Cone singularities in anti-de Sitter geometry

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Cortona, June 3-7, 2013

Motivations

Why study AdS 3-manifolds ?

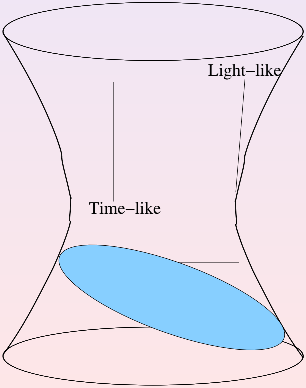
1. basic/simplest example of non-Riemannian symmetric space
2. similarities with hyperbolic geometry
    - quasifuchsian manifolds vs globally hyperbolic AdS manifolds
3. toy model for relativity
    - globally hyperbolic AdS manifolds
    - cone singularities (and more)
4. tool for Teichmüller theory
    - pleated surfaces in AdS_3 $\leftrightarrow$ earthquakes
    - maximal surfaces $\leftrightarrow$ minimal Lagrangian diffeos between hyperbolic surfaces

AdS_3 as a Lorentz analog of H^3

$$AdS_3 = \{x \in \mathbb{R}^{2,2} \mid \langle x, x \rangle = -1\} \ .$$

Constant curvature -1, $\pi_1(AdS_3) = \mathbb{Z}$.

- Conformal model, in a cylinder.
- Projective model, in a quadric.
- Space-like, time-like, light-like directions. Time-like geodesics are closed of length 2π.
- Totally geodesic space-like planes $\simeq H^2$.
- $Isom(AdS_3) = O(2,2)$.
- Boundary at ∞ with Lorentz-conformal structure.



AdS_3 as a Lorentz analog of S^3

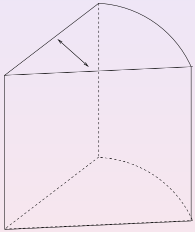
$AdS_3 = PSL(2,\mathbb{R})$ with its bi-invariant Killing metric ($S^3 \simeq O(3)$ with its bi-invariant Killing metric).
Left and right actions of $PSL(2,\mathbb{R})$,
$Isom_0(AdS_3) \simeq PSL(2,\mathbb{R}) \times PSL(2,\mathbb{R})$ (Left and right actions of $O(3)$ on itself, $O(4) \simeq O(3) \times O(3)$).

Not to be confused with de Sitter space dS_3, the 1-connected complete constant curvature 1 Lorentzian space.
dS_3 has a conformal model in the complement of the ball ($= H^3$) in $\mathbb{R}P^3$ resp. S^3.

Cone singularities in hyperbolic geometry

Def : glue isometrically two sides of a hyperbolic piece of cake (if angle $\leq 2\pi$).

Rigidity Thm (Hodgson, Kerckhoff '98) : for closed cone-mflds with singularities along closed curves and $\theta_i < 2\pi$, small deformations are parameterized by the variations of the cone angles.
Extension to geometrically finite cone-manifolds (Bromberg).
Powerfull tool, eg :



- Thurston's Orbifold Hyperbolization Thm (Boileau, Leeb, Porti, Cooper, Hodgson, Kerckhoff),
- Ahlfors' measure conjecture, the density conjecture (Brock, Bromberg, Evans, Souto, ...).

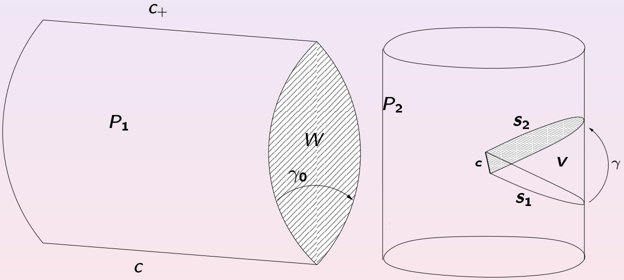
Cone singularities in AdS geometry

Richer situation, different motivations. Physical conditions : *degree* ≤ 2 (at most one future and past at each point), and *causal* neighborhood.
We *exclude singular light-like curves* for simplicity.
Four (remaining) types of cone singularities along curves, which can be :

- time-like : *massive particles*, $\theta < 2\pi$ if *positive mass*, i.e. *attracting* space-like geodesics.
- space-like of degree 2 : *tachyons*. Can be *positive* (attracting) time-like geodesics or *negative* (repelling).
- space-like of degree 1 (no future !), BTZ black holes.
- space-like of degre 0 (no future, no past) : Misner singularities.

Physical motivations (gravity in dim 2+1), cf Benedetti-Guadagnini '00 (massive particles in flat spaces of dim 2+1).
Complete classification Barbot-Bonsante-S (2011).

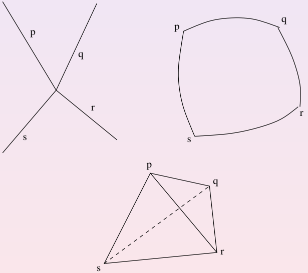
Local construction of singular lines



Graph singularities in hyperbolic cone-manifolds

Cone singularities can be along *graphs*.
The local description at a vertex v is given by its *link* – the space of geodesic rays starting from v. It is a spherical surface with cone singularities (corresponding to the singular segments at v).

If angles $< 2\pi$, those metrics are exactly those induced on convex polyhedra in S^3 (Alexandrov, '50).

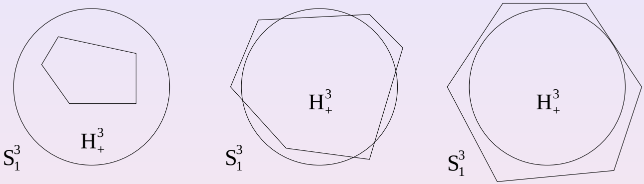


Thm (Mazzeo-Montcouquiol, Weiss 2011). For angles $< 2\pi$, closed hyperbolic cone-manifolds singular along a graph are locally rigid : small deformations are parameterized by small variations of the angles.

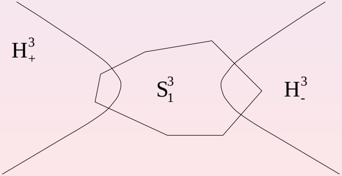
Interactions of particles in AdS geometry

Vertices of singular graphs are more complex than in the hyperbolic case.
The link of a vertex v is locally modeled on the space of rays from 0 in $\mathbb{R}^{2,1}$: HS^2, made of two copies of S^2 and one of dS_2.
So the link of v is a surface with an "HS-structure".
Def. *positive mass condition :* for all singular points (vertices or in particles), all simple closed curves in dS_2 part of link have length $< 2\pi$.
Thm (Barbot-Bonsante-S 2011). If tachyons are positive and the positive mass condition holds, the links of vertices (collisions points) are exactly the HS-structures induced on convex polyhedra in HS^3.
HS^3 : link of 0 in $\mathbb{R}^{3,1}$, two copies of H^3 and one of dS_3.
(Based on extension of Alexandrov thm to HS^3 in S 1998, 2001).

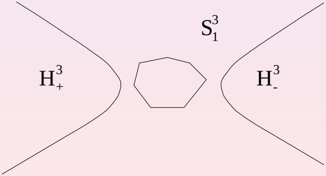
Examples



Polyhedra of hyperbolic type



Bi-hyperbolic type



Compact type

Closed AdS 3-manifolds

Thm (Kulkarni-Raymond). Torsion-free discrete subgroups Γ of $PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R})$ acting properly discont. and cocompactly on AdS_3 are of the form

$$\Gamma = \{(j(\gamma), \rho(\gamma)) : \gamma \in \pi_1(S)\},$$

where S is a closed surface and j and ρ are representations of $\pi_1(S)$ into $PSL_2(\mathbb{R})$ with j Fuchsian.

Thm (Kassel 2009). (j, ρ) is proper discont. iff

$$C_{length}(j, \rho) := \sup_{\gamma \in \pi_1(S) \setminus \{e\}} \frac{\lambda(\rho(\gamma))}{\lambda(j(\gamma))} < 1 \ .$$

Main difference with hyperbolic case : flexibility vs rigidity.

Recent work : Kassel-Guéritaud, Kassel-Guéritaud-Danciger, Goldman.

Transitions from hyperbolic to AdS with tachyons

Danciger (2013) explored the transition between hyperbolic and AdS geometry, for punctured torus bundles with Anosov monodromy, using the monodromy triangulation (Guéritaud).

Hyperbolic structures (with one cone singularity) correspond to solutions of Thurston's gluing equations for *complex* shape parameters $z_i \in \mathbb{C}$ associated to the simplices.

AdS structures (with one tachyon) correspond to **pseudo-complex** shape parameters, $z_i \in \mathbb{R} + \tau\mathbb{R}$ with $\tau^2 = 1$.

Real solutions correspond to a transitional geometry : transversely hyperbolic structures.

In both hyperbolic and AdS case, solutions for a 1-parameter family parameterized by the angle, and they are locally rigid (rel. angle).

Question (Danciger). Are closed AdS manifolds with tachyons locally rigid ?

Globally hyperbolic AdS manifolds

Def. An AdS manifold is globally hyperbolic maximal (GHM) if it contains a closed space-like surface S, any inextendible time-like curve intersects S exactly once, and it is maximal (for inclusion) under those conditions.

Def. A hyperbolic manifold is quasifuchsian if it is homeomorphic to $S \times \mathbb{R}$ and contains a non-empty compact convex subset.

Thm (Mess). The holonomy representation $\rho : S \rightarrow Isom(AdS_3)$ of M splits as $(\rho_L, \rho_R) \in PSL(2, \mathbb{R}) \times PSL(2, \mathbb{R})$, where ρ_L, ρ_R have maximal Euler number, so $\rho_L, \rho_R \in \mathcal{T}_S$. $(rho_L, \rho_R) \in \mathcal{T}_S \times \mathcal{T}_S$ uniquely determine M.

Thm (Bers). A quasifuchsian manifold M has a boundary at infinity $S_- \cup S_+$, each with a conformal structure c_-, c_+. $c_-, c_+ \in \mathcal{T}_S \times \mathcal{T}_S$ uniquely determine M.

Hyperbolic and AdS convex cores

A quasifuchsian manifold M contains a smallest $\neq \emptyset$ convex subset C, its *convex core*.
Same for GHM AdS manifold.
∂C is the union of surfaces, with hyperbolic induced metrics m_-, m_+, bend along measured geodesic laminations l_-, l_+.
Same for GHM AdS manifolds.

Prescribing the induced metric / bending lamination

Thurston conjectured that any $m_-, m_+ \in \mathcal{T}_S$ can be uniquely obtained as the induced metric on ∂C.

Thm. Any $m_-, m+$ can be obtained. Uniqueness ? (Folklore, based on results of Epstein-Marden 1986 or Labourie 1992).

Mess conjectured that any m_-, m_+ can be uniquely obtained.

Thm (Diallo 2013). Existence holds. Uniqueness ?

Thurston conjectured that any "reasonable" l_-, l_+ can be uniquely obtained.

Thm (Bonahon, Otal 2004). Existence holds. Uniqueness ?

Mess conjectured that any l_-, l_+ that fill S can be uniquely obtained.

Thm (Bonsante, S 2012). Existence holds. Uniqueness ?

Equivalent : existence and uniqueness of a fixed point for the composition of earthquakes along two laminations that fill.

Particles of angle $< \pi$

Def. A quasifuchsian manifold with particles M is defined as a quasifuchsian manifold, with n cone singularities along lines from $-\infty$ to $+\infty$. Conformal structure at ∞ marked by endpoints, $c_-, c_+ \in \mathcal{T}_{S,n}$.

Def. GHM AdS manifold with (massive) particles (cone sings along time-like lines, $\theta < \pi$) have left and right hyperbolic metrics h_-, h_+ (Krasnov-S 2007).

Thm. (Lecuire, Moroianu, S 2009) Any c_-, c_+ can be uniquely obtained (for fixed $\theta_1, \cdots, \theta_n$).

(Partial) results on convex core boundary also extend well.

Thm. (Bonsante, S 2012) Any h_-, h_+ can be uniquely obtained (for fixed $\theta_1, \cdots, \theta_n$.)

Angles $< 2\pi$ and colliding massive particles

When cone angles are $\theta_i < 2\pi$, collisions can (and do) occur.
Understanding of those GHM AdS manifolds remains limited.
To a "good" GHM AdS manifold with "interacting particles" one can associate a *sequence* of pairs of hyperbolic metrics with cone singularities, one for each slice without interaction.
For each collision, a surgery happens on both left and right hyperbolic cone-metric : a copy of the past component of the link is replaced by a copy of the future link.
Thm (Barbot, Bonsante, S 2013). This sequence of pairs of hyperbolic cone surfaces provides a local parameterization of the moduli space of GHM AdS metrics.
Question. Is this a global parameterization ?

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Some open questions on anti-de Sitter geometry,
T. Barbot, F. Bonsante, J. Danciger, W. M. Goldman, F. Guéritaud, F. Kassel, K. Krasnov, J.-M. Schlenker, A. Zeghib,
arXiv :1205.6103.

The end

Thanks for your attention –
and happy birthday Riccardo !