Open Problems on Densities II

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Abstract. This is a collection of open questions and problems concerning various density concepts on subsets of $\mathbb{N} := \{1,2,3,\dots\}$. It is a continuation of paper [10].
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1. INTRODUCTION

We deal with densities of subsets of the natural numbers. The common point in all densities is that they are functions f defined on $\mathscr{P}(\mathbb{N})$, the power set of $\mathbb{N}$, or on a part of $\mathscr{P}(\mathbb{N})$, such that $f(\emptyset) = 0$, $f(\mathbb{N}) = 1$ and if $A \subseteq B$ and $f(A)$, $f(B)$ are defined, then $f(A) \leq f(B)$. This definition is too general to be useful, thus many specific density concepts are used in Number Theory, Analysis, Combinatorics or Social Sciences.
We define in this section the most "classical" notions of density. We shall define more specific density concepts in later sections, when this will be necessary. Other density concepts, including axiomatic ones (see [4]), can be found in the survey [9].

In all density concepts that we use hereafter, there is a notion of *upper* density and a notion of *lower* density. Each density is denoted by a letter: d for the asymptotic density, δ for the logarithmic density, and so on (see below). The letter, if overlined, denotes the corresponding upper density. The letter, if underlined, denotes the corresponding lower density. If the upper and the lower densities are equal, we say that the set has a density. In this case, the corresponding letter is neither overlined nor underlined.

Some notations: $\mathbb{R}$ denotes the set of real numbers, $\mathbb{N}$ denotes the set of positive integers: $\mathbb{N} = \{1, 2, 3, \ldots\}$. For a subset A of $\mathbb{N}$ and real numbers x, y, $1 \leq x \leq y$, let $A(x,y) = |A \cap [x,y]|$, where $|X|$ denotes the cardinality of the finite set X. $A(x) = A(1,x)$ denotes the *counting function* of the set A.

1.1. Asymptotic (or **natural**) **density.**

The *upper* and *lower asymptotic densities* of a subset A of $\mathbb{N}$ are defined, respectively, as the upper and lower limits of $x^{-1}A(x)$ when x tends to $+\infty$. They are denoted by $\overline{d}(A)$ and $\underline{d}(A)$, respectively. If $\overline{d}(A) = \underline{d}(A)$, we say that A has an asymptotic density and we write $d(A)$ for the limit of $x^{-1}A(x)$ when x tends to $+\infty$.

1.2. Logarithmic density.

The *upper logarithmic density* is defined as :

$$\overline{\delta}(A) = \limsup_{x \to \infty} \left(\sum_{a \in A, a \leq x} \frac{1}{a} \right) \left(\sum_{a \leq x} \frac{1}{a} \right)^{-1} = \limsup_{x \to \infty} \left(\sum_{a \in A, a \leq x} \frac{1}{a} \right) (\ln\, x)^{-1}.$$

The *lower logarithmic density* is defined as :

$$\underline{\delta}(A) = \liminf_{x \to \infty} \left(\sum_{a \in A, a \leq x} \frac{1}{a} \right) \left(\sum_{a \leq x} \frac{1}{a} \right)^{-1} = \liminf_{x \to \infty} \left(\sum_{a \in A, a \leq x} \frac{1}{a} \right) (\ln\, x)^{-1}.$$

1.3. Weighted densities.

The so called *weighted* or *generalized* densities provide a generalization of both asymptotic and logarithmic densities. With respect to a positive "weight" sequence $a = (a_n)$, the $a-$*weighted densities* are defined, respectively, as the lower and the upper limits of

$$\left(\sum_{i \in A,\ i \leq n} a_i \right) \left(\sum_{k \leq n} a_k \right)^{-1}$$

as n tends to infinity. Usually one takes a sequence (a_n) satisfying

$$\sum_{n=1}^{\infty} a_n = +\infty$$

and

$$\lim_{n \to \infty} a_n \left(\sum_{k \leq n} a_k \right)^{-1} = 0.$$

See also [24] for further particular conditions.

The special case $a_n = n^\alpha,\ \alpha \geq -1$, generalizes both the asymptotic density $(\alpha = 0)$ and the logarithmic density $(\alpha = -1)$. Let us denote by $\underline{d}_\alpha(A)$ and $\overline{d}_\alpha(A)$ the lower and the upper densities, called $\alpha-$*densities*, of the set A, where the weight sequence is $a_n = n^\alpha,\ \alpha \geq -1$. See the next section for open problems on these densities in relation to the $\alpha-$analytic density.

1.4. Uniform (or **Banach) density.**

Let $A \subseteq \mathbb{N}$. For every real number s, $s \geq 1$, we define

$$\alpha_s = \liminf_{k\to\infty} \ A(k+1, k+s)$$

and

$$\alpha^s = \limsup_{k\to\infty} \ A(k+1, k+s).$$

Both α_s and α^s are integers belonging to $\{0, 1, \cdots, \lfloor s \rfloor\},\ \ \alpha_s \leq \alpha^s$. Each one, divided by s, **tends to a limit**, when s tends to infinity. A proof of this fact can be found in [23] or in [25].

The *lower uniform* (or *Banach*) *density* of $A \subseteq \mathbb{N}$ is defined as

$$\underline{u}(A) = \lim_{s\to\infty} \frac{\alpha_s}{s}$$

and its *upper uniform* (or *Banach*) *density* as

$$\overline{u}(A) = \lim_{s\to\infty} \frac{\alpha^s}{s}.$$

In [13], in addition to the existence of the above limits, several other properties of Banach density are proved and, in particular, that

$$\underline{u}(A) = \inf E \ \ ; \ \ \overline{u}(A) = \sup F$$

where

$$E := \{x \in [0,1] \ ; \ \forall \ell > 0, \exists \ell' > \ell \text{ and } k \text{ such that } \frac{A(k+1, k+\ell')}{\ell'} \leq x\}$$

and

$$F := \{x \in [0,1] \ ; \ \forall \ell > 0, \exists \ell' > \ell \text{ and } k \text{ such that } \frac{A(k+1, k+\ell')}{\ell'} \geq x\} \ .$$

1.5. Exponential density.

Let h be a positive increasing unbounded function defined on $\mathbb{R}_+ = [0, +\infty[$. We define the *lower h-density* of the set $A \subseteq \mathbb{N}$ by

$$\underline{\varepsilon}_h(A) = \liminf_{n\to\infty} \frac{h(A(n))}{h(n)}$$

and its *upper h-density* by

$$\overline{\varepsilon}_h(A) = \limsup_{n\to\infty} \frac{h(A(n))}{h(n)}.$$

The function $h(x) = \log x$ is frequently used and the corresponding h-density is called *exponential density*, here denoted by ε : $\overline{\varepsilon}(A)$ and $\underline{\varepsilon}(A)$ denote the upper and the lower exponential densities of A, respectively.

The exponential density ε acts as a magnifying glass on sets with asymptotic density zero. If $A = \{1^k, 2^k, 3^k, ...\}$, then $\varepsilon A = \frac{1}{k}$. Moreover, the preceding example motivates the name of this density ("exponential") since the exponent k determines εA.

The function $h(x) = x^\alpha$ with $\alpha > 0$ is not so interesting as it gives $\underline{\varepsilon}_h(A) = (\underline{d}A)^\alpha$ and similarly for upper densities.

1.6. Comparable and non comparable densities.

The density d is called *weaker* than the density d' if, for any set $A \subseteq \mathbb{N}$,

$$\underline{d'}(A) \le \underline{d}(A) \le \overline{d}(A) \le \overline{d'}(A)\ ;$$

the density d' is called *stronger* than the density d.
Two densities are called *comparable* if one of them is stronger than the other.
The uniform density is stronger than the $\alpha-$density for any $\alpha \ge -1$. It is well known (see for instance [15]) that the asymptotic density is stronger than the logarithmic one.
For further examples and for open problems on comparability of densities, see sections 2, 4, 6 and 7 below. The exponential density is not comparable to the uniform density, nor to the asymptotic density. An example is given in [9].

2. COMPARISON BETWEEN $\alpha-$DENSITY AND $\alpha-$ANALYTIC DENSITY

Let α be a real number greater than or equal to -1, and $A \subseteq \mathbb{N}$ a set of integers. We shall denote by 1_A the *characteristic function* of A. Put

$$A_\alpha(n) = \sum_{k\le n} k^\alpha 1_A(k)$$

and

$$D_{A,\alpha}(n) \doteq \frac{A_\alpha(n)}{\mathbb{N}_\alpha(n)}, \quad \Delta_{A,\alpha}(t) \doteq t \sum_{k\geq 1} k^\alpha e^{-t\mathbb{N}_\alpha(k)} 1_A(k).$$

Notice that $d_\alpha(A) = \lim_{n\to\infty} D_{A,\alpha}(n)$ and define the $\alpha-$analytic density of A as

$$\lim_{t\to 0^+} \Delta_{A,\alpha}(t) \doteq \delta_\alpha(A),$$

of course if these limits exist. Notice that $d_{-1}(A)$ and $\delta_{-1}(A)$ are, respectively, the classical logarithmic and analytic densities of A.
The following theorem (established in [5] in a more general context) links the two concepts of density introduced above (the case $\alpha = -1$ is well known, see for instance [21], p. 274).

Theorem 1. *Let $A \subseteq \mathbb{N}$ be a set of integers. For every real number $\ell \in [0,1]$, the two following conditions are equivalent:*

(a) A has α–density $d_\alpha(A) = \ell$.
(b) A has α–analytic density $\delta_\alpha(A) = \ell$.

When the $\alpha-$density (resp. the $\alpha-$analytic density) of A doesn't exist, one can consider the upper and lower densities.

Definition 1. *The lower and upper α–analytic densities of A are defined respectively as*

$$\underline{\delta}_\alpha(A) \doteq \liminf_{t\to 0^+} \Delta_{A,\alpha}(t), \qquad \overline{\delta}_\alpha(A) \doteq \limsup_{t\to 0^+} \Delta_{A,\alpha}(t).$$

In the paper [6], Prop. (2.1), it is proved that, in general, the upper $\alpha-$analytic density of A is not greater than the upper $\alpha-$density of A (concerning the lower densities, the inequality is obviously reversed).
A natural question is whether the same kind of result as Theorem 1 can be stated also for lower and upper $\alpha-$densities and lower and upper $\alpha-$analytic densities, *i.e.* whether the upper and the lower α–densities and the lower and upper α–analytic densities of A coincide. In Th. (3.1) of [6] it is shown that the answer is negative in general. On the other hand, in Section 4 of the same paper a class of subsets of $\mathbb{N}$ is identified for which the question can be answered affirmatively if $\alpha > -1$.

Open problem 1. *(a) Check the existence of such a class in the case $\alpha = -1$.*
(b) In the paper [22] *it is proved that, for $-1 \leq \alpha \leq \beta$*

$$\underline{d}_\beta(A) \leq \underline{d}_\alpha(A) \leq \overline{d}_\alpha(A) \leq \overline{d}_\beta(A).$$

Is it true that also

$$\underline{\delta}_\beta(A) \leq \underline{\delta}_\alpha(A) \leq \overline{\delta}_\alpha(A) \leq \overline{\delta}_\beta(A)?$$

3. PERMUTATIONS AND THE LÉVY GROUP

We focus our attention on the following situation. Let (a_n) be a sequence of positive numbers and $\pi : \mathbb{N} \to \mathbb{N}$ be an injective function; (a_n) and π will be fixed throughout. In the paper [8] a comparison is established between the d_a–densities (upper and lower) of a given set $A \subseteq \mathbb{N}$ and those of the transformed set $\pi(A)$ in terms of some suitable features of (a_n) and π. Put

$$e_k = \frac{a_{\pi(k)}}{a_k}, \qquad k \in \mathbb{N}, \quad e_0 = 0$$

and

$$\sigma'_\pi \doteq \limsup_{n\to\infty} \frac{1}{S_n} \sum_{\pi(k+1)\le n<\pi(k)} S_k e_{k+1} \ ,$$

where $S_n = \sum_{k=1}^{n} a_k$,

$$\sigma''_\pi \doteq \limsup_{n\to\infty} \frac{1}{S_n} \sum_{\substack{\pi(k)\le n\\ \pi(k+1)\le n}} S_k \big| e_k - e_{k+1} \big| \ ,$$

$$\sigma_\pi \doteq \sigma'_\pi + \sigma''_\pi .$$

Let $\pi(\mathbb{N})$ be the image of π and put

$$\underline{\ell} = \underline{d}_a(\pi(\mathbb{N})); \qquad \overline{\ell} = \overline{d}_a(\pi(\mathbb{N})).$$

In the paper [8] the following result is proved

Theorem 2. *(i) Assume that $\sigma_\pi < +\infty$. Then*

$$\sigma_\pi(\underline{d}_a(A) - \overline{d}_a(A)) + \underline{\ell}\,\underline{d}_a(A) \le \underline{d}_a(\pi(A)) \le \overline{d}_a(\pi(A)) \le \sigma_\pi(\overline{d}_a(A) - \underline{d}_a(A)) + \overline{\ell}\,\overline{d}_a(A).$$

(ii) Assume that the sequence (e_n) is non–increasing, and that

$$\sigma'''_\pi \doteq \limsup_{n\to\infty} \frac{1}{S_n} \sum_{\pi(k+1)\le n<\pi(k)} S_k < +\infty.$$

Then $\sigma'_\pi < +\infty$ and

$$\sigma'_\pi(\underline{d}_a(A) - \overline{d}_a(A)) + \underline{\ell}\,\underline{d}_a(A) \le \underline{d}_a(\pi(A)) \le \overline{d}_a(\pi(A)) \le \sigma'_\pi(\overline{d}_a(A) - \underline{d}_a(A)) + \overline{\ell}\,\overline{d}_a(A).$$

The following Corollary is an immediate consequence:

Corollary 1. *In addition to the assumptions of Theorem 2 (i) or those of Theorem 2 (ii), suppose that $\pi(\mathbb{N})$ has a d_a-density equal to ℓ. Then, if $d_a(A)$ exists, then also $d_a(\pi(A))$ exists and*

$$d_a(\pi(A)) = \ell\, d_a(A).$$

We point out the following particular case of Corollary 1, concerning the case of the classical asymptotic density d:

Corollary 2. *Let π be a permutation of the integers such that $d\big(\pi(\mathbb{N})\big)$ exists and is equal to 1 and*

$$(*)\qquad \limsup_{n\to+\infty}\frac{1}{n}\sum_{\pi(k+1)\le n<\pi(k)} k<+\infty.$$

Then π preserves the asymptotic density.

The *Lévy group* $\mathscr{G}$ is defined as the group of all permutations π of $\mathbb{N}$ satisfying

$$\lim_{n\to\infty}\frac{|k:k\le n<\pi(k)|}{n}=0.$$

A lemma from [2] says that a permutation π belongs to $\mathscr{G}$ if and only if it preserves the asymptotic density.

Open problem 2. *Establish*
1) whether the set $\mathscr{S}$ of permutations verifying $(*)$ is a subgroup of $\mathscr{G}$;
2) what is the relation between $\mathscr{S}$ and $\mathscr{G}$.

4. THE DIRICHLET'S WEIGHTED DENSITIES

Let $a=(a_n)$ and $S=(S_n)$ be two sequences of positive numbers, and let $\alpha\in\mathbb{R}$; put

$$w_n^{(\alpha)}=a_n\,e^{\alpha S_n}.$$

We make first some remarks (see [16] for details):
(a) The series

$$(1)\qquad \sum_n w_n^{(\alpha)}=\sum_n a_n\,e^{\alpha S_n}$$

is called a *Dirichlet's Series* (shortened D.S.).
(b) Put

$$\alpha_0\doteq-\limsup_{n\to\infty}\frac{\log(\sum_{k=1}^n a_k)}{S_n}.$$

The number α_0 is called the *abscissa of convergence* of the series (1): by a classical result, the series (1) converges for $\alpha<\alpha_0$ and diverges for $\alpha>\alpha_0$. Nothing can be said, in principle, for $\alpha=\alpha_0$, in the following sense: there are examples of D.S. which are divergent for $\alpha=\alpha_0$, and also examples of D.S. which are convergent for $\alpha=\alpha_0$. Also, it can happen that $\alpha_0=+\infty$ or $\alpha_0=-\infty$.

Example 1. *(i) The series*

$$\sum_n n^\alpha$$

has abscissa of convergence equal to -1*, and is divergent for* $\alpha = -1$.

(ii) *The series*

$$\sum_n n^\alpha / \log^2 n$$

has again abscissa of convergence equal to -1*, but is convergent for* $\alpha = -1$.

(iii) *Let a be a number, with* $0 < a < 1$*. The series*

$$\sum_n a^n n^\alpha$$

is convergent for all real numbers α.

(iv) *Let a be a number, with* $a > 1$*. The series*

$$\sum_n a^n n^\alpha$$

is divergent for all real numbers α.

Assume that $\alpha_0 > -\infty$. It is clear from the above remarks that, for $\alpha > \alpha_0$ (i. e. when the D.S. is divergent) it is possible to consider the weighted upper and lower densities defined by the weights $w_n^{(\alpha)}$, which we call *Dirichlet's weighted densities*, i. e. for every subset $A \subseteq \mathbb{N}$ we put

$$A_\alpha(n) = \sum_{k \in A, k \le n} w_k^{(\alpha)}$$

and define

$$\underline{d}_{a,S,\alpha}(A) = \liminf_{n\to\infty} \frac{A_\alpha(n)}{\mathbb{N}_\alpha(n)}, \qquad \overline{d}_{a,S,\alpha}(A) = \limsup_{n\to\infty} \frac{A_\alpha(n)}{\mathbb{N}_\alpha(n)}.$$

Notice that for $a_n = 1$ and $S_n = \log n$ we recover the classical $\alpha-$densities.
By a Theorem of [22], it is easy to see that the following result holds:

Proposition 1. *Let* $w_n^{(\alpha)} = a_n e^{\alpha S_n}$ *be fixed, and let* α_0 *be the abscissa of convergence of the associated D.S. Assume that* $\alpha_0 > -\infty$*. Then, for* $\beta > \alpha > \alpha_0$*, we have*

$$\underline{d}_{a,S,\beta}(A) \le \underline{d}_{a,S,\alpha}(A) \le \overline{d}_{a,S,\alpha}(A) \le \overline{d}_{a,S,\beta}(A).$$

In the paper [7] it is proved that the functions $\alpha \mapsto \underline{d}_{a,S,\alpha}(A)$ and $\alpha \mapsto \overline{d}_{a,S,\alpha}(A)$ with $a_n = 1$ and $S_n = \log n$ are continuous in $(\alpha_0, +\infty) = (-1, +\infty)$ and may be discontinuous at $\alpha_0 = -1$.

Open problem 3. *1) What can be said about the continuity of the Dirichlet's densities with respect to the parameter* α*? In particular, in which cases, if any, the continuity holds also for* $\alpha = \alpha_0$*?*

2) In which cases, if any, is it possible to define the associated Dirichlet analytic densities exactly as in the classical case (see the definition in [6]), i.e. to put

$$\Delta_{A,\alpha}(t) \doteq t \sum_{k\geq 1} a_n e^{\alpha S_n} e^{-t\mathbb{N}_{a,S,\alpha}(k)} 1_A(k),$$

$$\liminf_{t\to 0^+} \Delta_{A,\alpha}(t) \doteq \underline{\delta}_\alpha(A), \qquad \limsup_{t\to 0^+} \Delta_{A,\alpha}(t) \doteq \overline{\delta}_\alpha(A)$$

and to compare them with the weighted Dirichlet's densities?

5. DENSITY SETS

For $A \subseteq \mathbb{N}$ define

$$S(A) = \{(\overline{d}(B), \underline{d}(B));\ B \subseteq A\},$$

the *density set* of A. Denote by Tg the triangle $(0,0), (\overline{d}(A), 0), (\overline{d}(A), \underline{d}(A))$ and by Tz the trapezium $(0,0), (\overline{d}(A), 0), (\overline{d}(A), \underline{d}(A)), (\underline{d}(A), \underline{d}(A))$.

Theorem 3. [12] *For each $A \subseteq \mathbb{N}$ the set $S(A)$ is convex and closed with $Tg \subseteq S(A) \subseteq Tz$. On the other hand, for each convex closed set S with $Tg \subseteq S \subseteq Tz$, there exists $A \subseteq \mathbb{N}$ such that $S = S(A)$.*

In order to identify $S(A)$ it suffices to know its upper bound:

$$f\colon\ [0, \overline{d}(A)] \to [0, \underline{d}(A)];\ f(x) = \max\{y;\ (x,y) \in S(A)\}.$$

Gaps in A force $S(A)$ to be smaller but the opposite does not hold. To see this, for $A = \{a_1 < a_2 < \dots\} \subseteq \mathbb{N}$ define the value

$$\lambda(A) = \limsup_{n\to\infty} \frac{a_{n+1}}{a_n},$$

the *gap density* of A [14]. Its reciprocal value provides the upper bound for the right derivative of f at 0.

Theorem 4. [14] $f_r'(0) \leq \frac{1}{\lambda(A)} \leq 1$.

As a corollary we obtain that the whole set $S(A)$ lies below the line $y = \frac{1}{\lambda(A)} x$. We have seen that the occurrence of big gaps supplies partial information on $S(A)$. Gaps are extremal case of low local frequency of distribution of elements of A. What can be said in the case when no large gaps in A occur, i.e. $\lambda(A) = 1$? The following remark shows that nothing can be said.

Remark 1. *For each $A \subseteq \mathbb{N}$ there exists a $B \subseteq \mathbb{N}$ with $S(A) = S(B)$ and $\lambda(B) = 1$.*

In general, sudden decrease of the "local density", i.e. the value $\frac{n}{a_n}$, pushes down the upper bound of $S(A)$, the function f. How to measure decrease of the local density? There is no chance to do it by means of additive models like in uniform density, i.e. decrease of $A(n, n+m)$ w.r.t. $A(n-m, n)$ for fixed m and $n \to \infty$.

Perhaps there is some chance using

$$\gamma(A,\varepsilon) = \liminf_{n\to\infty} \frac{A(n,(1+\varepsilon)n)}{A((1-\varepsilon)n,n)}, \quad \gamma(A) = \lim_{\varepsilon\to 0^+} \gamma(A,\varepsilon)$$

where the $\liminf$ is taken through all values of n for which the denominator is positive. Notice that $\lambda(A) > 1$ implies $\gamma(A) = 0$.

Conjecture 1. *Perhaps* $\quad \gamma(A) = 1 \quad \Leftrightarrow \quad S(A) = Tz.$

Open problem 4. *Prove or disprove the conjecture. Notice that the opposite implication* $\gamma(A) = 0 \quad \Leftrightarrow \quad S(A) = Tg$ *does not hold.*

6. DENSITIES AND DISTRIBUTION FUNCTIONS

A non-decreasing function $g : [0,1] \to [0,1]$, $g(0) = 0$, $g(1) = 1$ is called a *distribution function*. We shall identify any two distribution functions coinciding at common points of continuity. It is well known that the set $\mathscr{D}$ of all distribution functions endowed with the $\mathscr{L}_2$ metric is a compact space. The following application of the theory of distribution functions, in order to study distribution properties of sets of positive integers, was started by Strauch and Tóth in [27].

Let $X = \{x_1 < x_2 < \ldots\} \subseteq \mathbb{N}$. We can form the *ratio block sequence* (X_n) where

$$X_n = \left(\frac{x_1}{x_n}, \frac{x_2}{x_n}, \ldots, \frac{x_n}{x_n}\right).$$

For each $n \in \mathbb{N}$ consider the *step distribution function*

$$F(X_n, x) = \frac{\#\{i \le n;\ \frac{x_i}{x_n} < x\}}{n}$$

and define the *set of distribution functions of the ratio block sequence*

$$G(X_n) = \{\lim_{k\to\infty} F(X_{n_k}, x)\}.$$

By compactness of $\mathscr{D}$, the set $G(X_n)$ is always nonempty and closed. The ratio block sequence (X_n) is uniformly distributed if $G(X_n) = \{id_{[0,1]}\}$.

There are results showing that information on $\underline{d}(X)$ and $\overline{d}(X)$ can substantially reduce the possible range of $G(X_n)$.

Theorem 5. [27] *Let $X \subseteq \mathbb{N}$ be such that $\underline{d}(X) > 0$. Then for every $g \in G(X_n)$ and $x \in [0,1]$*

$$\frac{\underline{d}(X)}{\overline{d}(X)} x \le g(x) \le \frac{\overline{d}(X)}{\underline{d}(X)} x.$$

In particular, if $d(X) > 0$ exists, then the ratio block sequence (X_n) is uniformly distributed.

The lower bound in the previous theorem can be slightly improved.

Theorem 6. [1] *Let $X \subseteq \mathbb{N}$ be such that $\underline{d} = \underline{d}(X) > 0$ and denote $\overline{d} = \overline{d}(X)$. Then for every $g \in G(X_n)$*

$$h_1(x) \le g(x) \le \frac{\overline{d}(X)}{\underline{d}(X)} x$$

where

$$h_1(x) = \begin{cases} x\frac{\underline{d}}{\overline{d}} & \text{if } x \in \left[0, \frac{1-\overline{d}}{1-\underline{d}}\right], \\ \frac{\underline{d}}{\frac{1}{x}-(1-\underline{d})} & \text{otherwise}, \end{cases}$$

and these bounds cannot be improved.

Notice that there is no information on $G(X_n)$ if $\underline{d}(X) = 0$.

Open problem 5. *In the case $\underline{d}(X) = 0$ try to find bounds for $G(X_n)$ in terms of the exponential density.*

The following kind of density was studied in a bit more general setting by Pólya [20]

$$\underline{\underline{d}}(A) = \lim_{\theta \to 1^-} \liminf_{n\to\infty} \frac{A(n) - A(\theta n)}{(1-\theta)n}, \quad \overline{\overline{d}}(A) = \lim_{\theta \to 1^-} \limsup_{n\to\infty} \frac{A(n) - A(\theta n)}{(1-\theta)n}$$

who calls them *minimal* and *maximal* densities of $A \subseteq \mathbb{N}$, respectively. It can be seen that

$$\underline{\underline{d}}(A) = \sup\{d(B),\ B \subseteq A\},\ \overline{\overline{d}}(A) = \inf\{d(B),\ B \supseteq A\}.$$

Evidently $\underline{\underline{d}}(A) \le \underline{d}(A) \le \overline{d}(A) \le \overline{\overline{d}}(A)$. There are examples of sets for which strict inequalities take place.

Example 2. *Let $A = \bigcup_{n=0}^{\infty} [2^{2n}, 2^{2n+1}) \cap \mathbb{N}$. Then*

$$0 = \underline{\underline{d}}(A) < \frac{1}{3} = \underline{d}(A) < \frac{2}{3} = \overline{d}(A) < 1 = \overline{\overline{d}}(A).$$

Open problem 6. *Find bounds of $G(X_n)$ in terms of densities $\underline{\underline{d}}$ and $\overline{\overline{d}}$.*

In general, intervals with decreasing frequency of elements in the set X produce functions $g \in G(X_n)$ with $g > id_{[0,1]}$ and intervals with increasing frequency of elements in X produce functions $g \in G(X_n)$ with $g < id_{[0,1]}$. Both big $S(X)$ and small $G(X_n)$ indicate regularity of distribution, but $G(X_n)$ is more sensitive to non-regularity than $S(X)$. A **decrease** of $S(X)$ is caused by a sudden **decrease** of frequency of elements of X while the **increase** of $G(X_n)$ is caused by sudden **changes** (i.e. not necessary decrease) of frequency of elements of X. If $G(X_n)$ indicates regularity of distribution then also $S(X)$ does. On the other hand, it may happen that $G(X_n)$ indicates non-regularity while $S(X)$ indicates regularity.

Conjecture 2. *If $\underline{d}(X) > 0$ then*

$$S(X) = Tz \qquad \Leftrightarrow \qquad \forall g \in G(X_n):\ g \le id_{[0,1]}.$$

Open problem 7. *Prove or disprove the conjecture.*

If the above conjecture holds then for $\underline{d}(X) > 0$ we have

$$G(X_n) = \{id_{[0,1]}\} \qquad \Rightarrow \qquad S(X) = Tz.$$

On the other hand, there exists $X \subseteq \mathbb{N}$ such that $S(X) = Tz$ and the graphs of $G(X_n)$ cover the whole triangle $(0,0),(1,0),(1,1)$.
We will conclude this section with two possible measures of "irregularity" of distribution of elements of $X \subseteq \mathbb{N}$.
Denote $\underline{g} = \inf G(X_n)$ and $\overline{g} = \sup G(X_n)$. Then

$$\delta(X) = \int_0^1 (\overline{g}(x) - \underline{g}(x))\, dx$$

can be a measure of "irregularity" of distribution of elements of X.

Conjecture 3. *If* $\overline{d}(X) > 0$ *then* $\quad \delta(X) = 0 \Leftrightarrow G(X_n) = \{id_{[0,1]}\}$.

Open problem 8. *Prove or disprove the conjecture.*

Define

$$\theta(X) = \inf\{\theta \in (0,1);\ \exists n_0(\theta)\ \forall n > n_0\ \forall k,l \in (\theta n, n) : \left|\frac{X(k)}{k} - \frac{X(l)}{l}\right| < \theta\}.$$

Conjecture 4. *If* $\overline{d}(X) > 0$ *then* $\quad \theta(X) = 0 \ \Leftrightarrow\ G(X_n) = \{id_{[0,1]}\}$.

Open problem 9. *Prove or disprove the conjecture and find relations between $\delta(X)$ and $\theta(X)$.*

7. DENSITY MEASURES

By a *density measure* we mean every additive measure μ on $\mathscr{P}(\mathbb{N})$ extending density, i.e. $\mu(A) = d(A)$ if $d(A)$ exists. Perhaps the simplest examples of density measures are measures of the kind $\mu_{\mathscr{U}}$, where $\mathscr{U}$ is a free ultrafilter on $\mathbb{N}$, defined by

$$\mu_{\mathscr{U}}(A) = \mathscr{U} - \lim \frac{A(n)}{n}, \quad A \subseteq \mathbb{N}.$$

Question 1. [3] *Does $\mu(A) \leq \overline{d}(A)$ hold for every density measure μ and $A \subseteq \mathbb{N}$?*

In [17] it is claimed that each density measure μ is of the form

$$\mu_\varphi(A) = \int_{\beta\mathbb{N}^*} \mathscr{U} - \lim \frac{A(n)}{n}\, d\,\varphi(\mathscr{U}),$$

for some probability Borel measure φ on $\beta\mathbb{N}^*$, the remainder in the Čech - Stone compactification of $\mathbb{N}$. In this case the answer to van Douwen's question would be evidently YES. Unfortunately, as we will see below, the above claim is not correct.

Theorem 7. [26] *For every fixed set $A \subseteq \mathbb{N}$*

$$\{\mu(A);\ \mu \text{ is a density measure}\} = [\underline{\underline{d}}(A), \overline{\overline{d}}(A)].$$

By this theorem and Example 2 we have the following.

Corollary 3. *The answer to van Douwen's question is NO.*

Open problem 10. *Characterize the sets $A \subseteq \mathbb{N}$ for which $\underline{\underline{d}}(A) = \underline{d}(A)$ and $\overline{d}(A) = \overline{\overline{d}}(A)$.*

A possible measure of "irregularity" of distribution of elements of A:

$$\eta(A) = \overline{\overline{d}}(A) - \overline{d}(A) + \underline{d}(A) - \underline{\underline{d}}(A).$$

Conjecture 5.

$$G(A_n) = \{id_{[0,1]}\} \quad \Rightarrow \quad \eta(A) = 0.$$

Notice that the opposite implication does not hold.

Open problem 11. *Big $\delta(A)$ implies big $\eta(A)$. Find more precise relations.*

8. VARIOUS PROBLEMS

8.1. More rapid convergence to a density, I.

The problem **4.1** in [10] has an easy negative answer : the set of positive even integers does not fulfil the requirements.

Actually, we propose the following new formulation. Suppose that the set $A \subseteq \mathbb{N}$ has asymptotic density $d > 0$. Let $f(n) = |A(n)n^{-1} - d|$ which tends to zero as n tends to $+\infty$. It can be proved that if $f(n)$ is not 0 for all $n \geq n_0$ (this is the case when $A = \mathbb{N}$), then there is $C > 0$ such that for infinitely many n, we have $f(n) \geq Cn^{-1}$.

Open problem 12. *Suppose, in addition, that*

$$\limsup_{n\to+\infty} nf(n) = +\infty\ .$$

Is there $d' \in]0,d]$ and $B \subseteq A$ such that, if we put $g(n) = |B(n)n^{-1} - d'|$, then

1) $g(n)$ tends to 0 , as n tends to $+\infty$; and

2) $\liminf_{n\to+\infty} \dfrac{g(n)}{f(n)} = 0$?

8.2. More rapid convergence to a density, II.
Suppose that the set $A \subseteq \mathbb{N}$ has asymptotic density d. Let $f(n) = |A(n)n^{-1} - d|$ which tends to zero as n tends to $+\infty$. The set A has logarithmic density also equal to d. Let

$$g(n) := |(\sum_{k \leq n} \frac{1}{k})^{-1}(\sum_{a \leq n, a \in A} \frac{1}{a}) \; - \; d| \; .$$

Is it true that $g(n)$ tends to zero, *on the mean* more rapidly than $f(n)$? For instance, does

$$\frac{\sum_{k \leq n} g(k)}{\sum_{k \leq n} f(k)}$$

tend to 0 as n tends to $+\infty$?

8.3. Three cubes. (Proposed by François Hennecart.)

Let C be the set of three cubes

$$C = \{n \in \mathbb{N} \; : \; n = a^3 + b^3 + c^3, (a,b,c) \in (\mathbb{N} \cup 0)^3\} \; .$$

Does $d(C)$ exist? Is so, evaluate dC.

8.4. Sets with prescribed densities.

In [19], see also [18], it is proved that given any quadruple $(\alpha, \beta, \gamma, \delta)$ of numbers such that

$$0 \leq \alpha \leq \beta \leq \gamma \leq \delta \leq 1,$$

there exists a set $A \subseteq \mathbb{N}$ so that

$$\underline{d}(A) = \alpha, \; \underline{\delta}(A) = \beta, \; \overline{\delta}(A) = \gamma, \; \overline{d}(A) = \delta.$$

Open problem 13. *Characterize the set of all* 8*-tuples* $(\alpha_1, \ldots, \alpha_8)$ *with*

$$0 \leq \alpha_1 \leq \alpha_2 < \cdots < \alpha_8 \leq 1$$

for which there exists $A \subseteq \mathbb{N}$ *such that*

$$\underline{u}(A) = \alpha_1, \; \underline{\underline{d}}(A) = \alpha_2, \; \underline{d}(A) = \alpha_3, \; \underline{\delta}(A) = \alpha_4$$

and

$$\overline{\delta}(A) = \alpha_5, \; \overline{d}(A) = \alpha_6, \; \overline{\overline{d}}(A) = \alpha_7, \; \overline{u}(A) = \alpha_8.$$

Notice that $\underline{d}(A) = \overline{d}(A) \; \Rightarrow \; \underline{\underline{d}}(A) = \overline{\overline{d}}(A)$, thus in this case the solution is a proper subset of the set of all nondecreasing 8-tuples from $[0,1]$.

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