Dynamiqs on free-by-cyclic groups.

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$\mathbb{H}$appy Birthday Riccardo!!

joint with S. Dowdall and I. Kapovich

Outline 1/17

$\mathbf{F}_N$ a free group of rank N, $\phi \in \mathrm{Out}(\mathbf{F}_N)$

$\Rightarrow G = G_\phi = \mathbf{F}_N \rtimes_\phi \mathbb{Z} \xrightarrow{u_0} \mathbb{Z}$

$\Rightarrow$ for u "close" to u_0 in $\mathbb{P}H^1(G;\mathbb{R}) = \mathbb{P}\mathrm{Hom}(G,\mathbb{R})$,

$\ker(u) \cong \mathbf{F}_{N(u)}$, $N(u) \in \mathbb{Z}_+$ and $G = \ker(u) \rtimes_{\phi_u} \mathbb{Z}$

[Neumann,Geoghegan-Mihalik-Sapir-Wise]

Goal: Describe geometric, topological, and dynamical relationships between ϕ_u and ϕ

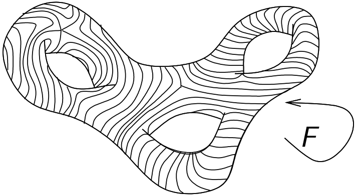
Motivation from fibered hyperbolic 3–manifolds.

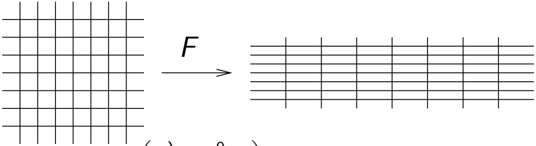
Motivation: Pseudo-Anosov homeomorphisms 2/17

$F\colon S \to S$ pseudo-Anosov on S, a closed surface of genus $g \geq 2$:

- $\exists$ invariant, transverse measured foliations $\mathcal{F}_S^{\pm}$ on S
- F stretches/contracts the measures
- $\lambda = \lambda(F) = \lim_{n\to\infty} \sqrt[n]{\mathrm{length}(F^n(\alpha))}$

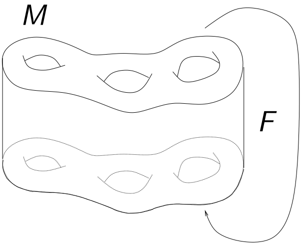
  $= \textit{dilatation}$ of F





$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

Motivation: The mapping torus 3/17



- $M = M_F = S \times [0,1]/(x,1) \sim (F(x),0)$
  $\cong \mathbb{H}^3/\Gamma$ [Thurston]

  $\Rightarrow \quad S \longrightarrow M \xrightarrow{\eta_0} S^1$ fibration

- $u_0 = (\eta_0)_* \in \mathrm{Hom}(\pi_1 M, \mathbb{R}) = H^1(M)$
  integral... $u_0 =$PD$[S]$

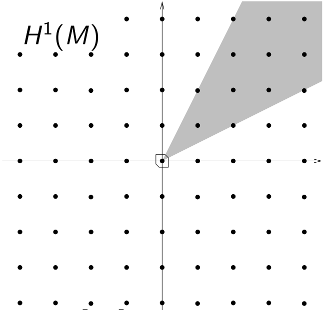
- Suspension flow $\psi_s \colon M \to M$, 1^{st} return $= F \colon S \to S$

- $\mathcal{F}$ foliation by fibers $\Rightarrow e = e(T\mathcal{F}) \in H^2(M)$ Euler class

- $\mathcal{O} = \{\gamma \subset M \mid \gamma$ closed orbit of singularity of $\mathcal{F}_S^{\pm}\}$

$$\Rightarrow \qquad \mathrm{PD}(e) = \frac{1}{2}\sum_{\gamma\in\mathcal{O}}(2 - \deg(\gamma))\gamma \in H_1(M)$$

Motivation: Thurston and Fried 4/17



$S \longrightarrow M = M_F \xrightarrow{\eta_0} S^1$ fibration

$u_0 = (\eta_0)_* = \mathrm{PD}[S] \in H^1(M)$ integral.

$u_0 \in \mathcal{C} \subset H^1(M)$, an (open) cone

on a fibered face of $||\cdot||_T$–ball,

Theorem [Thurston,Fried]

For all integral $u \in \mathcal{C} \Rightarrow$

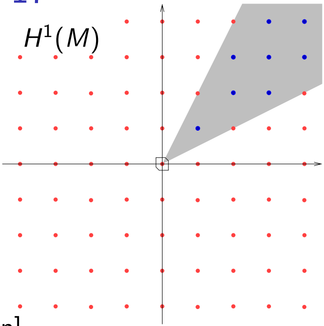
$\exists$ fibration $S_u \longrightarrow M \xrightarrow{\eta_u} S^1$ with $u = (\eta_u)_* = \mathrm{PD}[S_u]$ s.t.

- $\langle e, u\rangle = \chi(S_u) = -||u||_T$ and
- $\psi \pitchfork S_u$ and first return $F_u \colon S_u \to S_u$ is pseudo-Anosov.

$\exists!\ \mathfrak{H} \colon \mathcal{C} \to \mathbb{R}$ continuous, convex, homogeneous of degree -1 such that for all integral $u \in \mathcal{C}$

- $\log(\lambda(F_u)) = \mathfrak{H}(u)$ see also [Oertel, Long-Oertel, Matsumoto, McMullen]

Motivation: Dilatation asymptotics 5/17



Corollary Suppose $K \subset \mathcal{C}$ is compact and

$$\{u_n\}_{n=1}^{\infty} \subset \mathbb{R}_+ K$$

all u_n primitive integral, $u_n \to \infty$.

Then $g_n = \text{genus}(S_{u_n}) \to \infty$ and

$$\frac{c_0}{g_n} \leq \log(\lambda(F_{u_n})) \leq \frac{c_1}{g_n}$$

for some $0 < c_0 < c_1 < \infty$. [Penner,McMullen]

Theorem [Farb-L-Margalit] All pseudo-Anosov $F\colon S_g \to S_g$ with $\log(\lambda(F)) \leq c/g$ are monodromies of fibrations of one of a finite list of fibered, finite volume hyperbolic 3–manifolds, Dehn filled along the boundary of the fiber.

See also [Agol].

Transition: Group theory 6/17

$\pi_1 M = \pi_1 S \rtimes_{F_*} \mathbb{Z}$.

In fact, M is determined up to homeomorphism by $F_* \in \mathrm{Out}(\pi_1(S))$ [Dehn-Nielsen-Baer].

$\phi \in \mathrm{Out}(\pi_1(S))$ is represented by a pseudo-Anosov F if and only if ϕ has no nontrivial periodic conjugacy classes if and only if $\pi_1 S \rtimes_\phi \mathbb{Z}$ is word-hyperbolic. [Thurston]

$\lambda(F)$ = growth rate of word length in $\pi_1 S$ under iteration of F_*.

Integral $u \in \mathrm{Hom}(\pi_1 M, \mathbb{R}) = H^1(M)$ is induced by a fibration over S^1 if and only if $\ker(u)$ is finitely generated [Stallings]

Atoroidal and fully irreducible 7/17

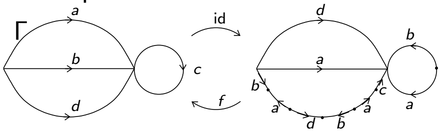
Theorem [Bestvina-Feign, Brinkmann, Bestvina-Handel] Let $\phi \in \mathrm{Out}(\mathbf{F}_N)$ be

- *Atoroidal*: no nontrivial periodic conjugacy classes, and
- *Fully irreducible*: no nontrivial periodic free factors.

Then

- $G = G_\phi = \mathbf{F}_N \rtimes_\phi \mathbb{Z}$ is word-hyperbolic, and
- ϕ is represented by an *irreducible train track map*.

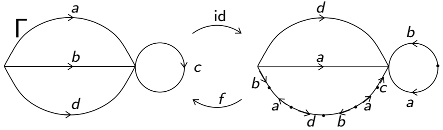
Example:



Many other examples [Clay-Pettet]

- A graph Γ, $\pi_1\Gamma \cong \mathbf{F}_N$,
- $f : \Gamma \to \Gamma$ a h.e. and $f_* = \phi$
- $f(V\Gamma) \subset V\Gamma$
- $f^n|_e$ is an immersion for all $n \geq 1$ and for all edges e
- irreducible transition matrix...

Dynamics and stretch factors 8/17



Transition matrix and Perron-Frobenius eigenvalue/eigenvector

$$A(f) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 1 \end{pmatrix}, \qquad \lambda \approx 2.4142, \qquad \mathbf{v} \approx \begin{pmatrix} .2265 \\ .0939 \\ .1327 \\ .5469 \end{pmatrix}$$

$\rightsquigarrow$ metric graph $(\Gamma, d_{\mathbf{v}})$, $f \simeq f_{\mathbf{v}} : (\Gamma, d_{\mathbf{v}}) \to (\Gamma, d_{\mathbf{v}})$, affine-stretch by λ on all edges.

$$\lambda = \lambda(f) = \lambda(\phi) = \lim_{n \to \infty} \sqrt[n]{\mathrm{length}(f^n(\alpha))} = \textit{stretch factor}.$$

depends only on $\phi = f_*$, not on f, α, or metric.

A model for free-by-cyclic group

Idea: Dynamics on branched surfaces in 3–manifolds [Williams,Christy,...,Benedetti-Petronio,...,Brinkmann-Schleimer,..]

Generalizations: Outside 3–manifolds [Gautero,Wang,...]

$$\phi \in \mathrm{Out}(\mathbf{F}_N) \quad \rightsquigarrow \quad (X_\phi, \psi, \mathcal{A})$$

- X_ϕ is a polyhedral 2–complex, $K(G,1)$ for $G = \mathbf{F}_N \rtimes_\phi \mathbb{Z}$.
- ψ is a semi-flow on X_ϕ.
- $\mathcal{A} = \{[z] \in H^1(X_\phi) \mid z \in Z^1(X_\phi)$ positive, cellular $\}$, open cone.
- $u_0 \in Hom(G, \mathbb{R}) = H^1(X_\phi)$, $u_0(x,n) = n \Rightarrow u_0 \in \mathcal{A}$.

Proposition. Given $\phi \in \mathrm{Out}(\mathbf{F}_N)$ let $(X_\phi, \psi, \mathcal{A})$ be as above. Then for all $u \in \mathcal{A}$ primitive integral there exists $\eta_u \colon X_\phi \to S^1$ with $(\eta_u)_* = u$ satisfying:

- $\Gamma_u = \eta_u^{-1}(*) \subset X_\phi$ is a graph for any $* \in S^1$;
- $\Gamma_u \hookrightarrow X_\phi$ induces an isomorphism $\pi_1(\Gamma_u) \cong \ker(u)$;
- $\Gamma_u \pitchfork \psi$, 1^{st} return $f_u \colon \Gamma_u \to \Gamma_u$ has $(f_u)_* = \phi_u \in \mathrm{Out}(\ker(u))$

Slightly different construction, but similar ideas as in [Gautero,Wang].

Theorem [Dowdall-Kapovich-L] 11/17

Given $\phi \in \mathrm{Out}(\mathbf{F}_N)$ *fully irreducible, atoroidal*. Let $(X_\phi, \psi, \mathcal{A})$, and $\Gamma_u \to X_\phi \xrightarrow{\eta_u} S^1$ and $f_u \colon \Gamma_u \to \Gamma_u$, for primitive integral $u \in \mathcal{A}$, all be as above.

Then $\exists! \; \mathfrak{H} \colon \mathcal{A} \to \mathbb{R}$ continuous, convex, homogeneous of degree -1, and the following hold for any primitive integral $u \in \mathcal{A}$

- f_u is an irreducible train track map,
- $\phi_u = (f_u)_*$ is fully irreducible and atoroidal,
- $\log(\lambda(f_u)) = \log(\lambda(\phi_u)) = \mathfrak{H}(u)$,
- $\chi(\Gamma_u) = \langle \epsilon, u \rangle$, where

$$\epsilon = \frac{1}{2} \sum_{e \in \mathcal{E}(X_\phi)} (2 - \deg(e))\, e \in H_1(X_\phi)$$

Theorem [Dowdall-Kapovich-L] – Remarks

$\phi \in \mathrm{Out}(\mathbf{F}_N)$ *fully irreducible, atoroidal*, then for $u \in \mathcal{A}$ primitive integral, $f_u \colon \Gamma_u \to \Gamma_u$ satisfies:

- f_u is an irreducible train track map,
- $\phi_u = (f_u)_*$ is fully irreducible and atoroidal,
- $\log(\lambda(f_u)) = \log(\lambda(\phi_u)) = \mathfrak{H}(u)$,
- $\chi(\Gamma_u) = \langle \epsilon, u \rangle$.

Remarks:

1. ϕ atoroidal implies all ϕ_u atoroidal by [Brinkmann,Bestvina-Feighn].
2. If we only assume ϕ is fully irreducible, then in general ϕ_u will not be fully irreducible... 3–manifolds.
3. Linearity of $u \mapsto \chi(\Gamma_u)$ also follows from Alexander norm considerations [McMullen,Button,Dunfield].

Small stretch factors 13/17

Corollary With the setup as above suppose $K \subset \mathcal{A}$ is compact and

$$\{u_n\}_{n=1}^{\infty} \subset \mathbb{R}_+ K$$

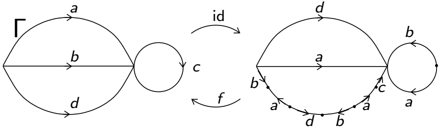
all u_n primitive integral, $u_n \to \infty$.

Then $N(n) = rk(\ker(u_n)) \to \infty$ and

$$\frac{c_0}{N(n)} \leq \log(\lambda(\phi_{u_n})) \leq \frac{c_1}{N(n)}$$

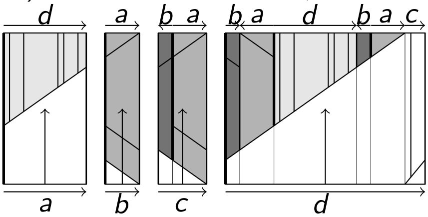
Theorem [Algom-Kfir–Rafi] All irreducible $\phi \in \mathrm{Out}(\mathbf{F}_N)$ with $\log(\lambda(\phi)) \leq c/N$ (over all $N \geq 2$) are monodromies of "surgeries" on a mapping torus of a graph map.

Idea of construction and proof.



$f : \Gamma \to \Gamma$ an irreducible train track representative for $\phi \in \mathrm{Out}(\mathbf{F}_N)$.

Build $M_f = \Gamma \times [0,1]/(x,1) \sim (f(x),0) \to S^1$ and semi-flow ψ...



Difficult to perturb $M_f \to S^1$ "nicely" since fibers are not transverse to 1–cells.

Take a quotient $M_f \to X_\phi$ so $\psi|_\Gamma$ descends to a "Stallings folding line", c.f. [Bestvina-Feighn,Francaviglia-Martino]

Cell structure w/ "vertical" and "skew" 1–cells, "trapazoid" 2–cells.
$\eta \colon X_\phi \to S^1$ can be perturbed to $f_u \colon X_\phi \to S^1$.
for $u \in \mathcal{A}$

Train track map 15/17

f_u an irreducible train track map?...

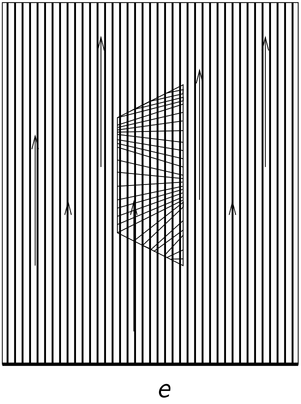
Lemma For every edge e of Γ, the characteristic map $\sigma\colon [0,1] \to e$ and the semi-flow ψ determine a map

$$[0,1] \times [0,\infty) \to X_\phi$$

by

$$(x,t) \mapsto \psi_t(\sigma(x)).$$

This map is locally injective.



$\phi_u = (f_u)_*$ fully irreducible:

• Use characterization of full irreducibility for irreducible train track maps of Kapovich, prove that this is inherited by f_u from f. Similar ideas from lemma.

Existence of $\mathfrak{H}$:

• Argue as Fried does, jumping through hoops...

Euler-like class calculates Euler characteristic

• Calculate.

Thanqs and
$\mathbb{H}$appy birthday!