C_k and b_{k-1} replacing a_k).

We shall prove only (13) and (14).

In order to get (13), put, for $x > 0$,

$$H(x) = \frac{1-\mathrm{e}^{-x}}{x},$$

and recall the inequality

$$|H(x)-1| \leqslant \frac{x}{2}.$$

Hence

$$\sup_n \left| \frac{b_n^{1+\alpha} - a_n^{1+\alpha}}{(1+\alpha) b_n^{1+\alpha} C_n} - 1 \right| = \sup_n \left| H\big((1+\alpha)C_n\big) - 1 \right| \leqslant \frac{1}{2}(1+\alpha) \sup_n C_n,$$

which concludes the proof of (13).

The proof of (14) is longer. We remark that, by assumption (H) and relation (6), we have

$$\liminf_{n\to\infty} \frac{\sum\limits_{k=1}^{n} C_k}{\log b_n} > 0.$$

This allows us to replace the term $\sum\limits_{k=1}^{n} C_k$ in the denominator of (14) by $\log b_n$; so, we shall prove that

$$\lim_{\alpha\to-1^+} \sup_n \frac{\sum\limits_{k=1}^{n-1} \Big(\sum\limits_{h=1}^{k} C_h \Big) \Big(\frac{b_{k+1}^{1+\alpha} - a_{k+1}^{1+\alpha}}{C_{k+1}} - \frac{b_k^{1+\alpha} - a_k^{1+\alpha}}{C_k} \Big)}{(1+\alpha) b_n^{1+\alpha} \log b_n} = 0. \tag{15}$$

We now need some lemmas.

Lemma 1. *The sequence $(n/\log b_n)_n$ is bounded.*

P r o o f. Recall that $M = \liminf\limits_{n\to\infty} B_n > 0$; fix ε, with $0 < \varepsilon < M$. There exists an integer n_0 such that, for $n \geqslant n_0$, we have

$$B_n \geqslant M - \varepsilon,$$

hence, for $n \geqslant k \geqslant n_0$, we get

$$\log b_n - \log b_k = \sum_{h=k+1}^{n} B_h \geqslant (M-\varepsilon)(n-k). \tag{16}$$

In particular, for $n \geqslant k = n_0$ we find

$$\log b_n \geqslant \log b_{n_0} + (M-\varepsilon)(n-n_0),$$

which completes the proof. □

Lemma 2. *Let $(D_k)_k$ and $(E_k)_k$ be any two positive bounded sequences and let m be a nonnegative integer. Then*

$$\sum_{k=1}^{n-1}\Bigl(\sum_{h=1}^{k} D_h\Bigr) b_{k+m}^{1+\alpha} E_k = O(b_n^{1+\alpha}\log b_n).$$

P r o o f. Since $(D_k)_k$ and $(E_k)_k$ are bounded, it is enough to prove the statement for $D_k = E_k = 1$ for every k, i.e. for the sequence

$$\sum_{k=1}^{n-1} k b_{k+m}^{1+\alpha}.$$

Let ε, n_0 be as in Lemma 1, and let $n \geqslant k \geqslant n_0$. Relation (16) can be written in the equivalent form

$$\frac{b_{k+m}}{b_n} \leqslant \mathrm{e}^{-(M-\varepsilon)(n-k-m)},$$

hence there exist positive constants c_1 and c_2 such that for $n > n_0 - 1$ we have

$$\begin{aligned}\frac{\sum\limits_{k=1}^{n-1} k b_{k+m}^{1+\alpha}}{b_n^{1+\alpha}\log b_n} &= \frac{\sum\limits_{k=1}^{n_0} k b_{k+m}^{1+\alpha}}{b_n^{1+\alpha}\log b_n} + \frac{\sum\limits_{k=n_0+1}^{n-1} k (b_{k+m}/b_n)^{1+\alpha}}{\log b_n}\\ &\leqslant \frac{c_1}{b_n^{1+\alpha}\log b_n} + n\frac{\sum\limits_{k=1}^{n-1}\mathrm{e}^{-(M-\varepsilon)(1+\alpha)(n-k-m)}}{\log b_n} \leqslant \frac{c_1}{b_n^{1+\alpha}\log b_n} + c_2\frac{n}{\log b_n},\end{aligned}$$

as

$$\sum_{k=1}^{n-1}\mathrm{e}^{-(M-\varepsilon)(1+\alpha)(n-k-m)} \leqslant \sum_{i=1-m}^{\infty}\mathrm{e}^{-(M-\varepsilon)(1+\alpha)i} = \frac{\mathrm{e}^{-(M-\varepsilon)(1+\alpha)(1-m)}}{(1-\mathrm{e}^{-(M-\varepsilon)(1+\alpha)})} = c_2.$$

An application of Lemma 1 completes the proof of Lemma 2. □

Lemma 3. *For every integer k we have*

$$\Bigl|\frac{b_{k+1}^{1+\alpha} - a_{k+1}^{1+\alpha}}{C_{k+1}} - \frac{b_k^{1+\alpha} - a_k^{1+\alpha}}{C_k}\Bigr| \leqslant (1+\alpha)^2 b_{k+1}^{1+\alpha} B_{k+1} + (1+\alpha)G(\alpha) b_k^{1+\alpha},$$

where G is a function such that

$$\lim_{\alpha\to -1^+} G(\alpha) = 0.$$

Proof. The reader can verify the equality

$$\frac{b_{k+1}^{1+\alpha}-a_{k+1}^{1+\alpha}}{C_{k+1}}-\frac{b_k^{1+\alpha}-a_k^{1+\alpha}}{C_k}=(1+\alpha)^2 b_{k+1}^{1+\alpha}H((1+\alpha)B_{k+1})H((1+\alpha)C_{k+1})B_{k+1} + (1+\alpha)\big(H((1+\alpha)C_{k+1})-H((1+\alpha)C_k)\big)b_k^{1+\alpha}$$

by substituting $H(x)=(1-\mathrm{e}^{-x})/x$ (with corresponding arguments) into its right hand side.

It is now enough to recall that $0\leqslant H(x)\leqslant 1$ and to put

$$G(\alpha)=\sup_k |H((1+\alpha)C_{k+1})-H((1+\alpha)C_k)|.$$

The fact that $\lim\limits_{\alpha\to-1^+}G(\alpha)=0$ follows from the Lagrange theorem:

$$|H((1+\alpha)C_{k+1})-H((1+\alpha)C_k)|\leqslant 2(1+\alpha)(\sup_k C_k)\sup_x|H'(x)|,$$

and it is easily verified that

$$\sup_x|H'(x)|=\sup_x\Big|\frac{x\mathrm{e}^{-x}-1+\mathrm{e}^{-x}}{x^2}\Big|=\sup_x\Big|\frac{1-\mathrm{e}^{-x}}{x}+\frac{1-\mathrm{e}^{-x}-x}{x^2}\Big|\leqslant\frac{3}{2}.$$

□

Relation (15) now follows by applying Lemma 3 and Lemma 2 with $m=0$ and $m=1$. This concludes the proof of Theorem 2.

Proof of Theorem 3. (i) is immediate, since

$$d_{-1}(A)=\lim_{n\to\infty}\frac{\sum\limits_{k=1}^{n}C_k}{\sum\limits_{k=1}^{n}B_k}=\lim_{n\to\infty}\frac{C_n}{B_n}=L$$

by Cesaro's theorem.

(ii) We need some algebra in order to write the first member of (9) in a suitable manner. By reducing to the common denominator we get that it is equal to

$$A=\frac{\sum\limits_{h,k=1}^{n}b_k^{1+\alpha}(1-\mathrm{e}^{-(1+\alpha)C_k})B_h-\sum\limits_{h,k=1}^{n}b_k^{1+\alpha}(1-\mathrm{e}^{-(1+\alpha)B_k})C_h}{\sum\limits_{h,k=1}^{n}b_k^{1+\alpha}(1-\mathrm{e}^{-(1+\alpha)B_k})B_h}. \tag{17}$$

Recall the definition of the function H (see Lemma 3). Then each summand in the denominator of the fraction A in (17) is equal to

$$(1+\alpha)b_k^{1+\alpha}H((1+\alpha)B_k)B_hB_k. \tag{18}$$

Moreover, in the first (second) parenthesis of the numerator of A we subtract and add the term $(1+\alpha)C_k$ ($(1+\alpha)B_k$, respectively) and separate the sums in order to split the fraction A of (17) into tree summands

$$A = R - S + T,$$

where (recall the expression (18))

$$R = \frac{\sum\limits_{h,k=1}^{n} b_k^{1+\alpha}(1-\mathrm{e}^{-(1+\alpha)C_k}-(1+\alpha)C_k)B_h}{(1+\alpha)\sum\limits_{h,k=1}^{n} b_k^{1+\alpha}H((1+\alpha)B_k)B_hB_k},$$

$$S = \frac{\sum\limits_{h,k=1}^{n} b_k^{1+\alpha}(1-\mathrm{e}^{-(1+\alpha)B_k}-(1+\alpha)B_k)C_h}{(1+\alpha)\sum\limits_{h,k=1}^{n} b_k^{1+\alpha}H((1+\alpha)B_k)B_hB_k},$$

$$T = \frac{\sum\limits_{h,k=1}^{n} b_k^{1+\alpha}C_kB_h - \sum\limits_{h,k=1}^{n} b_k^{1+\alpha}C_hB_k}{\sum\limits_{h,k=1}^{n} b_k^{1+\alpha}H((1+\alpha)B_k)B_hB_k}.$$

We recall the inequality ($x \geqslant 0$)

$$0 \leqslant \mathrm{e}^{-x} - 1 + x \leqslant \frac{x^2}{2}$$

and remark that, since (B_n) is positive and bounded, we have

$$\lim_{\alpha\to -1^+} \sup_k H((1+\alpha)B_k) = 1,$$

hence there exists $\alpha_0 > -1$ such that for $-1 < \alpha < \alpha_0$ we have

$$\sup_k H((1+\alpha)B_k) > \frac{1}{2},$$

so that

$$R \leqslant \frac{(1+\alpha)^2 \sum\limits_{h,k=1}^{n} b_k^{1+\alpha} C_k^2 B_h}{(1+\alpha) \sum\limits_{h,k=1}^{n} b_k^{1+\alpha} B_h B_k} \leqslant (1+\alpha)(\sup_k C_k)$$

(since $C_k \leqslant B_k$).

In an analogous way we also find

$$S \leqslant (1+\alpha)(\sup_k B_k).$$

Last, for $-1 < \alpha < \alpha_0$ we have

$$(19) \qquad |T| \leqslant \frac{\sum\limits_{h,k=1}^{n} b_k^{1+\alpha} |C_k B_h - C_h B_k|}{\sum\limits_{h,k=1}^{n} b_k^{1+\alpha} H((1+\alpha)B_k) B_h B_k} \leqslant \frac{\sum\limits_{h,k=1}^{n} b_k^{1+\alpha} B_h B_k |(C_k/B_k) - (C_h/B_h)|}{(1/2)\Big(\sum\limits_{k=1}^{n} b_k^{1+\alpha} B_k\Big) \log b_n}.$$

Fix $\varepsilon > 0$ and n_0 such that, for $n > n_0$, we have

$$L - \varepsilon < \frac{C_n}{B_n} < L + \varepsilon.$$

As the sequence (C_n/B_n) is bounded, the last term of (19) is not greater than

$$\frac{\sum\limits_{h,k=1}^{n_0} b_k^{1+\alpha} B_h B_k |(C_k/B_k) - (C_h/B_h)|}{(1/2)\Big(\sum\limits_{k=1}^{n} b_k^{1+\alpha} B_k\Big) \log b_n} + c_3 \frac{\sum\limits_{h=1}^{n_0} B_h}{\log b_n} + c_4 \frac{\sum\limits_{h=1}^{n_0} b_h^{1+\alpha} B_h}{\sum\limits_{h=1}^{n} b_h^{1+\alpha} B_h} + 4\varepsilon,$$

where c_3 and c_4 are suitable positive constants. Now the statement follows since $(\log b_n)$ and $\Big(\sum\limits_{h=1}^{n} b_h^{1+\alpha} B_h\Big)$ go to ∞. □

4. An open problem

Problem. *We have seen that to any given set $A \subset \mathbb{N}$ we can attach a pair of functions*

$$\underline{d}_A\colon\ [-1,\infty[\to[0,1] \quad \text{and} \quad \bar{d}_A\colon\ [-1,\infty[\to[0,1],$$

both continuous in the interval $]-1,\infty[$ and such that $\underline{d}_A$ is nonincreasing, $\bar{d}_A$ is nondecreasing and $\underline{d}_A(\alpha) \leqslant \bar{d}_A(\alpha)$ for all $\alpha \in [-1,\infty[$.

A natural question that arises is:

For which pairs of functions $\underline{d}, \bar{d}$ with properties listed above there exists a set $A \subset \mathbb{N}$ such that

$$\underline{d}_A = \underline{d} \quad \text{and} \quad \bar{d}_A = \bar{d}?$$

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