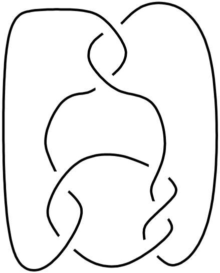
Jones polynomials and incompressible surfaces

joint with D. Futer and J. Purcell

Geometric Topology in Cortona (in honor of Riccardo Benedetti for his 60th birthday), Cortona, Italy, June 3-7, 2013

Given: Diagram of a knot or link



Talk Goal: Discuss a setting where, under certain knot diagrammatic hypothesis, we study both sides and derive relations between them.

Quantum Topology

- Knot invariants invariants esp. colored Jones polynomials

Geometric topology

- Incompressible surfaces in knot complements
- Geometric structures and data esp. hyperbolic geometry and volume

Outline

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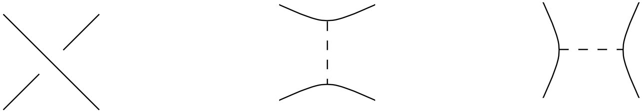
- **Method-Tools:**
  - Create ideal polyhedral decomposition of surface complements.
  - Use normal surface theory to get correspondence
topology of surface complement $\leftrightarrow$ state graph topology

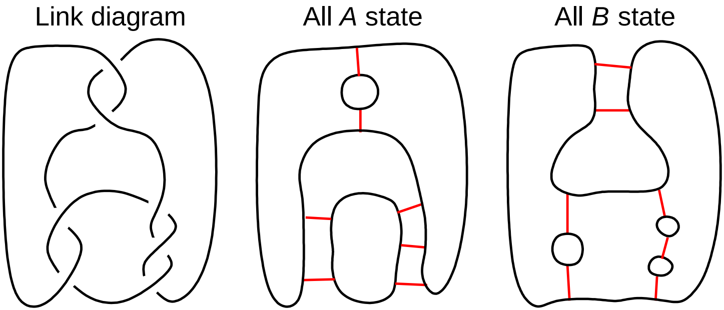
State Graphs

Two choices for each crossing, A or B resolution.



- Choice of A or B resolutions for all crossings: *state* σ.
- Result: Planar link without crossings. Components: *state circles*.
- Form a graph by adding edges at resolved crossings. Call this graph H_σ.
  (Note: n crossings $\rightarrow 2^n$ state graphs)

Example states



Above: H_A and H_B.

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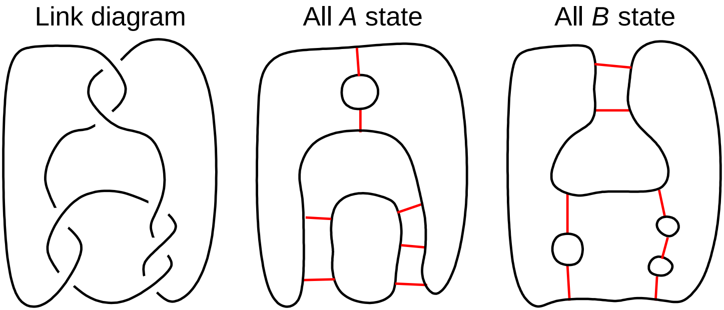

Above: H_A and H_B.

- The Jones polynomial of the knot can be calculated from H_A or H_B: *spanning graph expansion* arising from the Bollobas-Riordan *ribbon graph* polynomial (Turaev, Dasbach-Futer-K-Lin-Stoltzfus).

Colored Jones polynomial prelims

For a knot K, and $n = 1, 2, \ldots,$ we write its *n-colored Jones polynomial*:

$$J_{K,n}(t) := \alpha_n t^{m_n} + \beta_n t^{m_n - 1} + \cdots + \beta'_n t^{k_n + 1} + \alpha'_n t^{k_n}.$$

Some properties:

- $J_{K,n}(t)$ is determined by the Jones polynomials of certain cables of K.

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Remark. Properties manifest themselves in strong forms for knots with *state graphs* that have no edge with both endpoints on a single state circle—-That is when K is *A-adequate* (next)

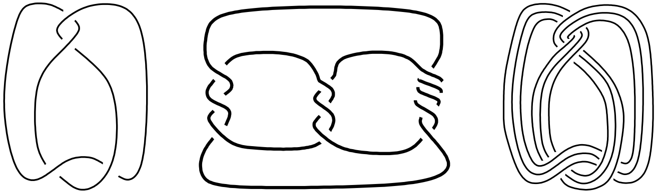
Semi-adequate links

Lickorish–Thistlethwaite 1987: Introduced *A–adequate* links (*B–adequate links*) in the context of Jones polynomials.

Definition

A link is A–adequate if has a diagram with its graph H_A has no edge with both endpoints on the same state circle. Similarly B-adequate.
Semi-adequate: A or B-adequate.

Some examples:



Semi-adequate links are abundant!

Some familiar classes and their geometry:

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- many arborescent knots (mostly *hyperbolic*),
- all closed 3-braids (prime are *torus knots* or *hyperbolic* (Stoimenow),
- large families of *hyperbolic* braid and plat closures (A. Giambrone),
- blackboard cables and Whitehead doubles of semi-adequate knots (*satellites*)

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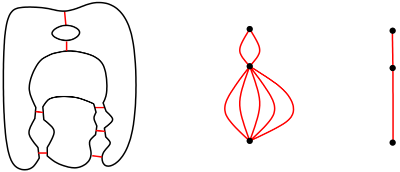
- blackboard cables and Whitehead doubles of semi-adequate knots (*satellites*)
- **Question:** Is there an algorithm to decide whether a given knot is semi-adequate?

CJP of semi-adequate links

- Collapse each state circle of H_A to a vertex to obtain the state graph $\mathbb{G}_A$.
- Remove redundant edges to obtain the *reduced state graph* $\mathbb{G}'_A$.



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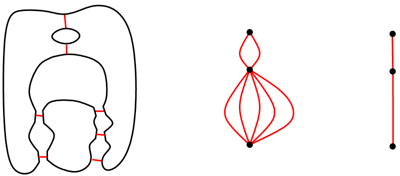

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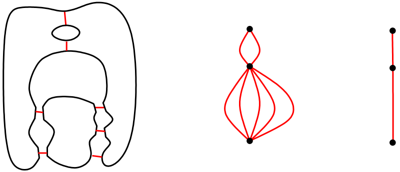

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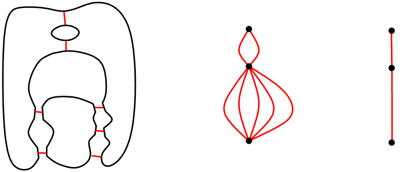

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$$k_n \geq -n^2 2c_-(D) + O(n),$$

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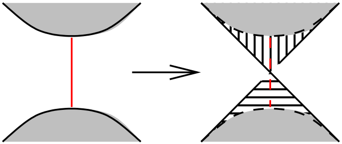
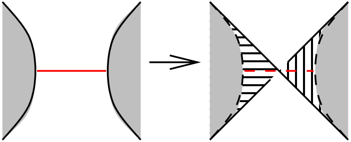
State surface

Given a state σ, using graph H_σ and link diagram, form the *state surface* S_σ.

- Each state circle bounds a disk in S_σ (nested disks drawn on top).
- At each edge (for each crossing) attach twisted band.

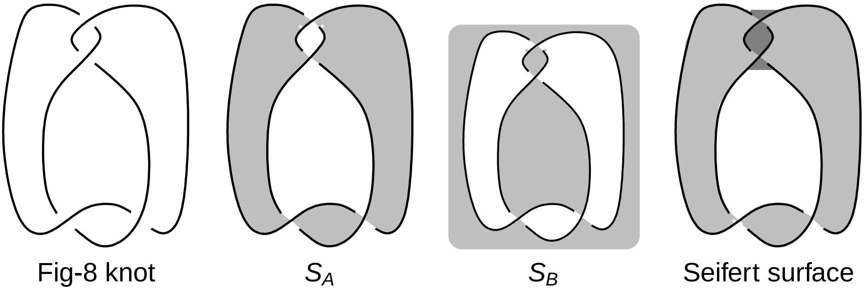


A–resolution



B–resolution

Example state surfaces



- For alternating knots: S_A and S_B are checkerboard surfaces.

The surface S_A and CJP: Boundary slopes

Theorem (Ozawa, FKP)

The surface $S_A = S_A(D)$ is essential in $S^3 \setminus K \iff D(K)$ is A–adequate.

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S_A= state surface with $K = \partial S_A$ an A-adequate knot (one component). The class $[K]$ in $H_1(\partial(S^3 \setminus K))$ is determined by an element in $\mathbf{Q} \cup \{\infty\}$, called *the boundary slope of S_A*.

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$$\textit{bdry slope of } S_A = \lim_{n\to\infty} \frac{-4}{n^2} k_n,$$

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The Jones Slopes Conjecture

- (Curtis-Taylor) Related ∂-slopes of checkerboard surfaces of alternating knots to degree of Jones polynomial.
- **Slopes Conjecture.** (Garoufalidis) For every knot K the sequence

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- **Remarks:**
- q-holonomicity implies that the sets of cluster points above are finite.
- (Hatcher) Every knot has finitely many ∂-slopes.

What's known

- For knots that are A and B-adequate slopes conjecture is know for "both sides".
- (Garoufalidis) torus knots, certain 3-string pretzel knots $P(-2, p, q)$ (A-adequate not B-adequate)
  For pretzel knots the boundary slopes are all known./ For torus knots CJP has been calculated.
- (Dunfield–Garoufalidis) Verified conjecture for the class of *2-fusion knots*.— (normal surface theory+character variety techniques to get the incompressible surface).
- (van der Veen) Formulated a Slopes conjecture for the *multi-colored* CP of links. Showed that S_A verifies it A-adequate links.

The surface S_A and CJP: Coefficients

For an A-adequate link, β'_K is the stabilized penultimate coefficient of CJP.

Theorem (Futer–K–Purcell)

For an A–adequate diagram $D(K)$, the following are equivalent:

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Stronger statements:

- (For a hyperbolic link K) S_A is *quasifuchsian* iff $\beta'_K \neq 0$
- when β'_K is large, S_A should be "far from being a fiber" (next).

Is there more in β'_K? How about in the whole tail?

- In general, β'_K measures the "size" (in the sense of Guts) of the hyperbolic part in Jaco-Shalen-Johannson decomposition S_A. This, combined with work of Agol- W. Thurston- Storm gives: large β'_K implies large volume for $S^3 \setminus K$.
- **What about the tail?**
- Recall $T_K(t) = 1 + \beta'_K t + O(t^2)$.

Theorem (Armond-Dasbach)

Suppose K A-adequate. Then, $T_K(t) = 1$ if and only if $\beta'_K = 0$.

Note: if $\beta'_K = 0$ then $\mathbb{G}'_A$ is a tree

Thus, $T_K(t) = 1$ if and only if S_A is a fiber in $S^3 \setminus K$.

- **Question.** If $T_K(t) \neq 1$ does it contain more information about the complement of S_A and the geometry of K than β'_K?

Topology of the state surface complement

- $M_A = S^3 \backslash\backslash S_A$ is obtained by removing a neighborhood of S_A from S^3.
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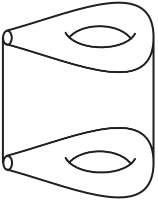
- On ∂M_A we have the *parabolic locus* P = remains from $\partial(S^3 \setminus K)$ after cutting along S_A.
- The annulus version of the JSJ decomposition for the pair (M_A, P) assures that M_A can be cut along along essential annuli, to obtain three kinds of pieces:

1. I–bundles (e.g. $\Sigma \times I$ for $\Sigma \subset S_A$, although $\Sigma \tilde{\times} I$ can also occur),
2. Seifert fibered solid tori,
3. *Guts* $(S^3 \setminus K, S_A)$. Thurston showed that the guts admit a hyperbolic metric.



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Theorem (Agol–Storm–Thurston)

Let M be a compact 3–manifold with hyperbolic interior of finite volume, and $S \subset M$ an embedded essential surface. Then

$$Vol(M) \geq -v_8\, \chi(Guts(M, S)),$$

where $v_8 \approx 3.6638$ is the volume of a regular ideal octahedron.

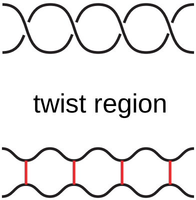
The meaning of β'_K: Special case

$D(K)$ =an A-adequate diagram with S_A the corresponding all-A state surface.

Theorem (F–Kalfagianni–Purcell)

Let $D(K)$ be an A–adequate diagram such that every 2–edge loop in G_A comes from a twist region. Then the surface S_A satisfies

$$\chi(Guts\,(S^3 \setminus K,\, S_A)) \;=\; \chi(G'_A) = 1 - \beta'_K$$



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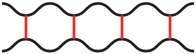

Corollary

Under the same hypotheses, if K is hyperbolic,

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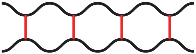

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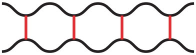

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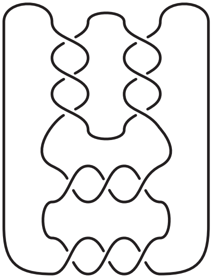
Under the same hypotheses, if K is hyperbolic,

$$Vol(S^3 \setminus K) \geq v_8(\beta'_K - 1).$$

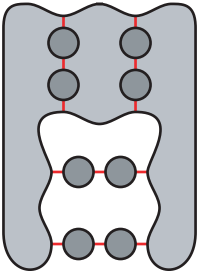
For alternating knots and links, this follows from work of Lackenby and Dasbach–Lin.
There are large families non-alternating knots satisfying the hypothesis (A. Giambrone)

A worked example

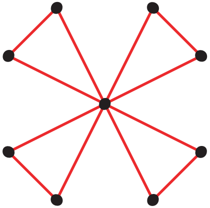
$D(K)$



all–*A* state



$G_A = G'_A$



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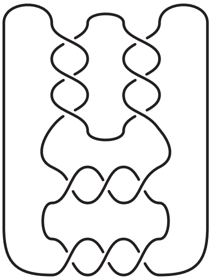

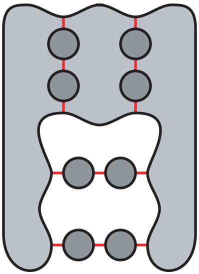

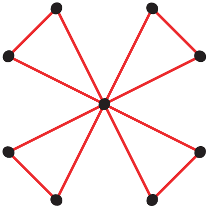

$$1 - |\beta'| = \chi(G_A)$$

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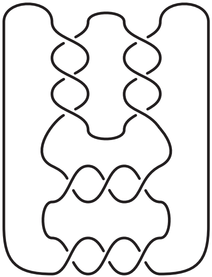

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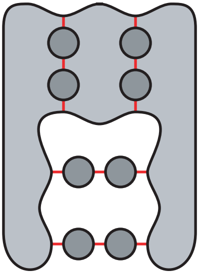

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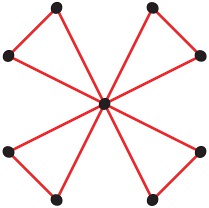

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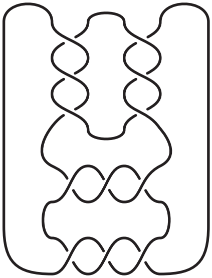

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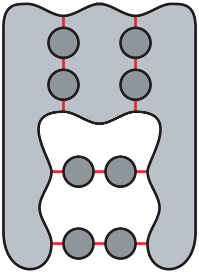

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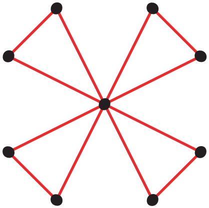

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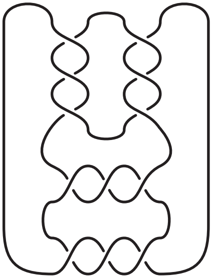

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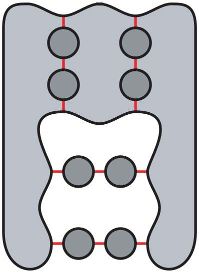

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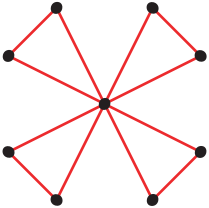

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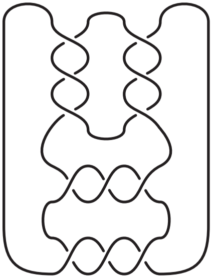

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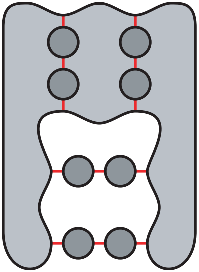

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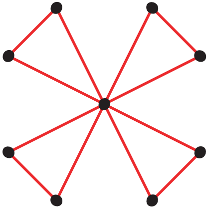

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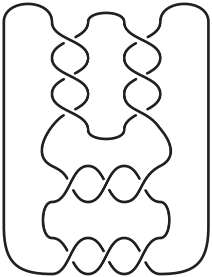

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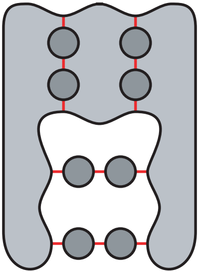

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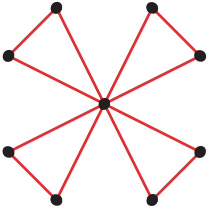

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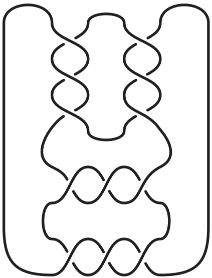

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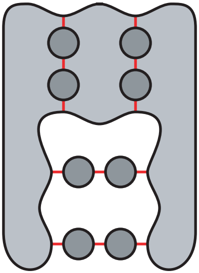

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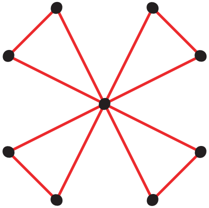

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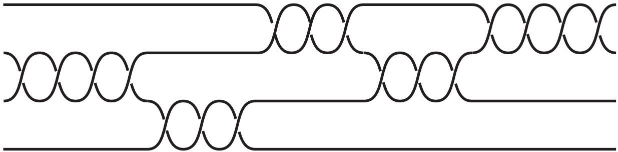
$$\text{Vol}\,(S^3 \smallsetminus K) = 13.64...$$

Sample family: positive braids



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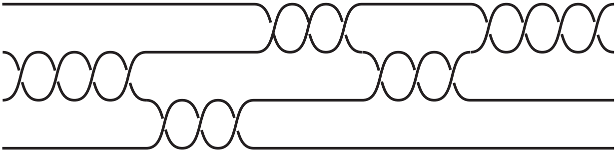

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Theorem (F–Kalfagianni–Purcell)

Suppose that K is the closure of a positive braid $b = \sigma_{i_1}^{r_1} \sigma_{i_2}^{r_2} \cdots \sigma_{i_k}^{r_k}$, where $r_j \geq 3$ for all j. In other words, there are k twist regions, each with at least 3 crossings.

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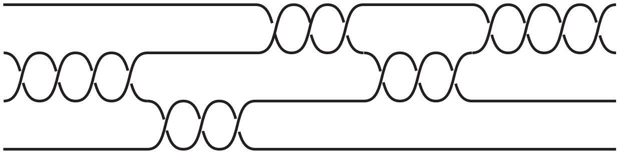

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$$\frac{2v_8}{3}k \;\leq\; Vol\,(S^3 \smallsetminus K) \;<\; 10v_3(k-1)\,.$$

Similarly,

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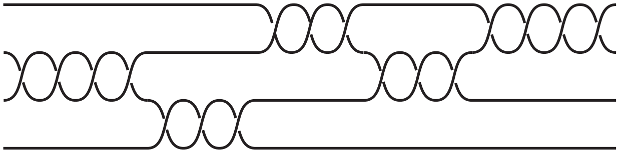

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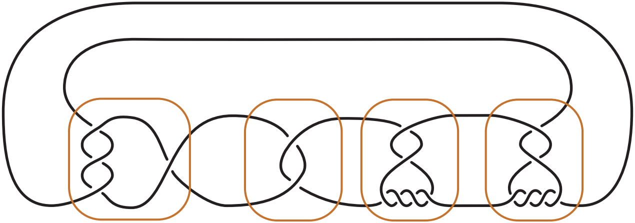
Here, $v_3 = 1.0149...$ is the volume of a regular ideal tetrahedron and $v_8 = 3.6638...$ is the volume of a regular ideal octahedron.

The gap between the upper and lower bounds is a factor of 4.155...

Sample family: Montesinos links

A Montesinos knot or link is constructed by connecting *n* rational tangles in a cyclic fashion.



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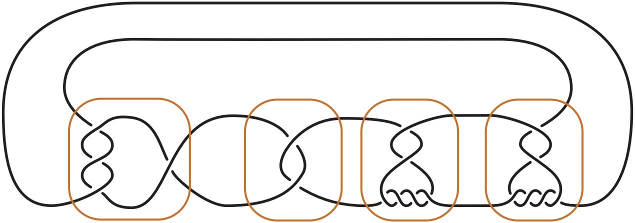

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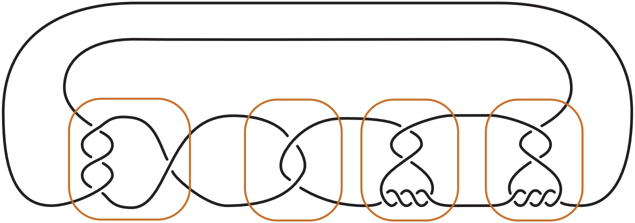

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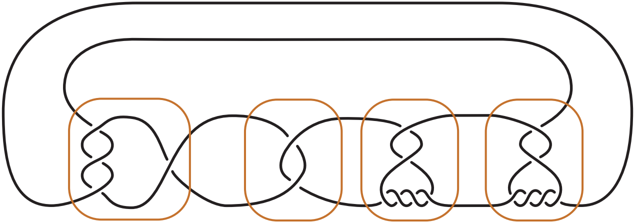

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If K has length at least four we get two-sided volume estimates:

$$v_8\,(\max\{\beta_K, \beta'_K\} - 2) \;\leq\; Vol\,(S^3 \smallsetminus K) \;<\; 4v_8\,(\beta'_K + \beta_K - 2) + 2v_8\,(\#K),$$

where $\#K$ is the number of link components of K.

A coarse Volume Conjecture?

Results and experimental evidence prompt:

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- Volume Conjecture (Kashaev, H. Murakami-J. Murakami) predicts relations between volume and coefficients of CJP
- Proven results and stabilization properties of CJP prompt more guided speculations as to where one might look for $B(K)$.

2–edge loops and I–bundles of $S^3 \backslash\backslash S_A$

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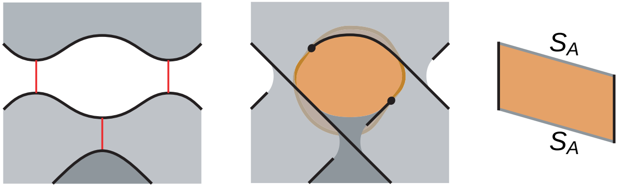

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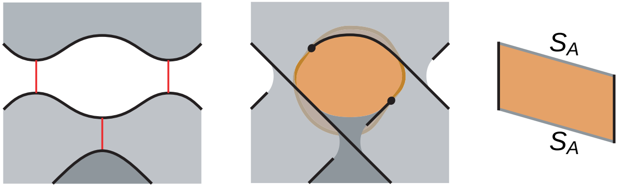

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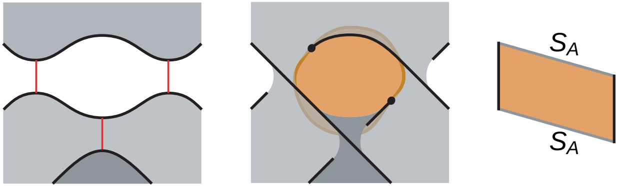

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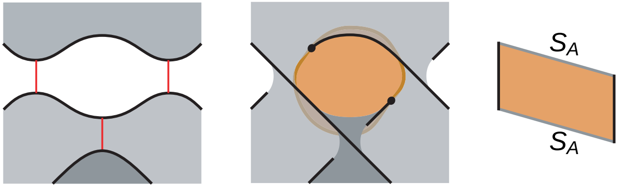

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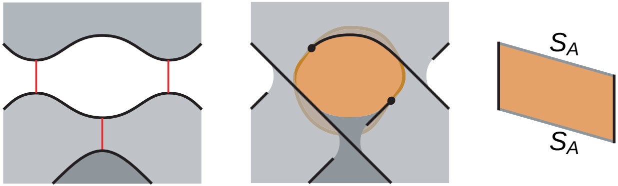

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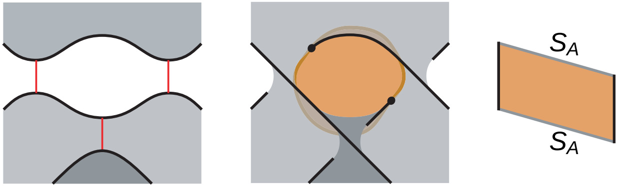

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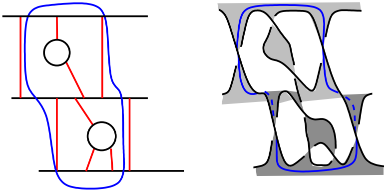
$$\chi(\text{Guts}) = \chi(S_A) + \#\text{EPDs} = \chi(G_A \setminus \text{extra edges}) = \chi(G'_A).$$

Topology of β'_K: most general form

A 2–edge loop in G_A may correspond to multiple product disks, some of which are *complex*. The number of complex disks is $||E_c|| \geq 0$.



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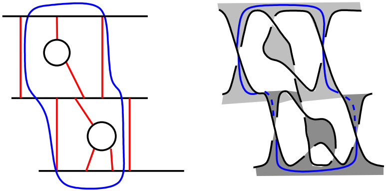

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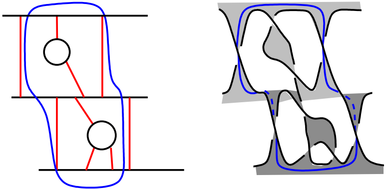

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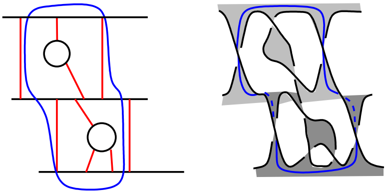

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Open problem: for each A–adequate link, is there a diagram with $||E_c|| = 0$?

Tool for the proof: a nice polyhedral decomposition

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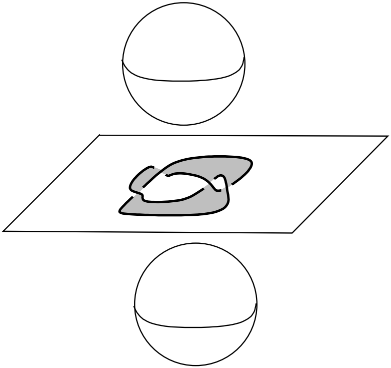

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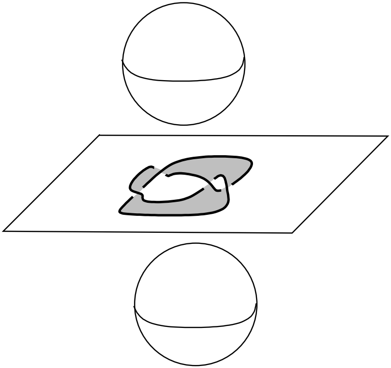

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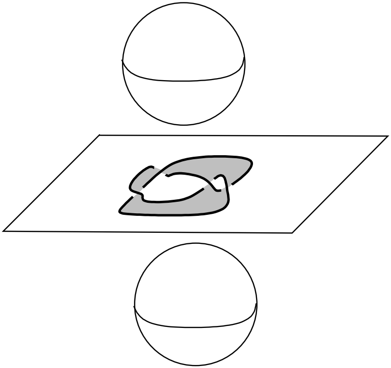

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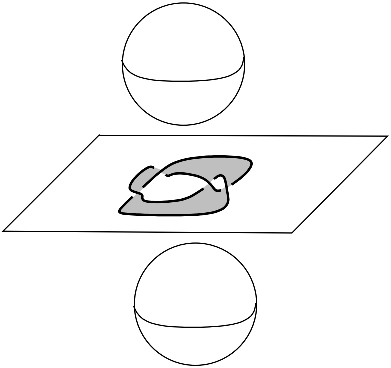

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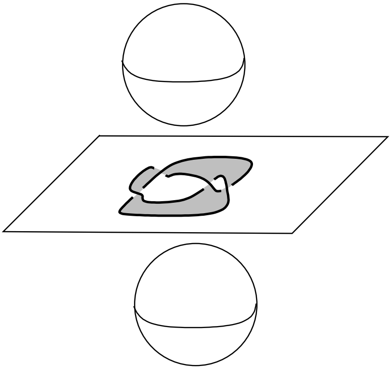

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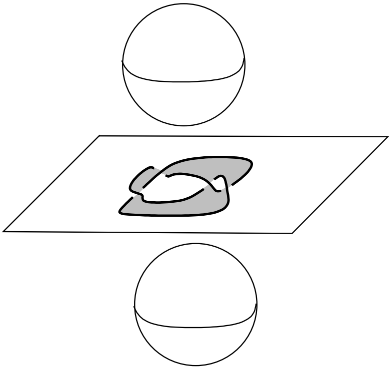

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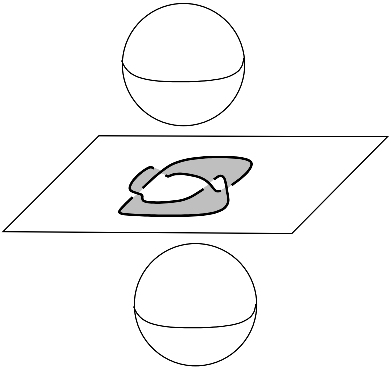
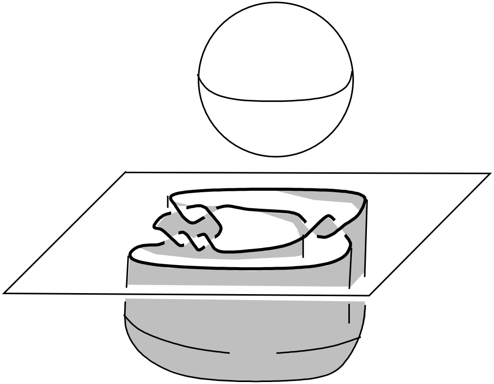
- Hence, gluing along white faces only produces a decomposition of $S^3 \backslash\backslash S_A$.



Polyhedral decomposition of the surface complement

Our surface S_A is layered below the plane of projection. We need more balloons to subdivide $S^3 \backslash\backslash S_A$.



3–cells:

- One “upper” 3–cell, above the plane of projection.
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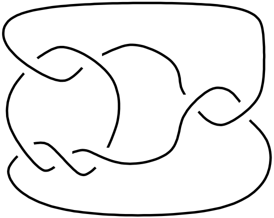

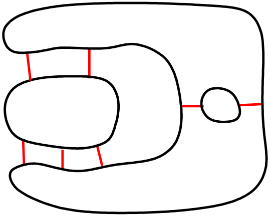

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Polyhedral decomposition of $S^3 \backslash\backslash S_A$: 3–cells

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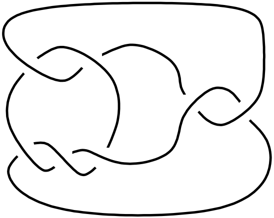

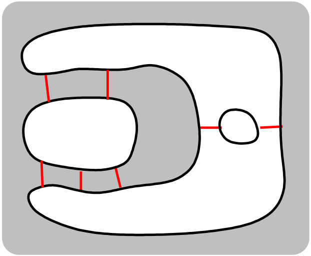

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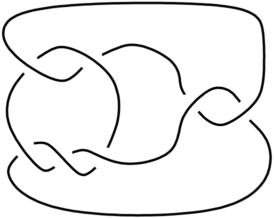
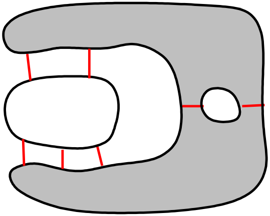


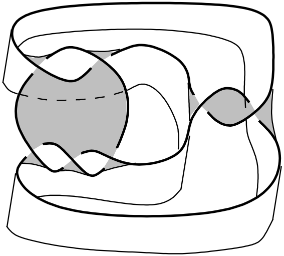


Two nontrivial components

Polyhedral decomposition of $S^3 \backslash\backslash S_A$: faces

Faces are checkerboard colored, and come in two distinct flavors:

- Portions of a 3–cell meeting S_A. These faces are shaded.
- Disks lying slightly below the plane of projection, with boundary on S_A.
  - One disk for each region of the graph H_A (state circles and red edges).
  - These faces are white.



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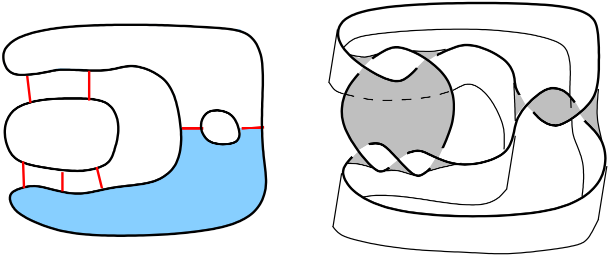

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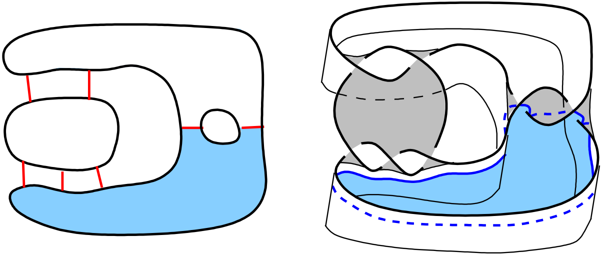
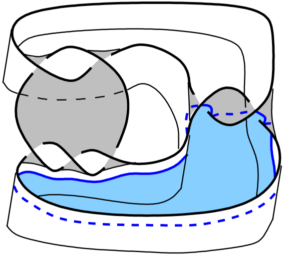


All polyhedra are glued to the upper polyhedron, along white faces only.

Ideal edges:

- Run from undercrossing to undercrossing, adjacent to region of H_A.

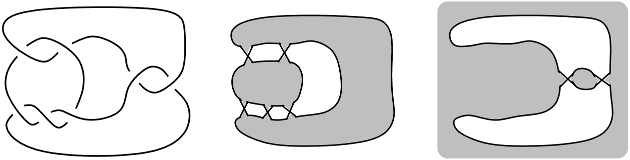


Ideal vertices:

- On the link. Portions of the link visible from inside the 3–cell.

Combinatorial descriptions of Polyhedra

Lower polyhedra are identical to checkerboard polyhedra of alternating sublinks.



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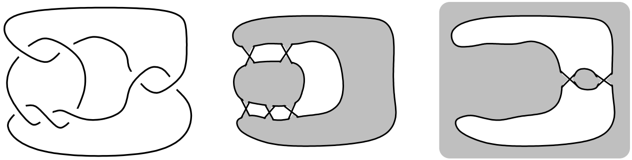
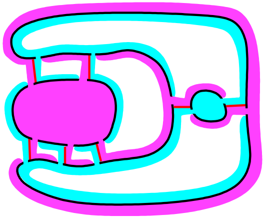
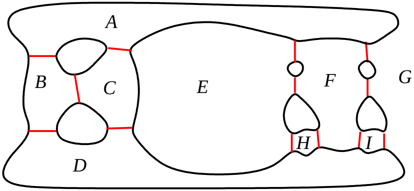
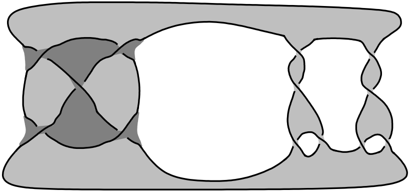


Upper polyhedron: Ideal edges and shaded faces are sketched by *tentacles* on projection of H_A



Properties of the polyhedra, summarized





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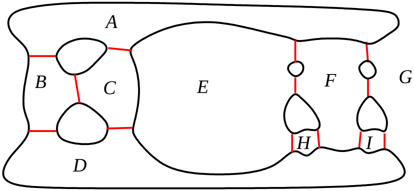

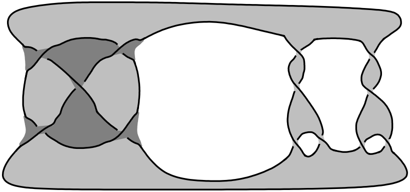

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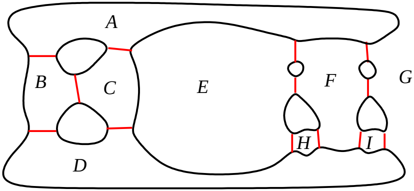

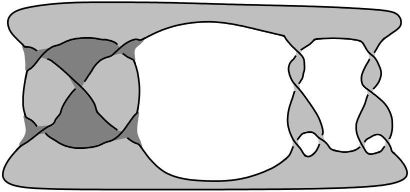

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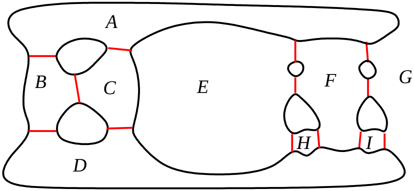

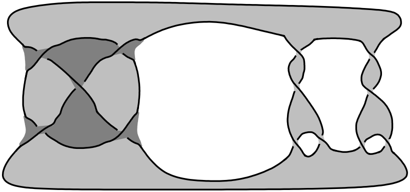

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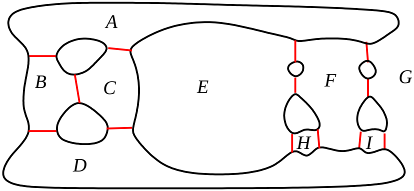

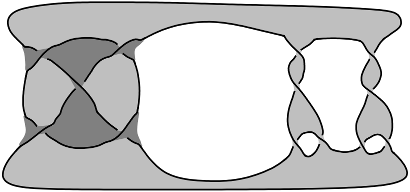

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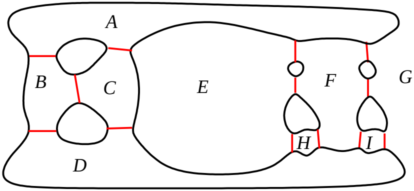

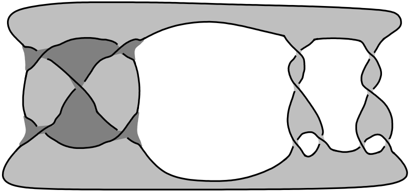

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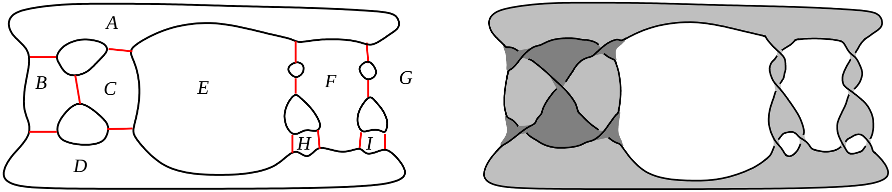

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- These product disks correspond to 2–edge loops of G_A, allowing us to detect fibering and compute $\chi(\text{Guts}\,(S^3 \backslash\backslash S_A))$.

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