Extra structures on 3-manifolds via extra structures on spines

Carlo Petronio

Plan

1. Triangulations and special spines of (oriented) 3-manifolds

2. Branched special spines, the normal flow, and the maw

3. Closed combed 3-manifolds [BP1997]

4. Spin 3-manifolds [BP1997]

5. Generic flows on 3-manifolds [P2012]

6. Spin structures via arbitrary spines [BP2013]

1. Triangulations and special spines

1.1. Triangulations

M compact 3-manifold with ∂ (possibly $\emptyset$)

An **(ideal) triangulation** of M is a realization of **the interior of M minus some points** as

(1) Take some copies of the standard 3-simplex

(2) Glue together the faces in pairs via simplicial maps

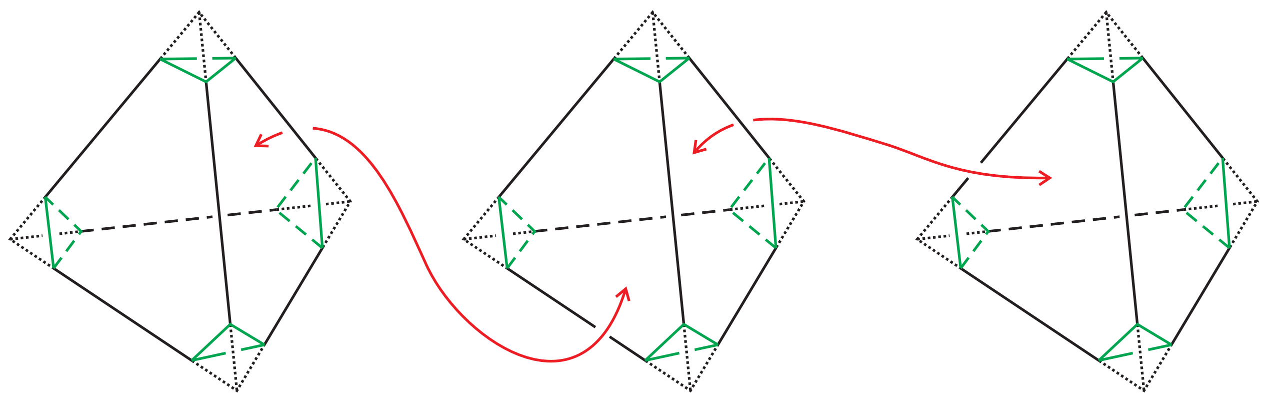
(3) Remove the vertices

OR a realization of M **minus some disjoint open balls** as

(1) Take some copies of the standard 3-simplex

(2) Glue together the faces in pairs via simplicial maps

(3′) Remove open regular neighbourhoods of the vertices



For **oriented** M:
face-pairings should be **orientation-reversing**

Often **sphere components of** ∂M **are forbidden**
$\Rightarrow$ **every triangulation defines a unique** M

and perhaps

- No balls removed if $\partial M \neq \emptyset$
- One ball removed if M is closed

1.2. Spines

$P \subset M$ is a **spine** of M if for $N = (M$ minus some balls$)$

- $N \searrow P$
- $N \setminus P \cong (\partial N) \times (0, 1]$
- $N \cong U_N(P)$

(equivalently)

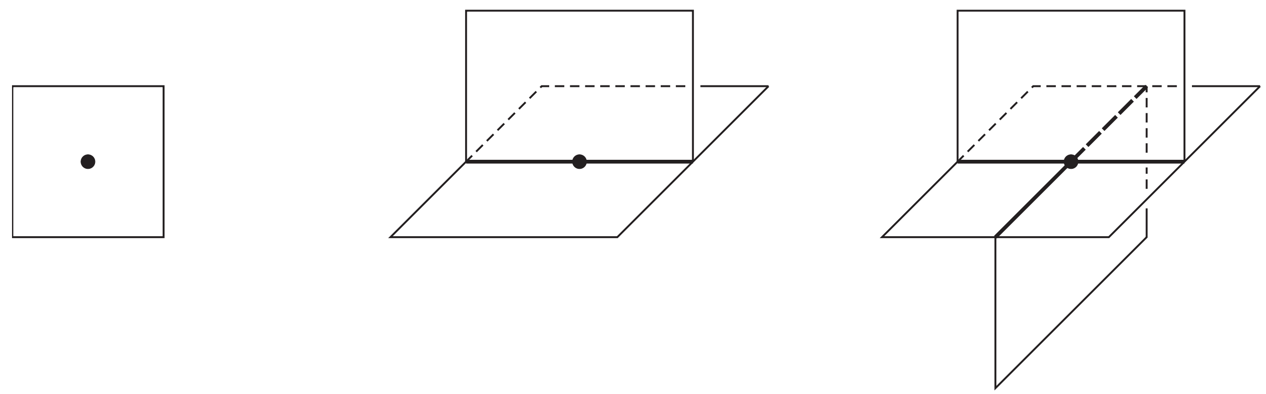
And perhaps

- $N = M \quad$ if $\partial M \neq \emptyset$
- $N = (M$ minus one ball$) \quad$ if M is closed

1.2. Special polyhedra

P is **special** if

1. Locally it appears as



regions **edges** **vertices**

2. Edges are segments
3. Regions are discs

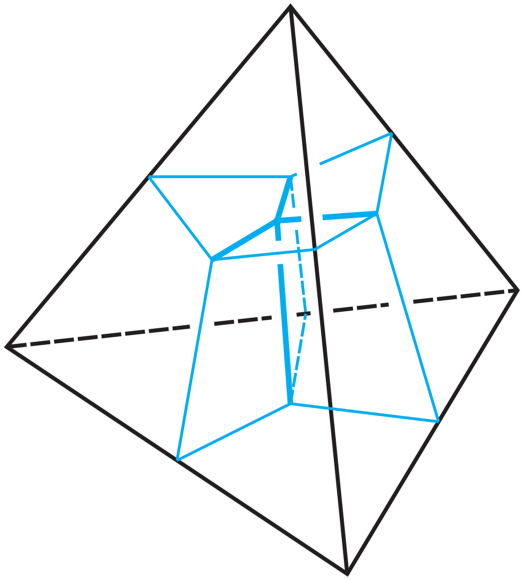
1.4 Duality

Fact

{triangulations of 3-manifolds}

$\updownarrow$ **duality**

{thickenable special polyhedra}



[BP1995] (combinatorial) **orientation** of a special polyhedron

Fact {triangulations of oriented 3-manifolds}

$\updownarrow$ **duality** (orientable $\Rightarrow$ thickenable)

{oriented special polyhedra}

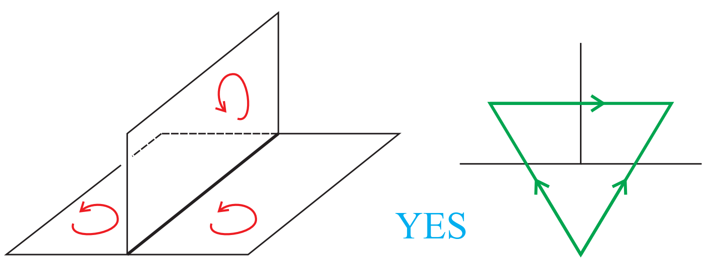
2. Branched spines, normal flow, maw

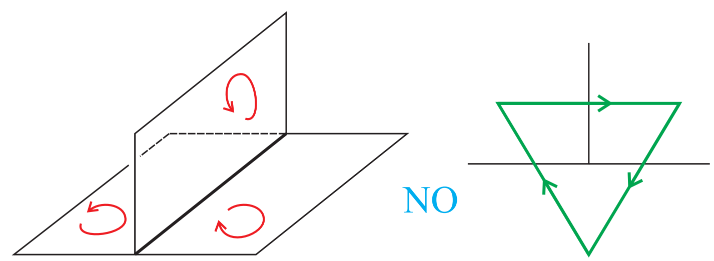
2.1. Branching

P oriented special polyhedron, $\mathcal{T}$ dual triangulation
A **branching** is

- An orientation for the regions of P such that no edge of P is induced 3 times the same orientation by the 3 incident regions
- An orientation for the edges of $\mathcal{T}$ such that the boundary of a triangle of $\mathcal{T}$ is never a cycle

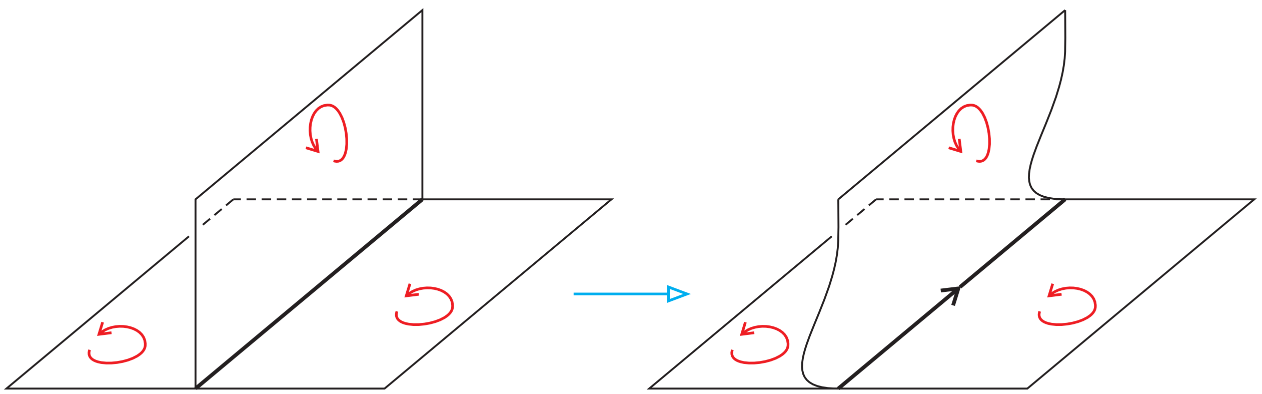
(equivalently)





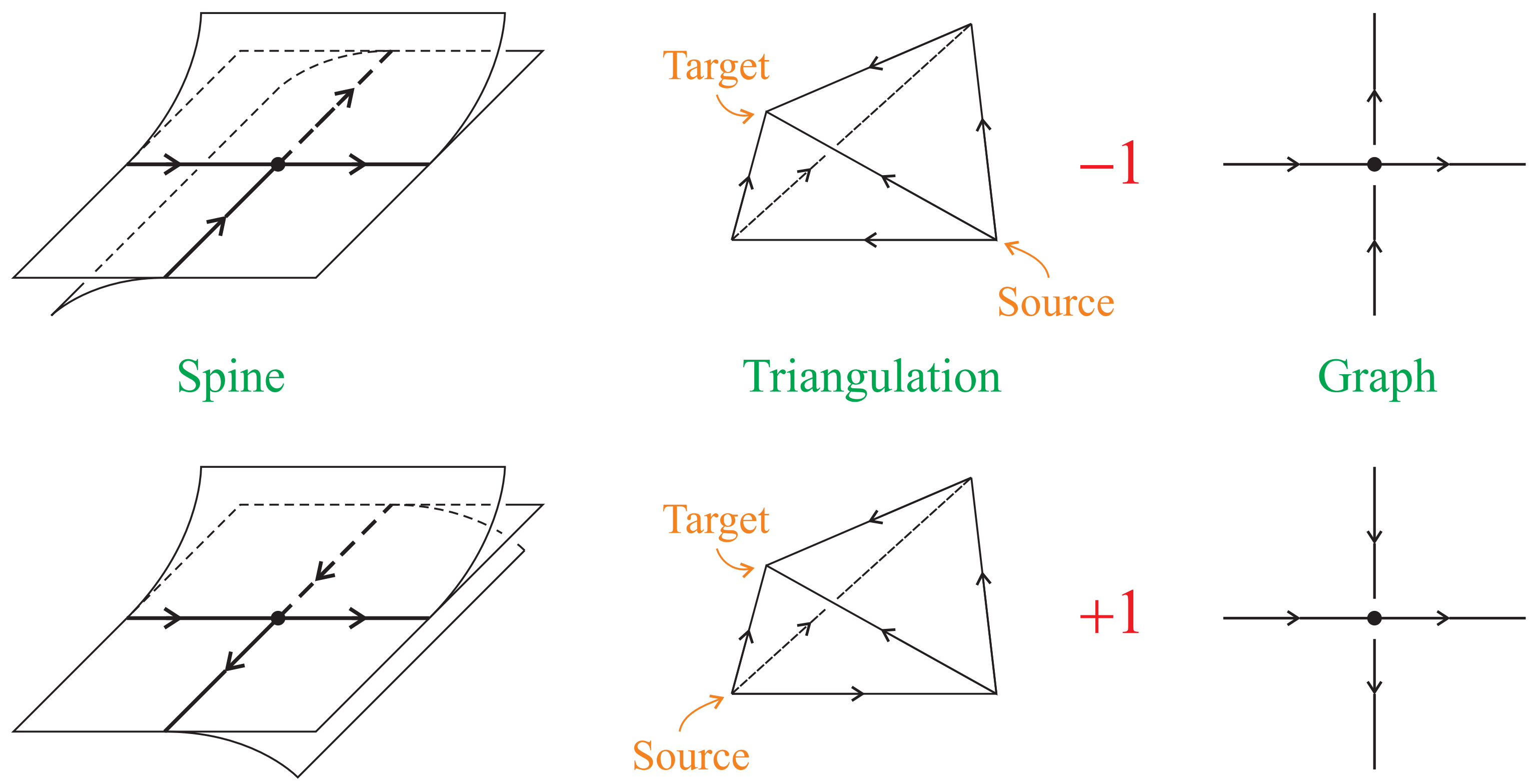
Fact Some P's **do not admit any branching**

Smoothing along edges induced by a branching

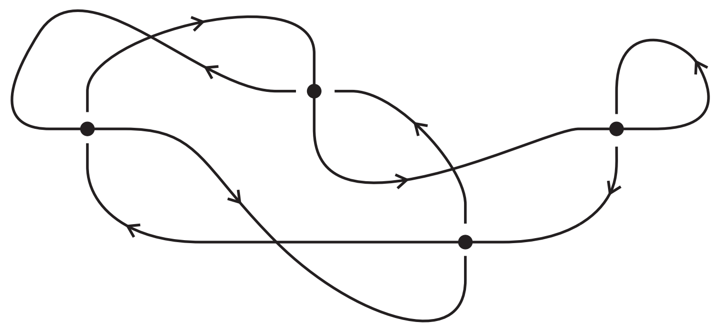


(and orientation of edges)

Fact The smoothing **extends at vertices**

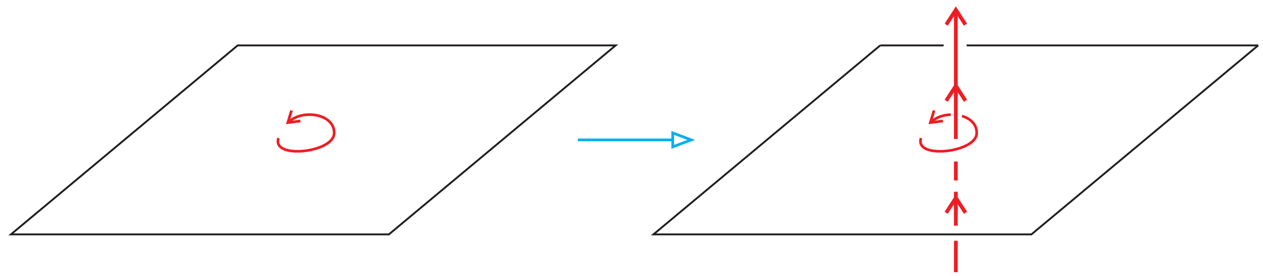


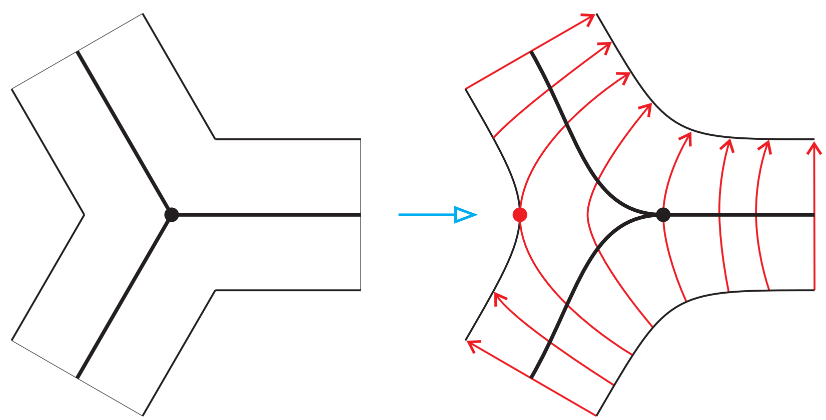
A branched triangulation



2.2. The normal flow ν

P branched (Ishii) positive normal flow $\nu(P)$ on $U(P)$





concave

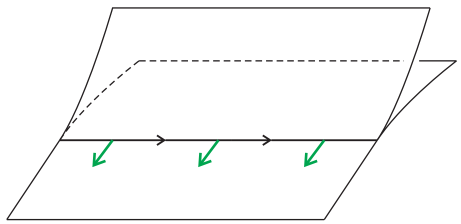
tangency to ∂

At vertices: a flow-line is **doubly tangent** to ∂

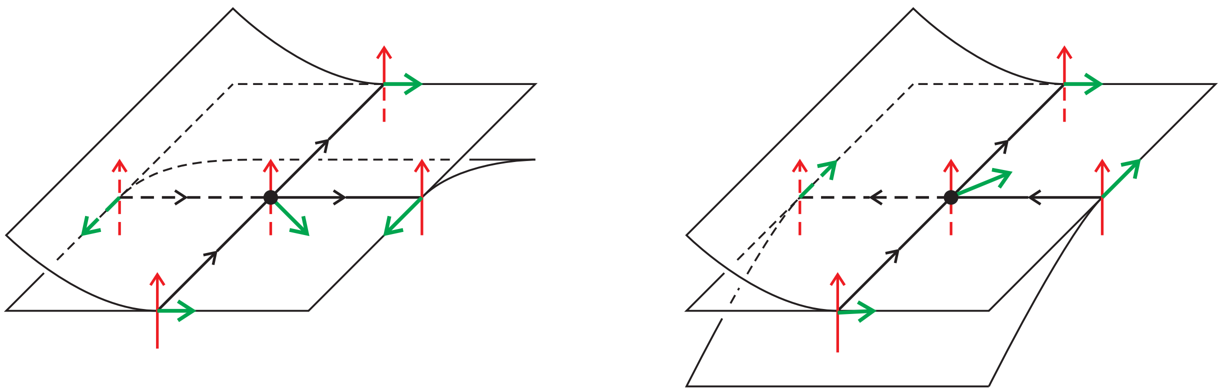
2.2. The maw ν

P branched

(Christy) descending field $\mu(P)$ on 1-skeleton $S(P)$ of P



ν and μ near a vertex



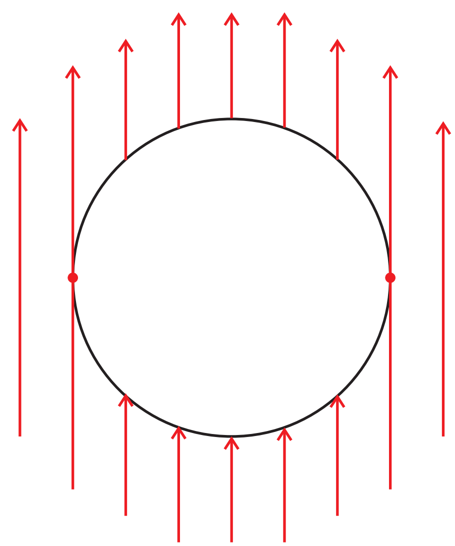
3. Closed combed 3-manifolds [BP1997]

Combinatorial realization of the set of pairs $(M, [v])$ with

- M closed oriented
- v non-zero vector field on M
- $[v]$ homotopy class (through non-zero vector fields)

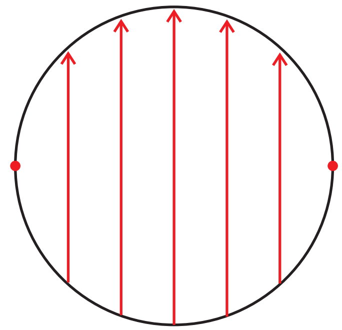
Objects P oriented branched special polyhedron with

$\partial U(P) \cong S^2$ and
$\nu(P)$ near $\partial U(P)$ given by



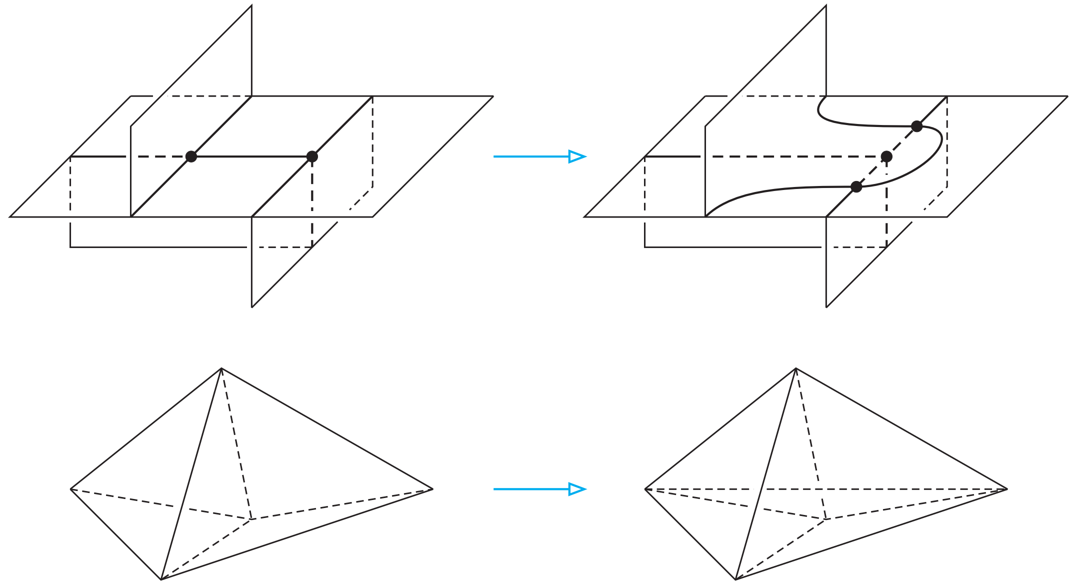
Reconstruction

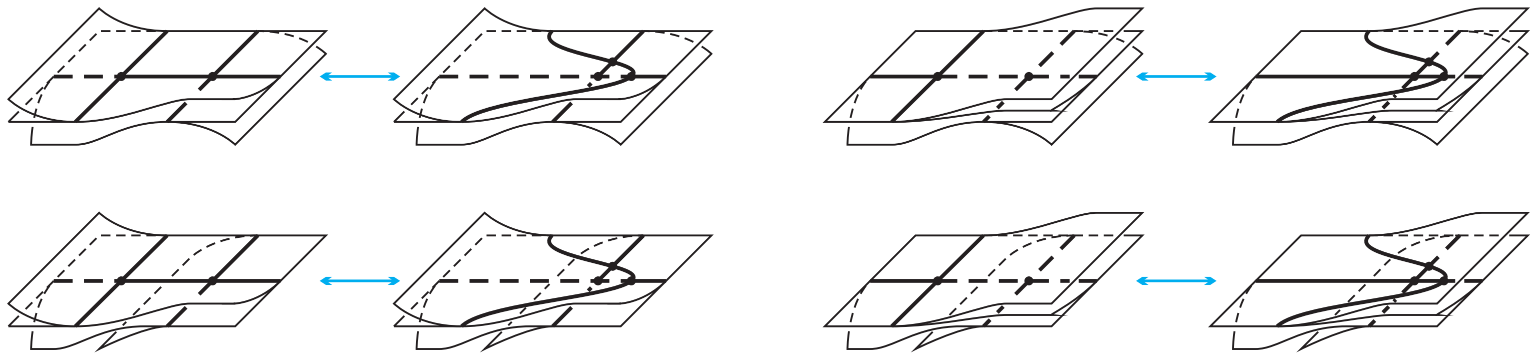
cap $\partial U(P)$ with



Moves

branched versions of the Matveev-Piergallini 2-3 move





Easy extension with same techniques

- allow $\partial M \neq \emptyset$
- allow v to have **concave** tangency to ∂M
- consider homotopy fixed on ∂M

4. Spin 3-manifolds [BP1997]

Combinatorial realization of the set of pairs (M, s) with

- M oriented
- s spin structure on M

Objects (P, β) with

- P oriented branched special polyhedron
- **weight** $\beta \in C^1(P; \mathbb{Z}/_{2\mathbb{Z}})$ such that $\delta\beta$ is the obstruction to extending $(\nu(P), \mu(P))$ from $S(P)$ to P

Reconstruction Use β to extend $(\nu(P), \mu(P))$ from $S(P)$ to P

Moves

- Add 1-coboundaries to β
- Weighted and branched versions of the MP move

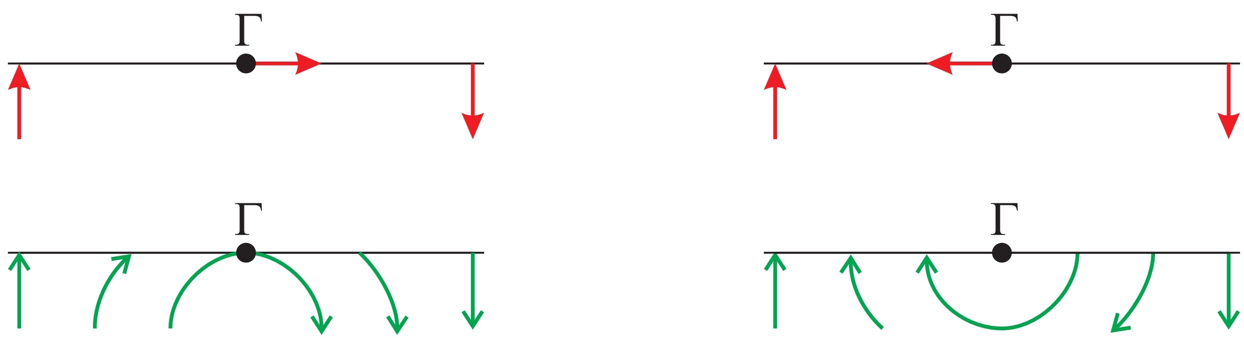
5. Generic flows [P2012]

5.1. Morin singularities

M compact, oriented, $\partial M \neq \emptyset$

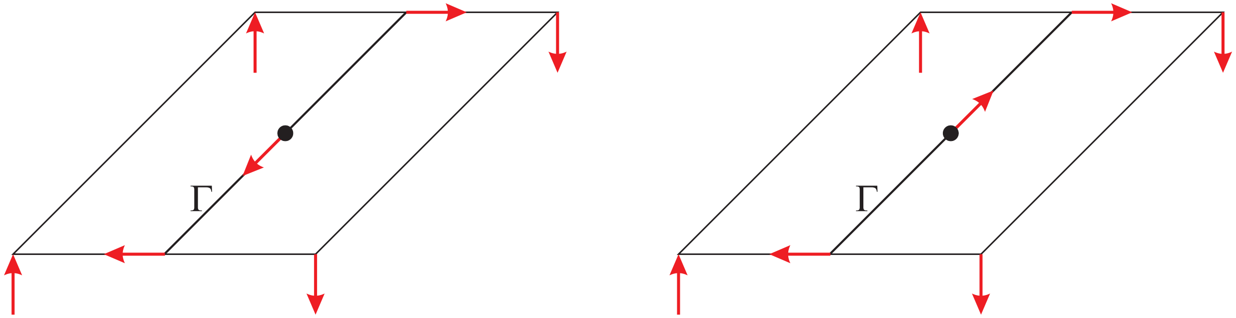
v nowhere-zero on M

generically: v tangent to ∂M along curve Γ
and tangent to Γ at essential isolated points

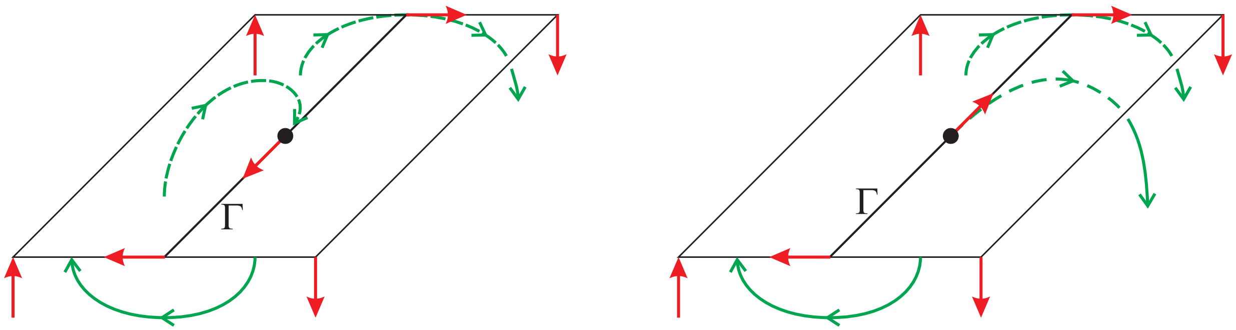


concave **convex**

Transition points



Transition orbits



concave-to-convex **convex-to-concave**

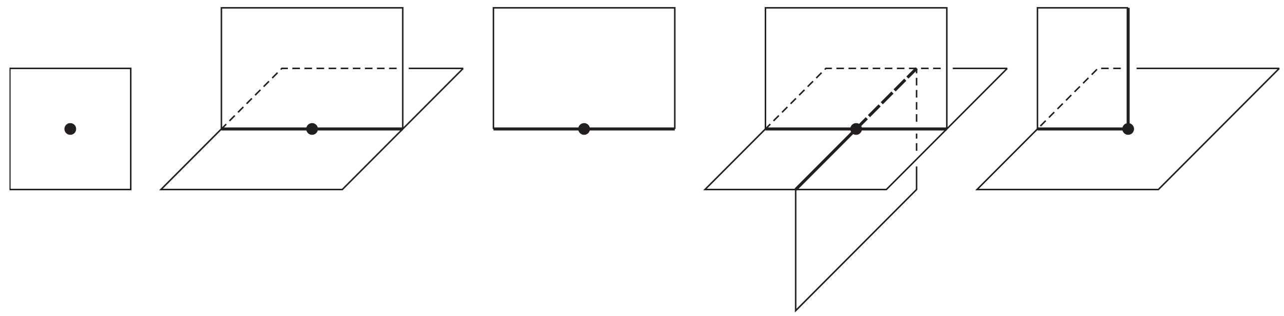
5.1. Combinatorial realization – objects

Set of pairs $(M, [v])$ with

- M oriented compact with ∂
- v non-zero vector field on M generic along ∂M
- $[v]$ homotopy class through generic fields
  ($\Rightarrow$ configuration on ∂M evolves isotopically)

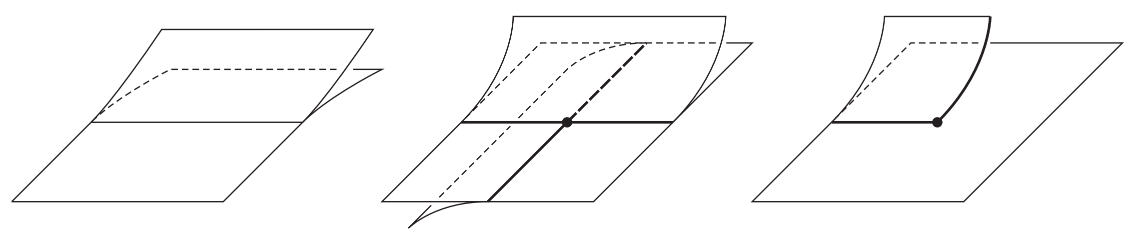
Objects P compact polyhedron

- locally



region edge **∂-edge** vertex **spike**

- oriented along edges
- oriented branching along edges

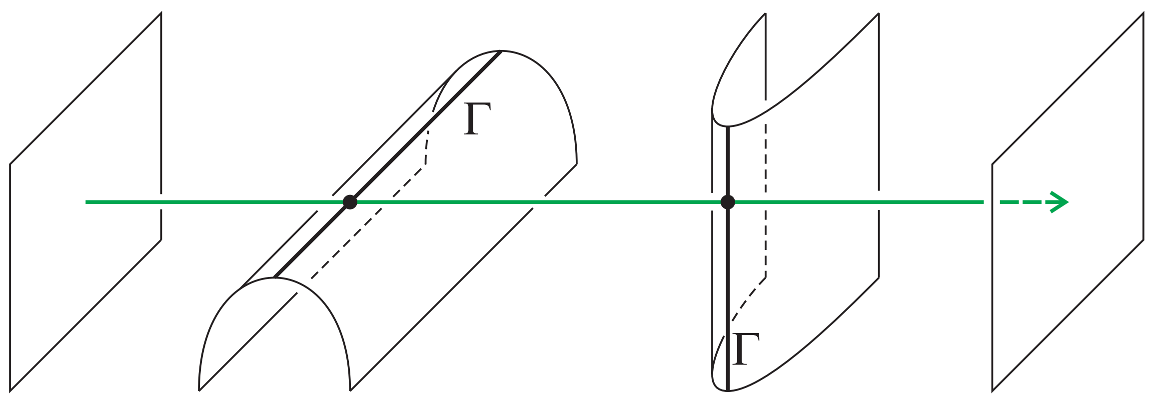


- boundary condition (discussed below – no cellularity!)

5.2. Reconstruction

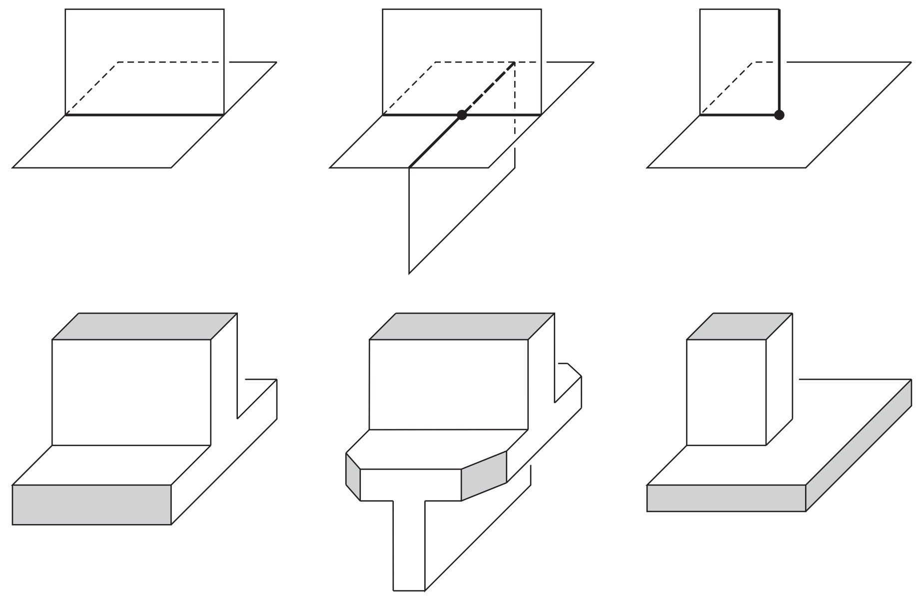
Theorem Each P as above thickens to unique $(U(P), \nu(P))$ with

- P a spine of $U := U(P)$
- $\nu := \nu(P)$ positively normal to P
- ν generic on ∂U
- Transition orbits of ν not elsewhere tangent to ∂U
- Each orbit of ν tangent to ∂U in at most two points, and transversely if so



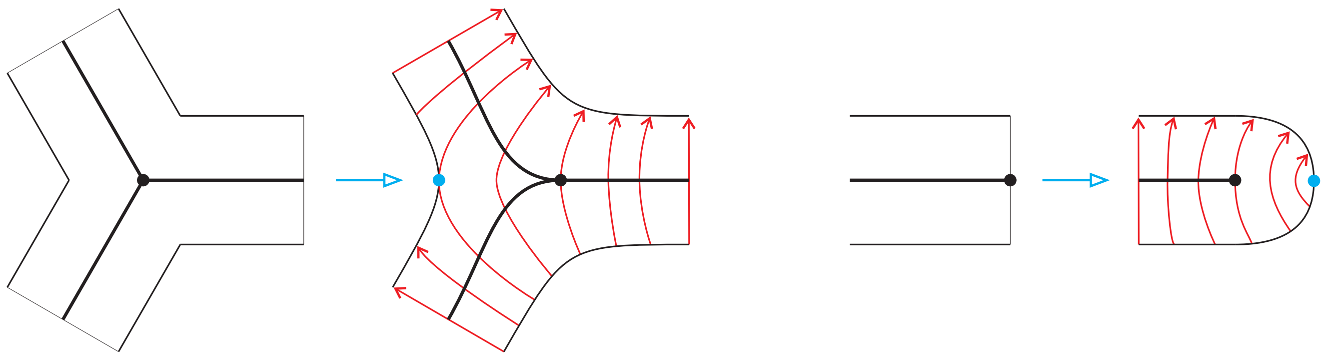
- All orbits of ν go from ∂U to ∂U

- Topological thickening

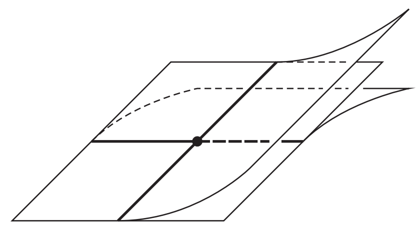


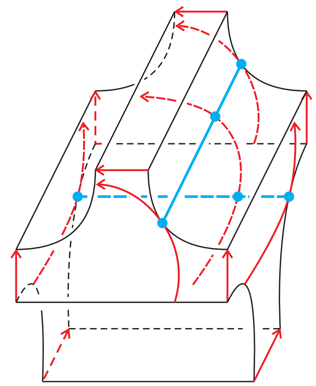
Unique because trivial I-bundle on each region

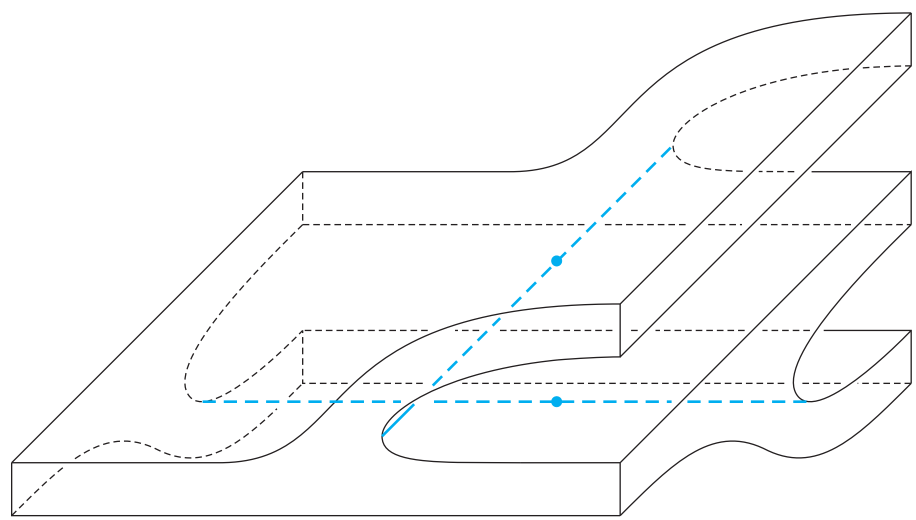
- Smooth thickening at edges and ∂-edges



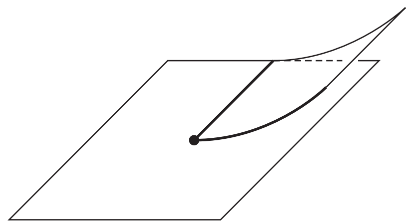
- Smooth thickening at vertices

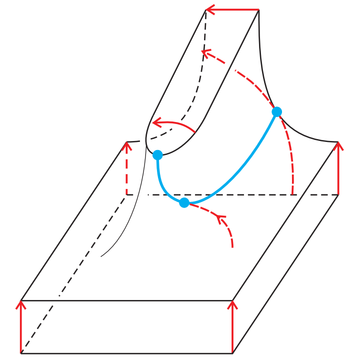


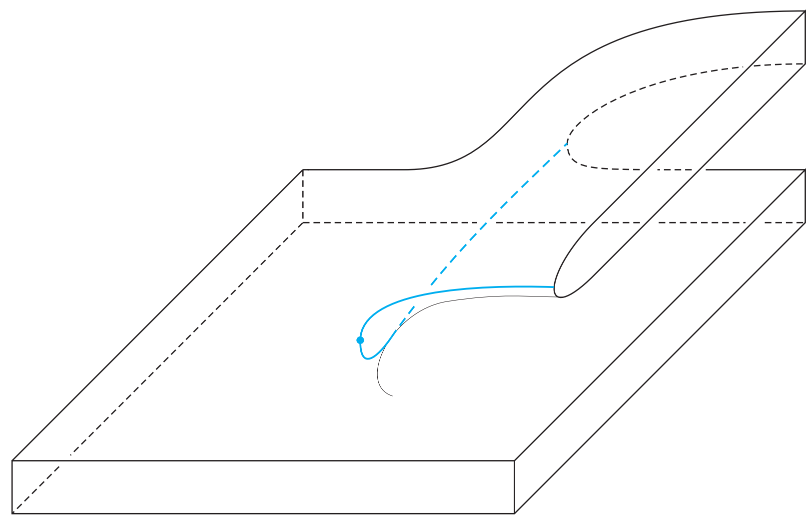




- Smooth thickening at spikes

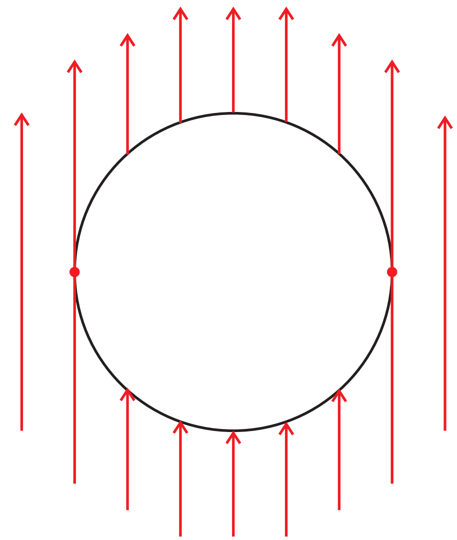






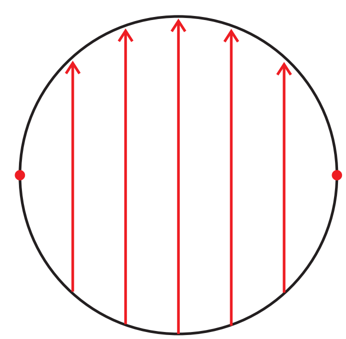
Extra condition on objects

(U, ν) must include at least one



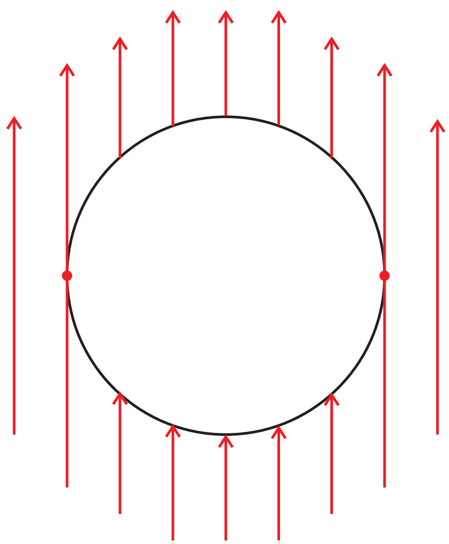
Reconstruction

cap (U, ν) with



Proposition Reconstruction well-defined

If there are two



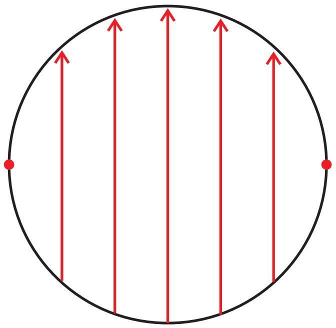
capping any of them
gives the same

Proposition Reconstruction surjective

I We want to obtain a given (M, v) by capping (U, ν) with

- ν generic on ∂U
- Transition orbits of ν not elsewhere tangent to ∂U
- Each orbit of ν tangent to ∂U in at most two points, and transversely if so
- All orbits of ν go from ∂U to ∂U

$*$ Choose U as M minus a "very big"

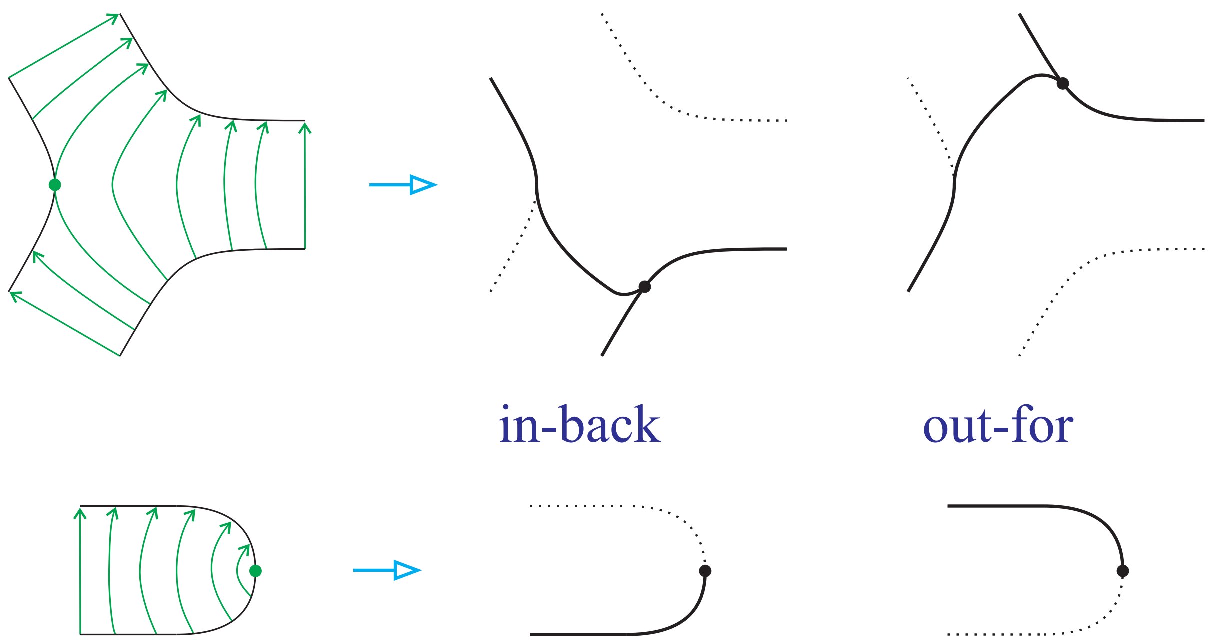


(trivially combed ball)
achieving first and last condition

Other two conditions true up to homotopy

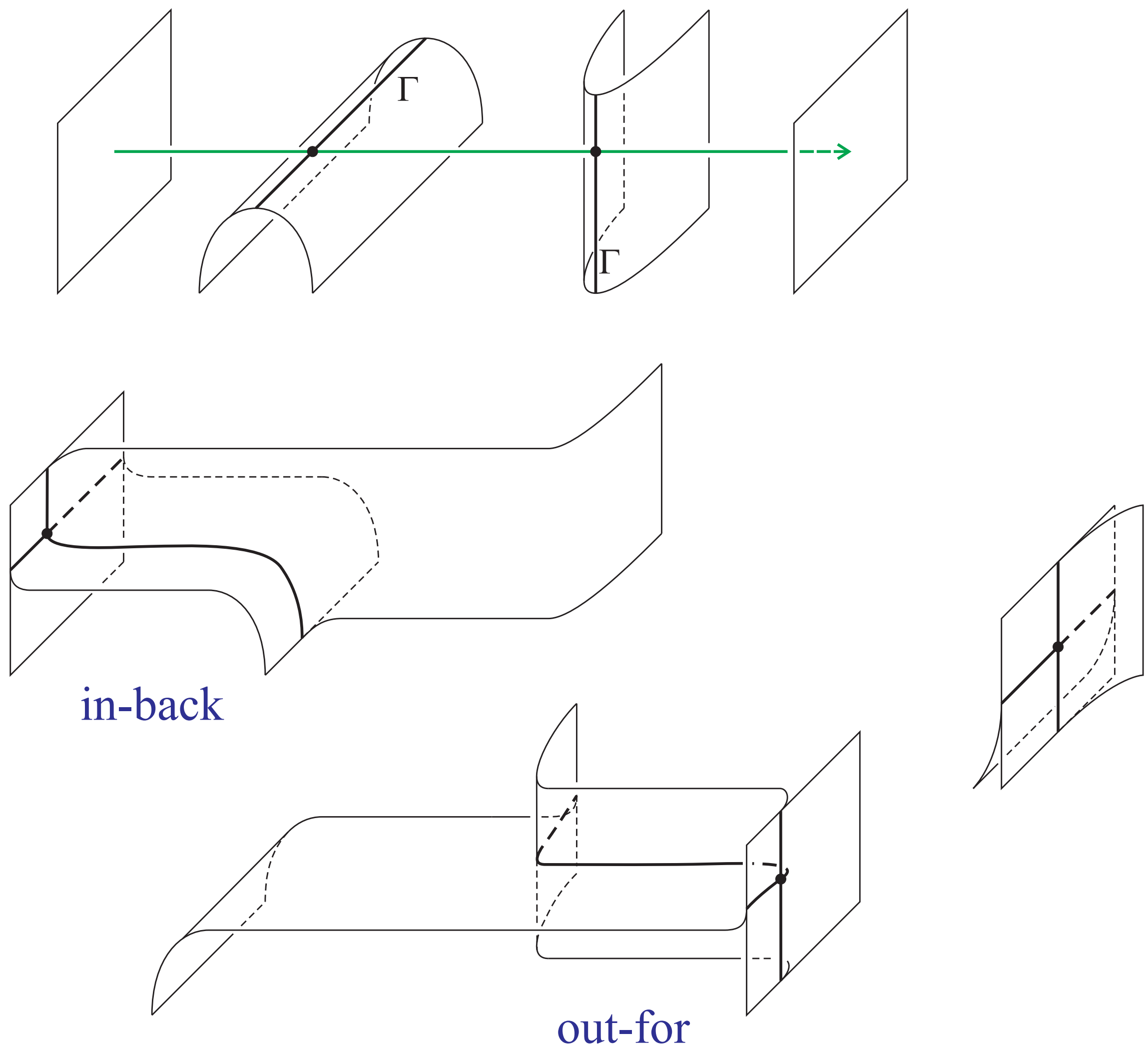
II Given (U, ν) find P with $U = U(P)$ and $\nu = \nu(P)$

- $*$ **in-backward** P: in-region of ∂U union orbits to concave or transition points
- $*$ **out-forward** P: out-region of ∂M union orbits from concave or transition points

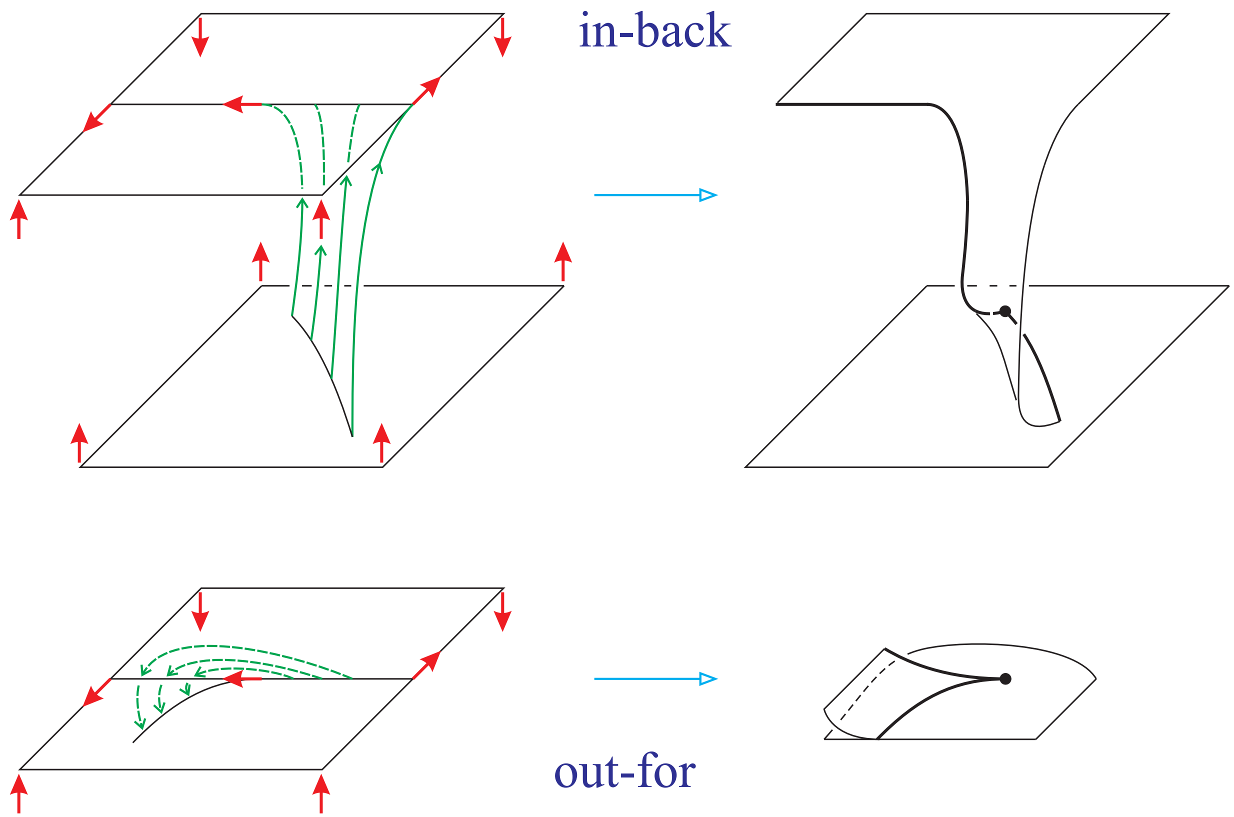


They are the same and they work

Birth of vertices

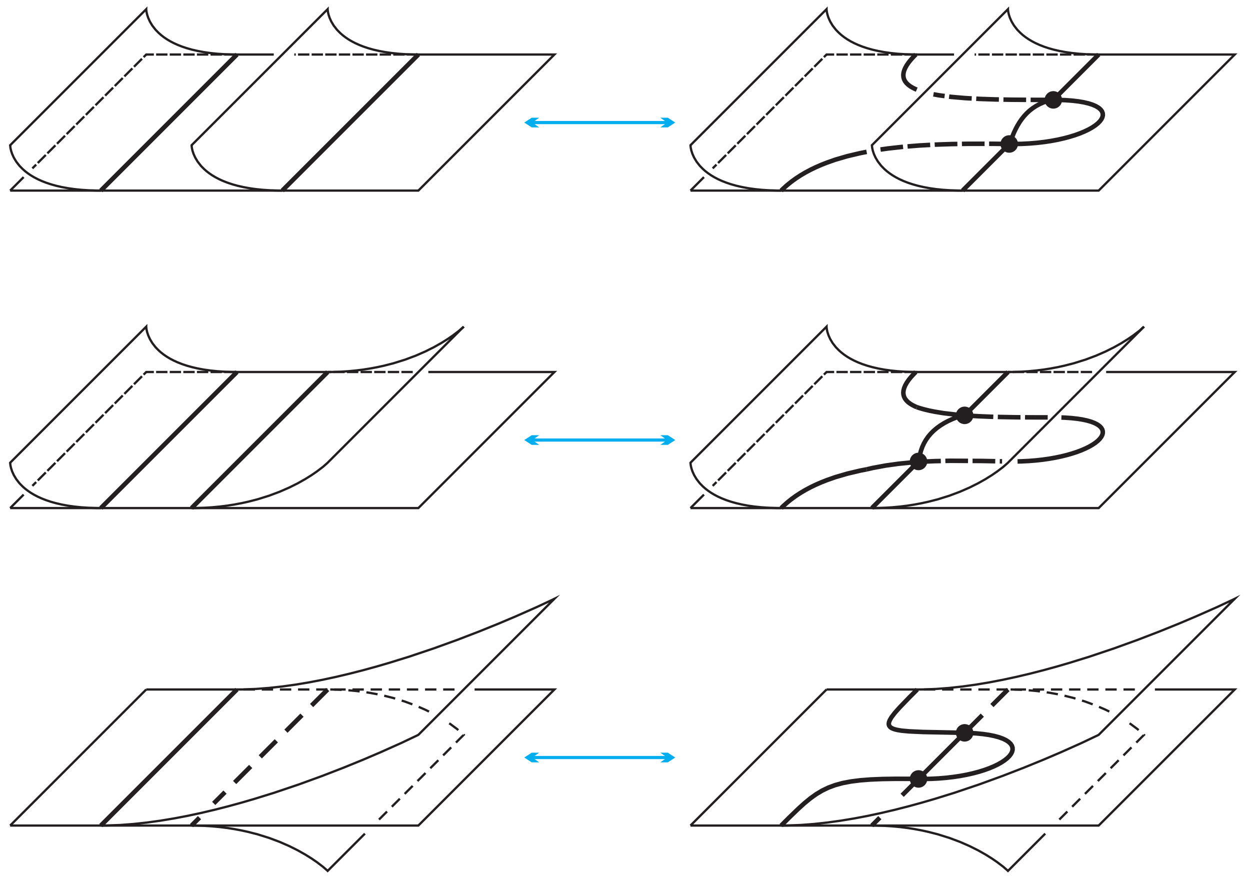


Birth of spikes

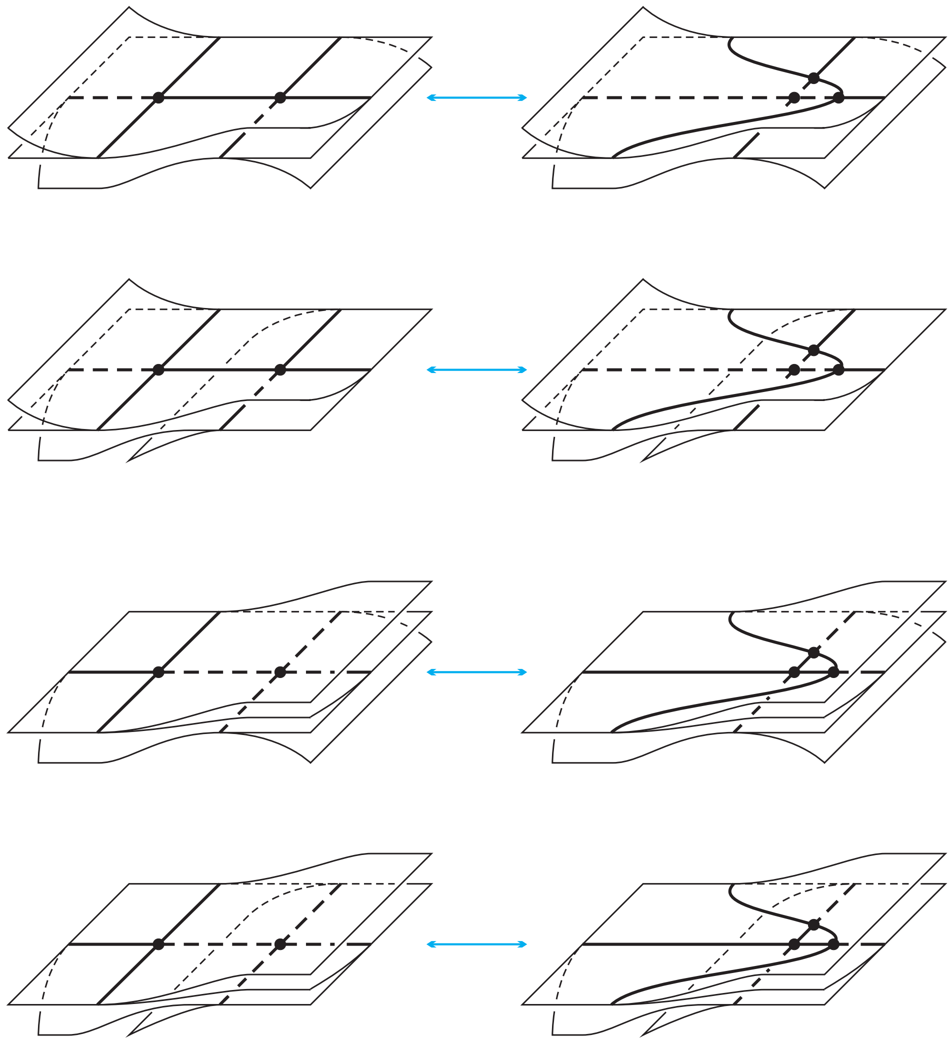


5.3. Moves

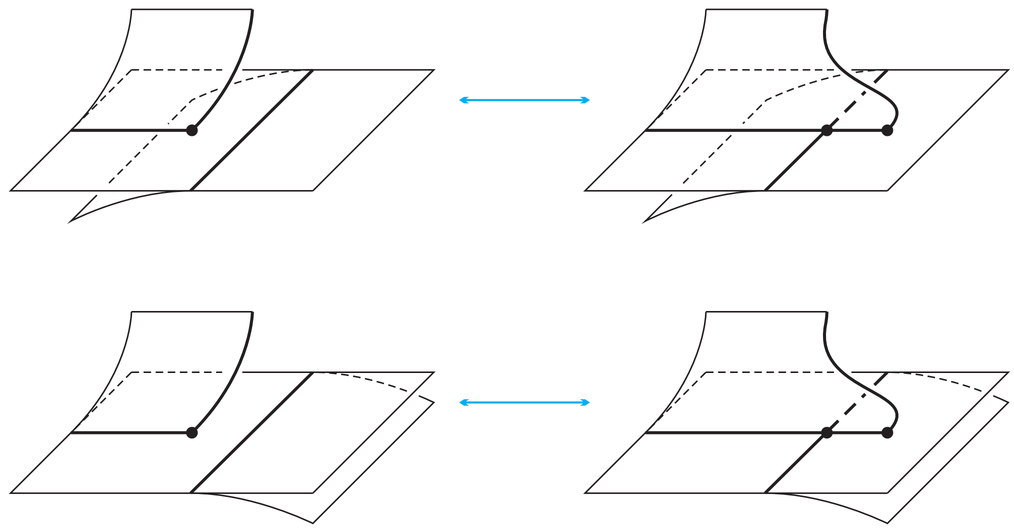
- $0 \leftrightarrow 2$ sliding moves



• $2 \leftrightarrow 3$ sliding moves



- spike-sliding moves



Idea of proof

I Express homotopy of ∂-to-∂ fields
on $U = (M$ minus trivially combed ball)
as composition of **elementary catastrophes**
reading effect on in-backward or out-forward spines

II Express isotopy of trivially combed ball
as composition of **elementary catastrophes**
reading effect on in-backward or out-forward spines

Fact Moves from **II** same as those from **I**

Catastrophe I.1

Orbit twice concavely tangent to ∂U but not transversely

Effect $0 \leftrightarrow 2$ sliding moves

Catastrophe I.2

Orbit thrice transversely and concavely tangent to ∂U

Effect $2 \leftrightarrow 3$ sliding moves

Catastrophe I.2

Transition orbit also concavely tangent to ∂U

Effect Spike-sliding moves

6. Spin structures via arbitrary spines

[BP1997] Combinatorial presentation of

$$\{(M, s) : \ s \text{ spin structure on } M\}$$

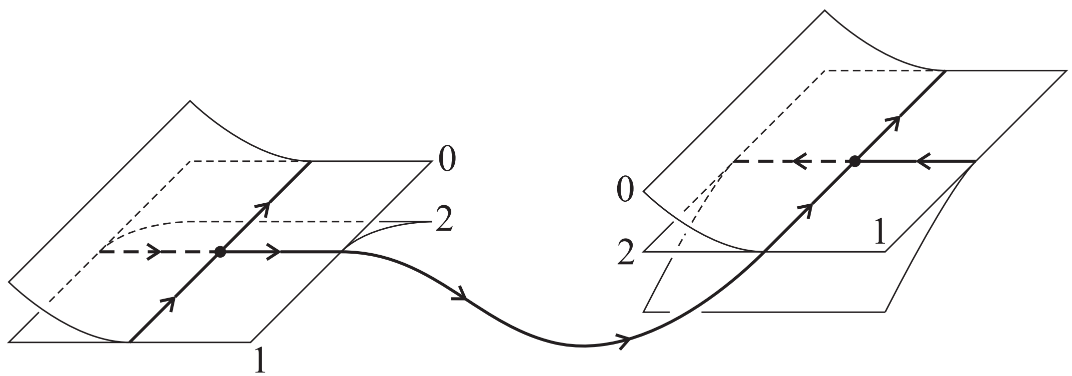
via branched spines — not all spines admit branching

Idea [BP2013] A weaker version of branching
that **exists on every** P
still allows to define $\nu(P), \mu(P)$ on
$S(P)$ = 4-valent gluing graph of triangulation dual to P

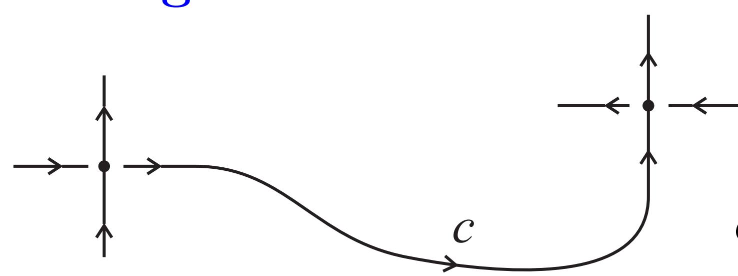
Pre-branching ω on P is an orientation of $S(P)$ with
2 edges in and 2 out at each vertex

Existence Express $S(P)$ as union of cycles

Weak branching b compatible with pre-branching ω is a branching at each vertex inducing ω



Graphic encoding

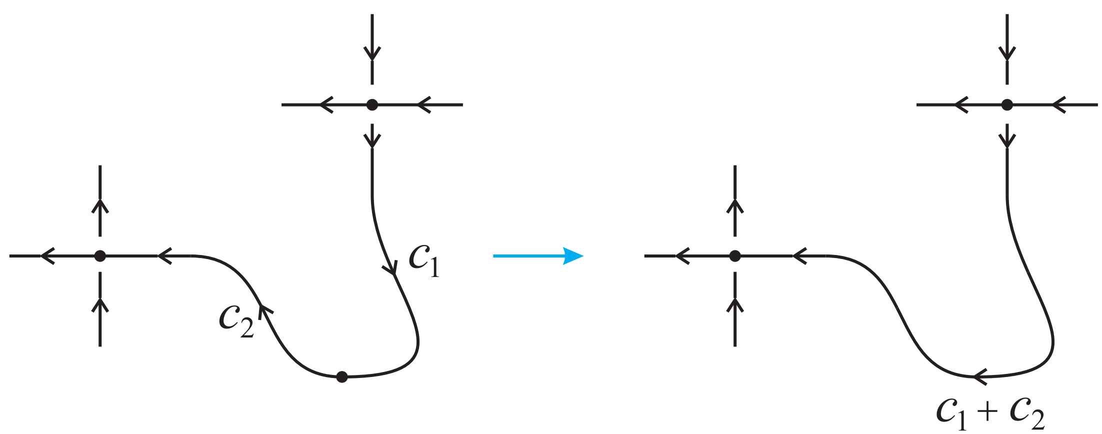


$c \in \mathbb{Z}/_{3\mathbb{Z}}$

Proposition

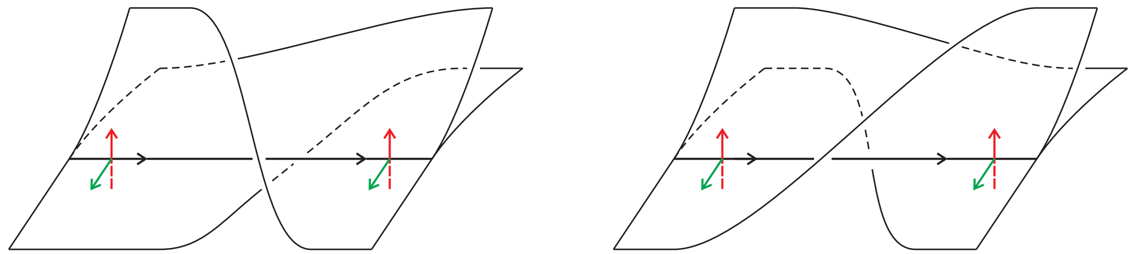
ω pre-branching on P $\quad b$ compatible weak branching

- They allow to define $\varphi(P) := (\nu(P), \mu(P))$ on $S(P)$
- The obstruction $\alpha(P, \omega, b) \in C^1(P; \mathbb{Z}/_{2\mathbb{Z}})$
  to extending $\varphi(P)$ on P
  can be computed explicitly
- φ and α are additive with respect to edge summation



Idea ν, μ defined at vertices

- obvious extension along branched edges (colour 0)
- extension along unbranched edges (colour ± 1)



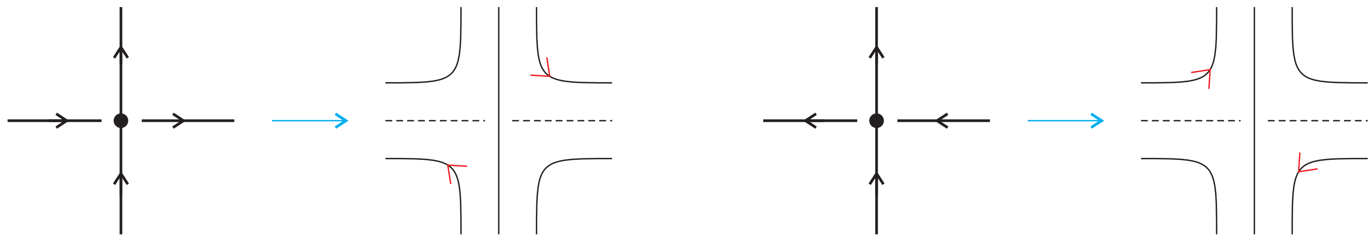
– extend ν vertical

– extend μ horizontal adding a full twist

Obstruction computation

$\alpha(P, \omega, b)$ on R is a sum of contributions in $\frac{1}{2}\mathbb{Z}/_{2\mathbb{Z}}$
(with final sum in $\mathbb{Z}/_{2\mathbb{Z}}$)

- from vertices — requires orientation of ∂R



- from edges

Proposition

$[\alpha(P,\omega,b)] = 0 \in H^2(P;\mathbb{Z}/_{2\mathbb{Z}})$ and there exists

$$\{\beta \in C^1(P;\mathbb{Z}/_{2\mathbb{Z}}) : \ \delta\beta = \alpha(P,\omega,b)\} \xrightarrow{s} \twoheadrightarrow \mathrm{Spin}(M)$$

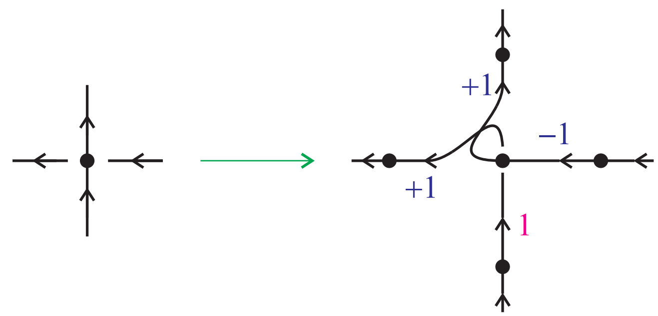
with $s(\beta_0) = s(\beta_1) \Leftrightarrow [\beta_0 + \beta_1] = 0 \in H^1(P;\mathbb{Z}/_{2\mathbb{Z}})$

$\beta \in C^1(P;\mathbb{Z}/_{2\mathbb{Z}})$ weight

Theorem

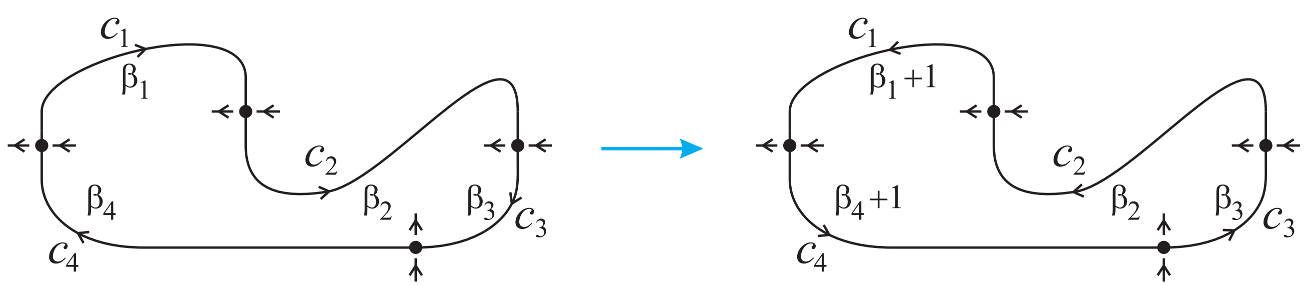
$s(P_0,\omega_0,b_0,\beta_0) = s(P_1,\omega_1,b_1,\beta_1) \ \Leftrightarrow \ldots$ **moves**

- P, ω, b fixed, β varies: $H^1(P; \mathbb{Z}/_{2\mathbb{Z}})$
- P, ω fixed, b changes: explicit local moves at vertices



$\pm 1 \in \mathbb{Z}/_{3\mathbb{Z}}$ edge colours $\qquad 1 \in \mathbb{Z}/_{2\mathbb{Z}}$ weight

- P fixed, ω varies: one global move (circuit)

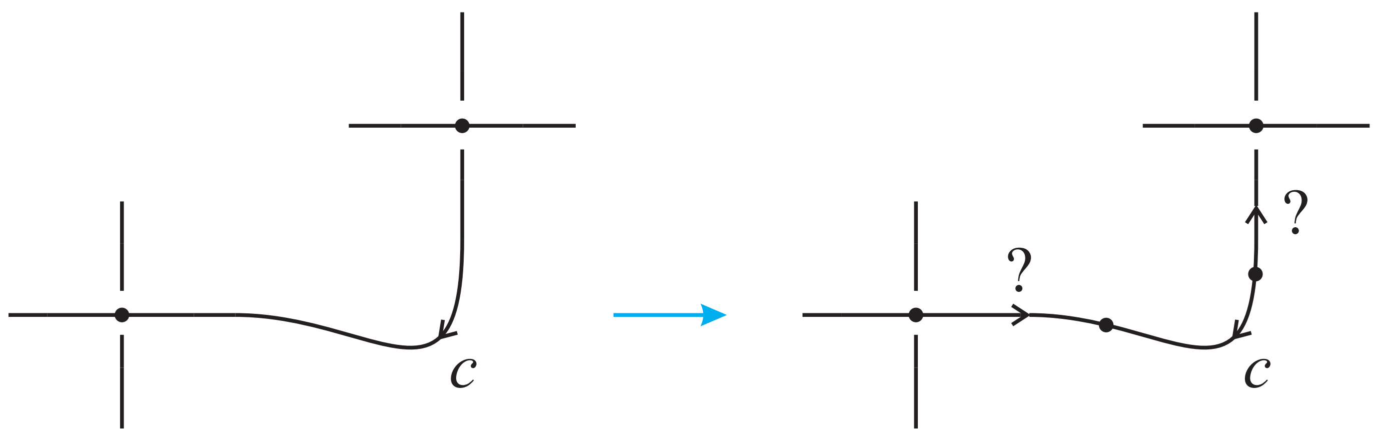


- P varies: weighted versions of 2-3 and bubble moves

Issue Replace global move by (semi-)local moves

Idea Allow branching to be "temporarily" arbitrary

- Start with weak branching b compatible with pre-branching ω
- Change branching at each vertex getting another b' compatible with some other ω'
- On some edges this will give



- To treat the change **locally** we need graphs encoding arbitrary branchings (even if globally we only want weak $\to$ weak changes)

- **New edges**

τ transposition

- **New weighted vertex moves**
- **New summation rules** for weighted edges
    - These rules are **not** associative for arbitrary branchings
    - When applied to weak $\to$ weak transitions
      they give well-defined result
    - The new weighted vertex moves
      when applied to weak $\to$ weak transitions
      generate the circuit move under the new summation rules
- Same weighted 2-3 and bubble moves