SECOND-ORDER ASYMPTOTICS OF THE FRACTIONAL PERIMETER AS $s \to 1$

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Abstract. In this note we provide a second-order asymptotic expansion of the fractional perimeter $\mathrm{P}_s(E)$, as $s \to 1^-$, in terms of the local perimeter and of a higher order nonlocal functional.

Contents

1. Introduction

The fractional perimeter of a measurable set $E \subseteq \mathbb{R}^d$ is defined as follows:

$$\mathrm{P}_s(E) = \int_E \int_{\mathbb{R}^d \setminus E} \frac{1}{|x-y|^{d+s}} dy dx \qquad s \in (0,1). \tag{1}$$

After being first considered in the pivotal paper [4] (see also [14] where the definition was first given), this functional has inspired a variety of literature both in the community of pure mathematics, regarding for instance existence and regularity of fractional minimal surfaces, and in view of applications to phase transition problems and to several models with long range interactions. We refer to [16], and references therein, for an introductory review on this subject.

The limits as $s \to 0^+$ or $s \to 1^-$ are critical, in the sense that the fractional perimeter (1) diverges to $+\infty$. Nevertheless, when appropriately rescaled, such limits give meaningful information on the set.

The limit of the (rescaled) fractional perimeter when $s \to 0^+$ has been considered in [10], where the authors proved the pointwise convergence of $s\mathrm{P}_s(E)$ to the volume functional $d\omega_d|E|$, for sets E of finite perimeter, where ω_d is the volume of the ball of radius 1 in $\mathbb{R}^d$. The corresponding second-order expansion has been recently considered in [7]. In particular it is shown that

$$\mathrm{P}_s(E) - \frac{d\omega_d}{s}|E| \xrightarrow{\Gamma} \int_E \int_{B_R(x)\setminus E} \frac{1}{|x-y|^d} dxdy - \int_E \int_{E \setminus B_R(x)} \frac{1}{|x-y|^d} dxdy - d\omega_d \log R|E|,$$

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with respect to the L^1-convergence of the corresponding characteristic functions, where the limit functional is independent of R, and it is called the 0-fractional perimeter.

The limit of $\mathrm{P}_s(E)$ as $s \to 1^-$, in pointwise sense and in the sense of Γ-convergence, has been studied in [1, 5], where it is proved that

$$(1-s)\mathrm{P}_s(E) \xrightarrow{\Gamma} \omega_{d-1}\mathrm{P}(E),$$

with respect to the L^1-convergence.

In this paper we are interested in the analysis of the next order expansion. In particular we will prove in Theorem 2.1 that

$$\frac{\omega_{d-1}\mathrm{P}(E)}{1-s} - \mathrm{P}_s(E) \xrightarrow{\Gamma} \mathcal{H}(E) \qquad \text{as } s \to 1^-,$$

with respect to the L^1-convergence, and the limit functional is defined as

$$\begin{aligned}\mathcal{H}(E) := &\int_{\partial^* E}\int_{E\Delta H^-(y)\cap B_1(y)} \frac{|(y-x)\cdot\nu(y)|}{|x-y|^{d+1}} dx d\mathcal{H}^{d-1}(y) \\ &- \int_{\partial^* E}\int_{E\setminus B_1(y)} \frac{(y-x)\cdot\nu(y)}{|x-y|^{d+1}} dx d\mathcal{H}^{d-1}(y) - \omega_{d-1}\mathrm{P}(E)\end{aligned} \tag{2}$$

for sets E with finite perimeter, and $\mathcal{H}(E) = +\infty$ otherwise. Here we denote by $\partial^* E$ the reduced boundary of E, by $\nu(y)$ the outer normal to E at $y \in \partial^* E$ and by $H^-(y)$ the hyperplane

$$H^-(y) := \{x \in \mathbb{R}^d \mid (y-x)\cdot\nu(y) > 0\}.$$

We observe that, in dimension $d = 2$, the functional $\mathcal{H}(E)$ coincides with the Γ-limit as $\delta \to 0^+$ of the nonlocal energy

$$2|\log\delta|\mathrm{P}(E) - \int_E\int_{\mathbb{R}^2\setminus E} \frac{\chi_{(\delta,+\infty)}(|x-y|)}{|x-y|^3} dx dy,$$

as recently proved by Muratov and Simon in [15, Theorem 2.3].

We also mention the recent work [6], where the authors establish the second-order expansion of appropriately rescaled nonlocal functionals approximating Sobolev seminorms, recently considered by Bourgain, Brezis and Mironescu [2].

As for the properties of the limit functional $\mathcal{H}$, first of all we observe that it is coercive in the sense that it provides a control on the perimeter of the set, see Proposition 3.1. Moreover it is bounded on $C^{1,\alpha}$ sets, for $\alpha > 0$, and on convex sets C such that for some $s \in (0,1)$ the boundary integral $\int_{\partial^* C} H_s(C,x) d\mathcal{H}^{d-1}(x)$ is finite, where $H_s(C,x)$ is the fractional mean curvature of C at x, see Proposition 3.3. In particular when E has boundary of class C^2, in Proposition 3.5 we show that the limit functional $\mathcal{H}(E)$ can be equivalently written as

$$\begin{aligned}\mathcal{H}(E) = &\frac{1}{d-1}\int_{\partial E}\int_{\partial E} \frac{(\nu(x)-\nu(y))^2}{2|x-y|^{d-1}} d\mathcal{H}^{d-1}(x) d\mathcal{H}^{d-1}(y) - \frac{d\omega_{d-1}}{d-1}\mathrm{P}(E) \\ &+ \frac{1}{d-1}\int_{\partial E}\int_{\partial E} \frac{1}{|x-y|^{d-1}} \left|\frac{(y-x)}{|y-x|}\cdot\nu(x)\right|^2 ((d-1)\log|x-y| - 1) d\mathcal{H}^{d-1}(x) d\mathcal{H}^{d-1}(y) \\ &+ \int_{\partial E}\int_{\partial E} \frac{H(E,x)\nu(x)\cdot(y-x)}{|y-x|^{d-1}} \log|x-y| \, d\mathcal{H}^{d-1}(x) d\mathcal{H}^{d-1}(y)\end{aligned}$$

where $H(E,x)$ denotes the (scalar) mean curvature at $x \in \partial E$, that is the sum of the principal curvatures divided by $d-1$. Notice that the first term in the expression above is the (squared)

L^2-norm of a nonlocal second fundamental form of ∂E. We recall also that an analogous representation formula for the same functional in dimension $d = 2$, has been given in [15].

Some interesting issues about the limit functional remain open, for instance existence and rigidity (at least for small volumes) of minimizers of $\mathcal{H}$ among sets with fixed volume, see the discussion in Remark 2.7.

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2. Second order asymptotics

We introduce the following functional on sets $E \subseteq \mathbb{R}^d$ of finite Lebesgue measure:

$$\mathcal{P}_s(E) = \begin{cases} \frac{\omega_{d-1}}{1-s}\mathrm{P}(E) - \mathrm{P}_s(E) & \text{if } \mathrm{P}(E) < +\infty \\ +\infty & \text{otherwise.} \end{cases} \tag{3}$$

We now state the main result of the paper.

Theorem 2.1. *There holds*

$$\mathcal{P}_s(E) \xrightarrow{\Gamma} \mathcal{H}(E) \qquad \text{as } s \to 1^-,$$

with respect to the L^1-topology, where the functional $\mathcal{H}(E)$ is defined in (2)*.*

Remark 2.2. Observe that $\mathcal{H}(E)$ can be also expressed as

$$\begin{aligned} \mathcal{H}(E) = &- \omega_{d-1}\mathrm{P}(E) + \int_{\partial^* E}\int_{E\Delta H^-(y)\cap B_1(y)} \frac{|(y-x)\cdot\nu(y)|}{|x-y|^{d+1}}dxd\mathcal{H}^{d-1}(y) \\ &+ \int_E\int_{E\setminus B_1(x)} \frac{1}{|x-y|^{d+1}}dydx - \int_E\int_{\partial B_1(x)\cap E} d\mathcal{H}^{d-1}(y)dx. \end{aligned} \tag{4}$$

Indeed by the divergence theorem and by the fact that $\mathrm{div}_y\left(\frac{y-x}{|y-x|^{d+1}}\right) = -\frac{1}{|y-x|^{d+1}}$ we get

$$\begin{aligned} &- \int_{\partial^* E}\int_{E\setminus B_1(y)} \frac{(y-x)\cdot\nu(y)}{|x-y|^{d+1}}dxd\mathcal{H}^{d-1}(y) \\ &= -\int_E\int_{\partial^* E\setminus B_1(x)} \frac{(y-x)\cdot\nu(y)}{|x-y|^{d+1}}d\mathcal{H}^{d-1}(y)dx \\ &= \int_E\int_{E\setminus B_1(x)} \frac{1}{|x-y|^{d+1}}dydx + \int_E\int_{\partial B_1(x)\cap E} \frac{(y-x)\cdot\frac{x-y}{|y-x|}}{|x-y|^{d+1}}d\mathcal{H}^{d-1}(y)dx \\ &= \int_E\int_{E\setminus B_1(x)} \frac{1}{|x-y|^{d+1}}dydx - \int_E\int_{\partial B_1(x)\cap E} d\mathcal{H}^{d-1}(y)dx. \end{aligned} \tag{5}$$

First of all we recall some properties of the functional $\mathcal{P}_s$.

Proposition 2.3 (Coercivity and lower semicontinuity)**.** *Let $s \in (0,1)$. If E_n is a sequence of sets such that $|E_n| \le m$ for some $m > 0$ and $\mathcal{P}_s(E_n) \le C$ for some $C > 0$ independent of n, then $\mathrm{P}(E_n) \le C'$ for some C' depending on C, s, d, m.*

In particular, the sequence E_n converges in L^1_{loc}, up to a subsequence, to a limit set E of finite perimeter, with $|E| \le m$.

Moreover, the functional $\mathcal{P}_s$ is lower semicontinuous with respect to the L^1-convergence.

Proof of Proposition 2.3. Let E with $|E| \leq m$. By the interpolation inequality proved in [3, Lemma 4.4] we get

$$\mathrm{P}_s(E) \leq \frac{d\omega_d}{2^s s(1-s)} \mathrm{P}(E)^s |E|^{1-s} \leq \frac{d\omega_d}{2^s s(1-s)} \mathrm{P}(E)^s m^{1-s}.$$

For a sequence E_n as in the statement, this gives

$$C(1-s) \geq \omega_{d-1}\mathrm{P}(E_n) - (1-s)\mathrm{P}_s(E_n) \geq \omega_{d-1}\mathrm{P}(E_n) - \frac{d\omega_d}{2^s s}\mathrm{P}(E_n)^s m^{1-s}. \tag{6}$$

From this we conclude that necessarily $\mathrm{P}(E_n) \leq C'$, where C' is a constant which depends on C, s, d, m. As a consequence, by the local compactness in L^1 of sets of finite perimeter (see [13]) we obtain the local convergence of E_n, up to a subsequence, to a limit set E of finite perimeter.

Now, assume that $E_n \to E$ in L^1 and that $\frac{c}{1-s}\mathrm{P}(E_n) - \mathrm{P}_s(E_n) \leq C$. By the previous argument, we get that $\mathrm{P}(E_n) \leq C'$, where C' is a constant which depends on $C, s, d, |E|$. By the compact embedding of BV in $H^{s/2}$, see [9, 14], we get that $\lim_n \mathrm{P}_s(E_n) = \mathrm{P}_s(E)$, up to passing to a suitable subsequence. This, along with the lower semicontinuity of the perimeter with respect to local convergence in L^1 (see [13]) gives the conclusion. □

The proof of Theorem 2.1 is based on some preliminary results. First of all we compute the pointwise limit, then we show that the functional $s\mathcal{P}_s(E)$ is given by the sum of the functional $\mathcal{F}_s(E)$, defined in (15), which is lower semicontinuous and monotone increasing in s, and of a continuous functional. This will permit to show that the pointwise limit coincides with the Γ-limit.

Proposition 2.4 (Pointwise limit)**.** *Let $E \subseteq \mathbb{R}^d$ be a measurable set such that $|E| < +\infty$ and $\mathrm{P}(E) < +\infty$. Then*

$$\lim_{s\to 1^-} \left[\frac{\omega_{d-1}}{1-s}\mathrm{P}(E) - \mathrm{P}_s(E)\right] = \begin{cases} \mathcal{H}(E) & \text{if } \int_{\partial^* E}\int_{E\Delta H^-(y)\cap B_1(y)} \frac{|(y-x)\cdot\nu(y)|}{|x-y|^{d+1}} dx d\mathcal{H}^{d-1}(y) < +\infty \\ +\infty & \text{otherwise.} \end{cases}$$

where $\mathcal{H}(E)$ is defined in (2) *and $H^-(y) := \{x \in \mathbb{R}^d \mid (y-x)\cdot\nu(y) > 0\}$.*

Proof. We can write $\mathrm{P}_s(E)$ as a boundary integral observing that for all $0 < s < 1$

$$\mathrm{div}_y\left(\frac{y-x}{|y-x|^{d+s}}\right) = -s\frac{1}{|y-x|^{d+s}}. \tag{7}$$

So, by the divergence theorem, (1) reads

$$\begin{aligned} \mathrm{P}_s(E) =& \frac{1}{s}\int_{\partial^* E}\int_E \frac{(y-x)\cdot\nu(y)}{|x-y|^{d+s}} dx d\mathcal{H}^{d-1}(y) \\ =& \frac{1}{s}\int_{\partial^* E}\int_{E\cap B_1(y)} \frac{(y-x)\cdot\nu(y)}{|x-y|^{d+s}} dx d\mathcal{H}^{d-1}(y) \\ &+ \frac{1}{s}\int_{\partial^* E}\int_{E\setminus B_1(y)} \frac{(y-x)\cdot\nu(y)}{|x-y|^{d+s}} dx d\mathcal{H}^{d-1}(y) \end{aligned} \tag{8}$$

where $\nu(y)$ is the outer normal at $\partial^* E$ in y and $R > 0$.

We fix now $y \in \partial^* E$ and we observe that, since $H^-(y) := \{x \in \mathbb{R}^d \mid (y-x)\cdot\nu(y) > 0\}$,

$$\begin{aligned}(9)\qquad &\int_{E\cap B_1(y)} \frac{(y-x)\cdot\nu(y)}{|x-y|^{d+s}}dx\\ &= \int_{H^-(y)\cap B_1(y)} \frac{(y-x)\cdot\nu(y)}{|x-y|^{d+s}}dx + \int_{E\setminus H^-(y)\cap B_1(y)} \frac{(y-x)\cdot\nu(y)}{|x-y|^{d+s}}dx\\ &\quad - \int_{H^-(y)\setminus E\cap B_1(y)} \frac{(y-x)\cdot\nu(y)}{|x-y|^{d+s}}dx\\ &= \int_{H^-(y)\cap B_1(y)} \frac{(y-x)\cdot\nu(y)}{|x-y|^{d+s}}dx - \int_{E\Delta H^-(y)\cap B_1(y)} \frac{|(y-x)\cdot\nu(y)|}{|x-y|^{d+s}}dx.\end{aligned}$$

Now we compute, denoting by B_1' the ball in $\mathbb{R}^{d-1}$ with radius 1,

$$\begin{aligned}(10)\qquad \int_{H^-(y)\cap B_1(y)} \frac{(y-x)\cdot\nu(y)}{|x-y|^{d+s}}dx &= \int_{\{x_d\geq 0\}\cap B_1} \frac{x_d}{|x|^{d+s}}dx\\ &= \int_{B_1'}\int_0^{\sqrt{1-|x'|^2}} \frac{x_d}{(x_d^2+|x'|^2)^{(d+s)/2}}dx_d\\ &= \int_{B_1'} \frac{1}{2-d-s}(1-|x'|^{2-d-s})dx' = \omega_{d-1}\frac{1}{1-s}.\end{aligned}$$

If we substitute (10) in (9) we get

$$(11)\qquad \int_{E\cap B_1(y)} \frac{(y-x)\cdot\nu(y)}{|x-y|^{d+s}}dx = \frac{\omega_{d-1}}{1-s} - \int_{E\Delta H^-(y)\cap B_1(y)} \frac{|(y-x)\cdot\nu(y)|}{|x-y|^{d+s}}dx.$$

By (8) and (11) we obtain

$$\begin{aligned}(12)\qquad \omega_{d-1}\frac{\mathrm{P}(E)}{(1-s)} - \mathrm{P}_s(E) =&\omega_{d-1}\frac{\mathrm{P}(E)}{(1-s)} - \omega_{d-1}\frac{\mathrm{P}(E)}{s(1-s)}\\ &+ \frac{1}{s}\int_{\partial^*E}\int_{E\Delta H^-(y)\cap B_1(y)} \frac{|(y-x)\cdot\nu(y)|}{|x-y|^{d+s}}dxd\mathcal{H}^{d-1}(y)\\ &- \frac{1}{s}\int_{\partial^*E}\int_{E\setminus B_1(y)} \frac{(y-x)\cdot\nu(y)}{|x-y|^{d+s}}dxd\mathcal{H}^{d-1}(y).\end{aligned}$$

Now we observe that, by Lebesgue's dominated convergence theorem, there holds
(13)

$$\lim_{s\to 1^-}\frac{1}{s}\int_{\partial^*E}\int_{E\setminus B_1(y)} \frac{(y-x)\cdot\nu(y)}{|x-y|^{d+s}}dxd\mathcal{H}^{d-1}(y) = \int_{\partial^*E}\int_{E\setminus B_1(y)} \frac{(y-x)\cdot\nu(y)}{|x-y|^{d+1}}dxd\mathcal{H}^{d-1}(y).$$

Moreover, by the monotone convergence theorem,

$$(14)\qquad \lim_{s\to 1^-}\int_{E\Delta H^-(y)\cap B_1(y)} \frac{|(y-x)\cdot\nu(y)|}{|x-y|^{d+s}}dx = \int_{E\Delta H^-(y)\cap B_1(y)} \frac{|(y-x)\cdot\nu(y)|}{|x-y|^{d+1}}dx$$

if $\frac{|(y-x)\cdot\nu(y)|}{|x-y|^{d+1}} \in L^1(E\Delta H^-(y)\cap B_1(y))$ and $\lim_{s\to 1^-}\int_{E\Delta H^-(y)\cap B_1(y)} \frac{|(y-x)\cdot\nu(y)|}{|x-y|^{d+s}}dx = +\infty$ otherwise. The conclusion then follows from (12), (13), (14) sending $s \to 1^-$. $\square$

Lemma 2.5. *For $s \in (0,1)$ and $E \subseteq \mathbb{R}^d$ of finite measure, we define the functional*

$$\mathcal{F}_s(E) := \begin{cases} s\left[\frac{\omega_{d-1}}{1-s}\mathrm{P}(E) - \mathrm{P}_s(E) - \int_E \int_{E\setminus B_1(x)} \frac{1}{|x-y|^{d+s}} dy dx\right] & \text{if } \mathrm{P}(E) < +\infty \\ +\infty & \text{otherwise.} \end{cases} \tag{15}$$

Then the following holds:

(1) *The map $s \mapsto \mathcal{F}_s(E)$ is monotone increasing as $s \to 1^-$. Moreover, for every E of finite perimeter*

$$\lim_{s\to 1^-} \mathcal{F}_s(E) = -\,\omega_{d-1}\mathrm{P}(E) + \int_{\partial^* E}\int_{E\Delta H^-(y)\cap B_1(y)} \frac{|(y-x)\cdot\nu(y)|}{|x-y|^{d+1}} dx d\mathcal{H}^{d-1}(y) - \int_E \int_{\partial B_1(x)\cap E} d\mathcal{H}^{d-1}(y) dx.$$

(2) *For every family of sets E_s such that $\mathcal{F}_s(E_s) \le C$, for some $C > 0$ independent of s, and $E_s \to E$ in L^1, there holds*

$$\liminf_{s\to 1} \mathcal{F}_s(E_s) \ge -\,\omega_{d-1}\mathrm{P}(E) + \int_{\partial^* E}\int_{E\Delta H^-(y)\cap B_1(y)} \frac{|(y-x)\cdot\nu(y)|}{|x-y|^{d+1}} dx d\mathcal{H}^{d-1}(y) - \int_E \int_{\partial B_1(x)\cap E} d\mathcal{H}^{d-1}(y) dx.$$

Proof. (1) Arguing as in (5) and using (7), we get

$$\mathcal{F}_s(E) = s\Big[\frac{\omega_{d-1}}{1-s}\mathrm{P}(E) - \mathrm{P}_s(E) + \frac{1}{s}\int_{\partial^* E}\int_{E\setminus B_1(y)} \frac{(y-x)\cdot\nu(y)}{|x-y|^{d+s}} dx d\mathcal{H}^{d-1}(y) - \frac{1}{s}\int_E \int_{\partial B_1(x)\cap E} d\mathcal{H}^{d-1}(y) dx\Big].$$

Therefore from (8), and (11), we get for $0 < \bar{s} < s < 1$

$$\begin{aligned} &\frac{\mathcal{F}_s(E) + \int_E \int_{\partial B_1(x)\cap E} d\mathcal{H}^{d-1}(y) dx}{s} \\ =&\omega_{d-1}\frac{\mathrm{P}(E)}{(1-s)} - \mathrm{P}_s(E) + \frac{1}{s}\int_{\partial^* E}\int_{E\setminus B_1(y)} \frac{(y-x)\cdot\nu(y)}{|x-y|^{d+s}} dx d\mathcal{H}^{d-1}(y) \\ =&\omega_{d-1}\frac{\mathrm{P}(E)}{(1-s)} - \frac{1}{s}\int_{\partial^* E}\int_{E\cap B_1(y)} \frac{(y-x)\cdot\nu(y)}{|x-y|^{d+s}} dx d\mathcal{H}^{d-1}(y) \\ =&-\frac{\omega_{d-1}}{s}\mathrm{P}(E) + \frac{1}{s}\int_{\partial^* E}\int_{E\Delta H^-(y)\cap B_1(y)} \frac{|(y-x)\cdot\nu(y)|}{|x-y|^{d+s}} dx d\mathcal{H}^{d-1}(y) \\ >&-\frac{\omega_{d-1}}{s}\mathrm{P}(E) + \frac{1}{s}\int_{\partial^* E}\int_{E\Delta H^-(y)\cap B_1(y)} \frac{|(y-x)\cdot\nu(y)|}{|x-y|^{d+\bar{s}}} dx d\mathcal{H}^{d-1}(y) \\ =&\frac{\mathcal{F}_{\bar{s}}(E) + \int_E \int_{\partial B_1(x)\cap E} d\mathcal{H}^{d-1}(y) dx}{s}, \end{aligned}$$

which gives the desired monotonicity.

Now we observe that by the dominated convergence for every E with $|E| < +\infty$ and $\mathrm{P}(E) < +\infty$,

$$\lim_{s\to 1} \frac{1}{s} \int_{\partial^* E} \int_{E\setminus B_1(y)} \frac{(y-x)\cdot \nu(y)}{|x-y|^{d+s}} dx d\mathcal{H}^{d-1}(y) - \frac{1}{s} \int_E \int_{\partial B_1(x)\cap E} d\mathcal{H}^{d-1}(y) dx$$
$$= \int_{\partial^* E} \int_{E\setminus B_1(y)} \frac{(y-x)\cdot \nu(y)}{|x-y|^{d+1}} dx d\mathcal{H}^{d-1}(y) - \int_E \int_{\partial B_1(x)\cap E} d\mathcal{H}^{d-1}(y) dx$$

So, we conclude by Proposition 2.4.

(2) We fix a family of sets E_s such that $\mathcal{F}_s(E_s) \le C$ and $E_s \to E$ in L^1 as $s \to 1^-$. Fix $\bar{s} < 1$ and observe that by the monotonicity property proved in item (i), we get

$$\liminf_{s\to 1} \mathcal{F}_s(E_s) \ge \liminf_{s\to 1} \mathcal{F}_{\bar{s}}(E_s)$$
$$\ge \liminf_{s\to 1} \bar{s} \left[\frac{\omega_{d-1}}{1-\bar{s}} \mathrm{P}(E_s) - \mathrm{P}_{\bar{s}}(E_s)\right] - \lim_{s\to 1} \bar{s} \int_{E_s} \int_{E_s\setminus B_1(x)} \frac{1}{|x-y|^{d+\bar{s}}} dy dx$$
$$\ge \bar{s} \left[\frac{\omega_{d-1}}{1-\bar{s}} \mathrm{P}(E) - \mathrm{P}_{\bar{s}}(E)\right] - \bar{s} \int_E \int_{E\setminus B_1(x)} \frac{1}{|x-y|^{d+\bar{s}}} dy dy = \mathcal{F}_{\bar{s}}(E)$$

where we used for the first limit the lower semicontinuity proved in Proposition 2.3, and the dominated convergence theorem for the second limit.

We conclude by item (i), observing that $\mathcal{F}_{\bar{s}}(E) < C$, and sending $\bar{s} \to 1^-$.

□

We are now ready to prove our main result.

Proof of Theorem 2.1. We start with the Γ-liminf inequality. Let E_s be a sequence of sets such that $E_s \to E$ in L^1. We will prove that

$$\liminf_{s\to 1} s \left[\frac{\omega_{d-1}}{1-s} \mathrm{P}(E_s) - \mathrm{P}_s(E_s)\right] \ge \mathcal{H}(E),$$

which will give immediately the conclusion. Recalling the definition of $\mathcal{F}_s(E)$ given in (15), we have that

$$\liminf_{s\to 1} s \left[\frac{\omega_{d-1}}{1-s} \mathrm{P}(E_s) - \mathrm{P}_s(E_s)\right] \ge \liminf_{s\to 1} \mathcal{F}_s(E_s) + \liminf_{s\to 1} s \int_{E_s} \int_{E_s\setminus B_1(x)} \frac{1}{|x-y|^{d+s}} dy dx.$$

By Proposition 2.5, item (ii) and by Fatou lemma, we get

$$\liminf_{s\to 1} s \left[\frac{\omega_{d-1}}{1-s} \mathrm{P}(E_s) - \mathrm{P}_s(E_s)\right] \ge -\omega_{d-1} \mathrm{P}(E)$$
$$+ \int_{\partial^* E} \int_{E\Delta H^-(y)\cap B_1(y)} \frac{|(y-x)\cdot \nu(y)|}{|x-y|^{d+1}} dx d\mathcal{H}^{d-1}(y) - \int_E \int_{\partial B_1(x)\cap E} d\mathcal{H}^{d-1}(y) dx$$
$$+ \int_E \int_{E\setminus B_1(x)} \frac{1}{|x-y|^{d+1}} dy dx = \mathcal{H}(E)$$

where the last equality comes from (5).

The Γ-limsup is a consequence of the pointwise limit in Proposition 2.4. □

We conclude this section with the equi-coercivity of the family of functionals $\mathcal{P}_s$, which is a consequence of the monotonicity property of $\mathcal{F}_s$ obtained in Lemma 2.5.

Proposition 2.6 (Equi-coercivity). *Let s_n be a sequence of positive numbers with $s_n \to 1^-$, let $m, C \in \mathbb{R}$ with $m > 0$, and let E_n be a sequence of measurable sets such that $|E_n| \le m$ and $\mathcal{P}_{s_n}(E_n) \le C$ for all $n \in \mathbb{N}$.*

Then $\mathrm{P}(E_n) \le C'$ for some $C' > 0$ depending on C, d, m, and the sequence E_n converges in L^1_{loc}, up to a subsequence, to a limit set E of finite perimeter, with $|E| \le m$.

Proof. Reasoning as in Proposition 2.3, we get that E_n has finite perimeter, for every $n \in \mathbb{N}$. Recalling (15), we get that

$$|C| \ge s_n \mathcal{P}_{s_n}(E_n) = \mathcal{F}_{s_n}(E_n) + s_n \int_{E_n} \int_{E_n \setminus B_1(x)} \frac{1}{|x-y|^{d+s_n}} dydx \ge \mathcal{F}_{s_n}(E_n).$$

We fix now $\bar{n}$ such that $s_{\bar{n}} > \frac{1}{2}$ and we claim that there exists C', depending on m, d but independent of n, such that $\mathrm{P}(E_n) \le C'$ for every $n \ge \bar{n}$. If the claim is true, then it is immediate to conclude that eventually enlarging C', $\mathrm{P}(E_n) \le C'$ for every n.

For every $n \ge \bar{n}$, we use the monotonicity of the map $s \mapsto \mathcal{F}_s(E_n)$ proved in Lemma 2.5, and the fact that $|E_n| \le m$, to obtain that

$$\begin{aligned} |C| \ge \mathcal{F}_{s_n}(E_n) \ge \mathcal{F}_{s_{\bar{n}}}(E_n) &= s_{\bar{n}} \mathcal{P}_{s_{\bar{n}}}(E_n) - s_{\bar{n}} \int_{E_n} \int_{E_n \setminus B_1(x)} \frac{1}{|x-y|^{d+s_{\bar{n}}}} dydx \\ &\ge s_{\bar{n}} \mathcal{P}_{s_{\bar{n}}}(E_n) - s_{\bar{n}} \int_{E_n} \int_{E_n \setminus B_1(x)} dydx \ge s_{\bar{n}} \mathcal{P}_{s_{\bar{n}}}(E_n) - s_{\bar{n}} |E_n|^2 \ge s_{\bar{n}} \mathcal{P}_{s_{\bar{n}}}(E_n) - s_{\bar{n}} m^2. \end{aligned}$$

This implies in particular that $\mathcal{P}_{s_{\bar{n}}}(E_n) \le \frac{|C|}{s_{\bar{n}}} + m^2 \le 2|C| + m^2$, and we conclude by Proposition 2.3. □

Remark 2.7 (Isoperimetric problems). Let us consider the following isoperimetric-type problem for the functionals $\mathcal{P}_s$ and $\mathcal{H}$:

$$\min_{|E|=m} \mathcal{P}_s(E) \tag{16}$$

$$\min_{|E|=m} \mathcal{H}(E), \tag{17}$$

where $m > 0$ is a fixed constant. Observe that $\widetilde{E}$ is a minimizer of (16) if and only if the rescaled set $m^{-\frac{1}{d}} \widetilde{E}$ is a minimizer of

$$\min_{|E|=1} \frac{\omega_{d-1}}{1-s} \mathrm{P}(E) - m^{\frac{1-s}{d}} \mathrm{P}_s(E).$$

Note in particular that the functional $\mathcal{P}_s$ is given by the sum of an attractive term, which is the perimeter functional, and a repulsive term given by the fractional perimeter with a negative sign.

In general we cannot expect existence of solutions to these problems for every value of m. However, from [8, Thm 1.1, Thm 1.2] it follows that there exist $0 < m_2(s) \le m_1(s)$ such that, for all $m < m_1(s)$, Problem (16) admits a solution and moreover, if $m < m_2(s)$, the unique solution (uo to translations) is the ball of volume m. Actually, the bounds $m_1(s), m_2(s)$ tend to 0 as $s \to 1^-$, hence these results cannot be extended directly to Problem (17).

A weaker notion of solution, introduced in [12], are the so-called generalized minimizers, that is, minimizers of the functional $\sum_i \mathcal{P}_s(E_i)$ (resp. of $\sum_i \mathcal{H}(E_i)$), among sequences of sets $(E_i)_i$ such that $|E_i| > 0$ and $P(E_i) < +\infty$ for finitely many i's, and $\sum_i |E_i| = m$. Note that, if E_n is a minimizing sequence for (16) or (17), by reasoning as in Proposition 2.6, we get that there exists a constant $C = C(m) > 0$ such that $\mathrm{P}(E_n) \le C$ for every n. Then, as it is proved in [11, Proposition 2.1], there exists $C' = C'(m) > 0$, depending on C and m, such that $\sup_x |E_n \cap B_1(x)| \geq C'$. Using these facts, reasoning as in [12], it is possible to show existence of generalized minimizers both for (16) and (17), for every value of $m > 0$.

3. Properties of the limit functional

In this section we analyze the main properties of the limit functional $\mathcal{H}$. Note that, since it is obtained as a Γ-limit, it is naturally lower semicontinuous with respect to L^1 convergence.

First of all we observe that by the representation of $\mathcal{H}$ in (4), for every E with finite perimeter there holds

(18)
$$\begin{aligned} -\omega_{d-1}\mathrm{P}(E) - d\omega_d|E| \leq \mathcal{H}(E) &\leq \int_{\partial^* E}\int_{E\Delta H^-(y)\cap B_1(y)} \frac{|(y-x)\cdot\nu(y)|}{|x-y|^{d+1}}dxd\mathcal{H}^{d-1}(y) + d\omega_d|E| \\ &\leq \int_{\partial^* E}\int_{E\Delta H^-(y)\cap B_1(y)} \frac{1}{|x-y|^{d}}dxd\mathcal{H}^{d-1}(y) + d\omega_d|E|. \end{aligned}$$

We start with a compactness property in L^1 for sublevel sets of $\mathcal{H}$, which follows from a lower bound on $\mathcal{H}$ in terms of the perimeter.

Proposition 3.1. *Let $E \subseteq \mathbb{R}^d$ be such that $\mathcal{H}(E) \leq C$. Then there exists a constant C' depending on $C, |E|, d$ such that $\mathrm{P}(E) \leq C'$.*

In particular, if E_n is a sequence of sets such that $\mathcal{H}(E_n) \leq C$, then there exists a limit set E of finite perimeter such that $\mathcal{H}(E) \leq C$ and $E_n \to E$ in L^1_{loc} as $n \to +\infty$, up to a subsequence.

Proof. By Lemma 2.5, for $s \in (0,1)$ there holds

$$\mathcal{F}_s(E) \leq \mathcal{H}(E) - \int_E\int_{E_n\setminus B_1(y)} \frac{1}{|x-y|^{d+1}}dxdy \leq \mathcal{H}(E) \leq C.$$

The estimate on $P(E_n)$ then follows by Proposition 2.6.

The second statement is a direct consequence of the lower semicontinuity of $\mathcal{H}$, and of the local compactness in L^1 of sets of finite perimeter. □

We point out the following rescaling property of the functional $\mathcal{H}$, the will allow us to consider only sets with diameter less than 1.

Proposition 3.2. *For every $\lambda > 0$ there holds*

$$\mathcal{H}(\lambda E) = \lambda^{d-1}\mathcal{H}(E) - \omega_{d-1}\lambda^{d-1}\log\lambda\mathrm{P}(E). \tag{19}$$

Proof. We observe that for every $R > 0$, with the same computation as in (10) we get

$$\begin{aligned} \int_{E\cap B_R(y)} \frac{(y-x)\cdot\nu(y)}{|x-y|^{d+s}}dx &= \int_{H^-(y)\cap B_R(y)} \frac{(y-x)\cdot\nu(y)}{|x-y|^{d+s}}dx - \int_{E\Delta H^-(y)\cap B_R(y)} \frac{|(y-x)\cdot\nu(y)|}{|x-y|^{d+s}}dx \\ &= \omega_{d-1}\frac{R^{1-s}}{1-s} - \int_{E\Delta H^-(y)\cap B_R(y)} \frac{|(y-x)\cdot\nu(y)|}{|x-y|^{d+s}}dx. \end{aligned}$$

Therefore, arguing as in Proposition 2.4, we can show that $\mathcal{H}(E)$ can be equivalently defined as follows, for all $R > 0$

$$\mathcal{H}(E) = -\,\omega_{d-1}\mathrm{P}(E)(1+\log R) + \int_{\partial^* E}\int_{E\Delta H^-(y)\cap B_R(y)} \frac{|(y-x)\cdot\nu(y)|}{|x-y|^{d+1}}dx d\mathcal{H}^{d-1}(y) - \int_{\partial^* E}\int_{E\setminus B_R(y)} \frac{(y-x)\cdot\nu(y)}{|x-y|^{d+1}}dx d\mathcal{H}^{d-1}(y). \tag{20}$$

This formula immediately gives the desired rescaling property (19). □

Now, we identify some classes of sets where $\mathcal{H}$ is bounded.

Proposition 3.3. *Let E be a measurable set with $|E| < +\infty$ and $P(E) < +\infty$.*

(1) *If ∂E is uniformly of class $C^{1,\alpha}$ for some $\alpha > 0$, then $\mathcal{H}(E) < +\infty$.*
(2) *If E is a convex set then, for every $s \in (0,1)$, there holds*

$$\mathcal{H}(E) \le \frac{(\mathrm{diam}E)^s}{2}\int_{\partial^* E} H_s(E,y)d\mathcal{H}^{d-1}(y) - \omega_{d-1}\mathrm{P}(E)\left(\frac{1}{s} + \log(\mathrm{diam}E)\right)$$

where $\mathrm{diam}E := \sup_{x,y\in E}|x-y|$, *and $H_s(E,y)$ is the fractional mean curvature of E at y, which is defined as*

$$H_s(E,y) := \int_{\mathbb{R}^d} \frac{\chi_{\mathbb{R}^d\setminus E}(x) - \chi_E(x)}{|x-y|^{d+s}}dx,$$

in the principal value sense.

Proof. (1) If ∂E is uniformly of class $C^{1,\alpha}$, then there exists $\eta > 0$ such that for all $y \in \partial E$, $\partial E \cap B_\eta(y)$ is a graph of a $C^{1,\alpha}$ function h, such that $\|\nabla h\|_{C^{0,\alpha}(B'_\eta(y))} \le C$, for some C independent of y. Up to a rotation and translation, we may assume that $y = 0$, $h(0) = 0$ and $\nabla h(0) = 0$ and moreover $-C|x'|^{1+\alpha} \le h(x') \le C|x'|^{1+\alpha}$ for all $x' \in B'_\eta$. Therefore recalling that $E \cap B_\eta = \{(x, x_d) \mid x_d \le h(x')\}$ and that $H^-(0) = \{(x', x_d) \mid x_d \le 0\}$, there holds

$$E\Delta H^-(0) \cap B_\eta \subseteq C_\eta := \{(x', x_d) \mid -C|x'|^{1+\alpha} \le x_d \le C|x'|^{1+\alpha}, |x'| \le \eta\}.$$

We compute

$$\begin{aligned}
&\int_{E\Delta H^-(0)\cap B_1} \frac{1}{|x|^d}dx = \int_{E\Delta H^-(0)\cap B_\eta} \frac{1}{|x|^d}dx + \int_{E\Delta H^-(0)\cap B_1\setminus B_\eta} \frac{1}{|x|^d}dx \\
&\le \int_{C_\eta} \frac{1}{|x|^d}dx + \frac{1}{2}\int_{B_1\setminus B_\eta} \frac{1}{|x|^d}dx \le \int_{C_\eta} \frac{1}{|x'|^d}dx + \frac{1}{2}\int_{B_1\setminus B_\eta} \frac{1}{|x|^d}dx \\
&\le 2C\int_{B'_\eta} \frac{|x'|^{1+\alpha}}{|x'|^d}dx' - \frac{1}{2}d\omega_d\log(\eta\wedge 1) = \frac{2C(d-1)\omega_{d-1}\eta^\alpha}{\alpha} - \frac{1}{2}d\omega_d\log(\eta\wedge 1).
\end{aligned}$$

Then, recalling (18) we get that

$$\mathcal{H}(E) \le \left(\frac{2C(d-1)\omega_{d-1}\eta^\alpha}{\alpha} - \frac{1}{2}d\omega_d\log(\eta\wedge 1)\right)\mathrm{P}(E) + d\omega_d|E| < +\infty.$$

(2) Let $R = \mathrm{diam}E$. Then by (20), we get

$$\begin{aligned}\mathcal{H}(E) &= -\omega_{d-1}\mathrm{P}(E)(1+\log R) + \int_{\partial^* E}\int_{E\Delta H^-(y)\cap B_R(y)} \frac{|(y-x)\cdot\nu(y)|}{|x-y|^{d+1}} dx d\mathcal{H}^{d-1}(y)\\ &\le -\omega_{d-1}\mathrm{P}(E)(1+\log R) + \int_{\partial^* E}\int_{E\Delta H^-(y)\cap B_R(y)} \frac{1}{|x-y|^{d}} dx d\mathcal{H}^{d-1}(y)\\ &\le -\omega_{d-1}\mathrm{P}(E)(1+\log R) + \int_{\partial^* E}\int_{E\Delta H^-(y)\cap B_R(y)} \frac{R^s}{|x-y|^{d+s}} dx d\mathcal{H}^{d-1}(y).\end{aligned}$$

By convexity for every $y \in \partial^* E$, recalling that $E \subseteq B_R(y)$, there holds

$$\begin{aligned}&\int_{E\Delta H^-(y)\cap B_R(y)} \frac{R^s}{|x-y|^{d+s}} dx = \frac{R^s}{2}\int_{B_R(y)} \frac{\chi_{\mathbb{R}^d\setminus E}(x) - \chi_E(x)}{|x-y|^{d+s}} dx\\ &= \frac{R^s}{2} H_s(E,y) - \frac{R^s}{2}\int_{\mathbb{R}^d\setminus B_R(y)} \frac{1}{|x-y|^{d+s}} dx = \frac{R^s}{2} H_s(E,y) - \frac{d\omega_d}{2s}.\end{aligned}$$

Therefore, substituting this equality in the previous estimate, we get

$$\mathcal{H}(E) \le \frac{R^s}{2}\int_{\partial^* E} H_s(E,y) d\mathcal{H}^{d-1}(y) - \omega_{d-1}\mathrm{P}(E)(1+\log R) - \frac{d\omega_d}{2s}\mathrm{P}(E).$$

□

Remark 3.4. Note that by Proposition 3.3, $\mathcal{H}(Q) < +\infty$ for every cube $Q = \Pi_{i=1}^d [a_i, b_i]$. Indeed for $y \in \partial^* Q$, there holds that $H_s(Q,y) \sim \frac{1}{(d(y,(\partial Q\setminus\partial^* Q)))^s}$ for $s \in (0,1)$ and so $\int_{\partial^* Q} H_s(Q,y) d\mathcal{H}^{d-1}(y) < +\infty$.

Finally we provide some useful equivalent representations of the functional $\mathcal{H}$.

Proposition 3.5.

(i) *Let E be a set with finite perimeter such that $\mathcal{H}(E) < +\infty$. Then*

$$\begin{aligned}\mathcal{H}(E) = &-\frac{d\omega_{d-1}}{d-1}\mathrm{P}(E)\\ &- \lim_{\delta\to 0^+}\Big[\frac{1}{d-1}\int_{\partial^* E}\int_{\partial^* E\setminus B_\delta(y)} \frac{\nu(y)\cdot\nu(x)}{|x-y|^{d-1}} d\mathcal{H}^{d-1}(x) d\mathcal{H}^{d-1}(y) + \omega_{d-1}\log\delta \mathrm{P}(E)\Big].\end{aligned}$$

(ii) *Let E be a compact set with boundary of class C^2. Then*

$$\begin{aligned}\mathcal{H}(E) = &\frac{1}{d-1}\int_{\partial E}\int_{\partial E} \frac{(\nu(x)-\nu(y))^2}{2|x-y|^{d-1}} d\mathcal{H}^{d-1}(x) d\mathcal{H}^{d-1}(y) - \frac{d\omega_{d-1}}{d-1}\mathrm{P}(E)\\ &+ \frac{1}{d-1}\int_{\partial E}\int_{\partial E} \frac{1}{|x-y|^{d-1}}\left|\frac{(y-x)}{|y-x|}\cdot\nu(x)\right|^2 ((d-1)\log|x-y| - 1) d\mathcal{H}^{d-1}(x) d\mathcal{H}^{d-1}(y)\\ &+ \int_{\partial E}\int_{\partial E} \frac{H(E,x)\nu(x)\cdot(y-x)}{|y-x|^{d-1}} \log|x-y| d\mathcal{H}^{d-1}(x) d\mathcal{H}^{d-1}(y).\end{aligned}$$

Proof. (i) If the diameter of E is less than 1, then $E \setminus B_1(y) = \emptyset$ for all $y \in \partial E$, and so

$$\mathcal{H}(E) = -\omega_{d-1}\mathrm{P}(E) + \int_{\partial^* E}\int_{E\Delta H^-(y)\cap B_1(y)} \frac{|(y-x)\cdot\nu(y)|}{|x-y|^{d+1}} dx d\mathcal{H}^{d-1}(y).$$

Using that

$$\frac{1}{d-1}\mathrm{div}_x\left(\frac{\nu(y)}{|x-y|^{d-1}}\right)=\frac{(y-x)\cdot\nu(y)}{|x-y|^{d+1}}$$

we compute the second inner integral for $y\in\partial^*E$, recalling that $E\subset B_1(y)$,

$$\begin{aligned}
&\int_{E\Delta H^-(y)\cap B_1(y)}\frac{|(y-x)\cdot\nu(y)|}{|x-y|^{d+1}}dx\\
&=\int_{H^-(y)\setminus E\cap B_1(y)}\frac{(y-x)\cdot\nu(y)}{|x-y|^{d+1}}dx-\int_{(E\setminus H^-(y))}\frac{(y-x)\cdot\nu(y)}{|x-y|^{d+1}}dx\\
&=\lim_{\delta\to0}\Big[\int_{H^-(y)\setminus E\cap B_1(y)\setminus B_\delta(y)}\frac{(y-x)\cdot\nu(y)}{|x-y|^{d+1}}dx-\int_{(E\setminus H^-(y))\setminus B_\delta(y)}\frac{(y-x)\cdot\nu(y)}{|x-y|^{d+1}}dx\Big]\\
&=\lim_{\delta\to0}\Big[-\frac{1}{d-1}\int_{\partial^*E\setminus B_\delta(y)}\frac{\nu(x)\cdot\nu(y)}{|x-y|^{d-1}}d\mathcal{H}^{d-1}(x)+\frac{1}{d-1}\int_{\partial B_1(y)\cap H^-(y)}\nu(x)\cdot\nu(y)d\mathcal{H}^{d-1}(x)\\
&\quad+\frac{1}{d-1}\int_{\partial H^-(y)\cap B_1(y)\setminus B_\delta(y)}\frac{1}{|x-y|^{d-1}}d\mathcal{H}^{d-1}(x)-\frac{1}{\delta^{d-1}}\int_{\partial B_\delta(y)\cap H^-(y)\Delta E}\nu(x)\cdot\nu(y)d\mathcal{H}^{d-1}(x)\Big].
\end{aligned}$$

Now we observe that

$$\begin{aligned}
&\lim_{\delta\to0}\frac{1}{\delta^{d-1}}\int_{\partial B_\delta(y)\cap H^-(y)\Delta E}|\nu(x)\cdot\nu(y)|d\mathcal{H}^{d-1}(x)\le\lim_{\delta\to0}\frac{1}{\delta^{d-1}}\int_{\partial B_\delta(y)\cap H^-(y)\Delta E}d\mathcal{H}^{d-1}(x)\\
&=\lim_{\delta\to0}\int_{\partial B_1\cap H^-(y)\Delta\frac{(E-y)}{\delta}}d\mathcal{H}^{d-1}(x)=0
\end{aligned}$$

since, for $y\in\partial^*E$, there holds that $\frac{(E-y)}{\delta}\to H^-(y)$ locally in L^1 as $\delta\to0$, see [13, Thm II.4.5]. We compute

$$\frac{1}{d-1}\int_{\partial B_1(y)\cap H^-(y)}\nu(x)\cdot\nu(y)d\mathcal{H}^{d-1}(x)=\frac{1}{d-1}\int_{x_d=-\sqrt{1-|x'|^2}}x_d d\mathcal{H}^{d-1}(x)=-\frac{\omega_{d-1}}{d-1}$$

and

$$\frac{1}{d-1}\int_{\partial H^-(y)\cap B_1(y)\setminus B_\delta(y)}\frac{1}{|x-y|^{d-1}}d\mathcal{H}^{d-1}(x)=\frac{1}{d-1}\int_{B_1'\setminus B_\delta'}\frac{1}{|x'|^{d-1}}dx'=-\omega_{d-1}\log\delta.$$

Therefore

$$\begin{aligned}
\mathcal{H}(E)=&-\frac{d\omega_{d-1}}{d-1}\mathrm{P}(E)\\
&-\lim_{\delta\to0^+}\Big[\frac{1}{d-1}\int_{\partial^*E}\int_{\partial^*E\setminus B_\delta(y)}\frac{\nu(y)\cdot\nu(x)}{|x-y|^{d-1}}d\mathcal{H}^{d-1}(x)d\mathcal{H}^{d-1}(y)+\omega_{d-1}\log\delta\mathrm{P}(E)\Big].
\end{aligned}$$

If ∂E has diameter greater or equal to 1, we obtain the formula by rescaling, using (19).

(ii) Let us fix $y\in\partial E$ and define for all $x\in\partial E$, $x\neq y$, the vector field

$$\eta(x)=f(|x-y|)(y-x)\qquad\text{where } f(r):=\frac{\log r}{r^{d-1}}.$$

By the Gauss-Green Formula (see [13, I.11.8]), for $\delta > 0$ there holds

$$
\begin{aligned}
&\frac{1}{d-1}\int_{\partial E\setminus B_\delta(y)} \mathrm{div}_\tau \eta(x) d\mathcal{H}^{d-1}(x) \\
&= \int_{\partial E\setminus B_\delta(y)} H(E,x)\nu(x)\cdot\eta(x) d\mathcal{H}^{d-1}(x) + \frac{1}{d-1}\int_{\partial B_\delta(y)\cap\partial E} \eta(x)\cdot\frac{x-y}{|x-y|} d\mathcal{H}^{d-2}(x) \\
&= \int_{\partial E\setminus B_\delta(y)} H(E,x)\nu(x)\cdot\eta(x) d\mathcal{H}^{d-1}(x) - \omega_{d-1}\log\delta
\end{aligned}
$$

where $\mathrm{div}_\tau\eta(x)$ is the tangential divergence, that is $\mathrm{div}_\tau\eta(x) = \mathrm{div}\eta(x) - \nu(x)^T\nabla\eta(x)\nu(x)$. Therefore integrating the previous equality on ∂E, we get that

$$
\begin{aligned}
\omega_{d-1}\log\delta \mathrm{P}(E) = &\int_{\partial E}\int_{\partial E\setminus B_\delta(y)} H(E,x)\nu(x)\cdot\eta(x) d\mathcal{H}^{d-1}(x) d\mathcal{H}^{d-1}(y) \\
&- \frac{1}{d-1}\int_{\partial E}\int_{\partial E\setminus B_\delta(y)} \mathrm{div}_\tau\eta(x) d\mathcal{H}^{d-1}(x) d\mathcal{H}^{d-1}(y).
\end{aligned} \tag{21}
$$

Now we compute

$$
\begin{aligned}
&\mathrm{div}_\tau\eta(x) = \mathrm{tr}\nabla\eta(x) - \nu(x)^T\nabla\eta(x)\nu(x) \\
&= -\mathrm{tr}\left(f(|x-y|)\mathbf{I} + f'(|x-y|)|x-y|\frac{y-x}{|x-y|}\otimes\frac{y-x}{|x-y|}\right) \\
&+ \nu(x)^T\left(f(|x-y|)\mathbf{I} + f'(|x-y|)|x-y|\frac{y-x}{|x-y|}\otimes\frac{y-x}{|x-y|}\right)\nu(x) \\
&= -f(|x-y|)d - f'(|x-y|)|x-y| + f(|x-y|) + f'(|x-y|)|x-y|\left|\frac{y-x}{|y-x|}\cdot\nu(x)\right|^2 \\
&= -\frac{1}{|x-y|^{d-1}} + \frac{1-(d-1)\log|x-y|}{|x-y|^{d-1})}\left|\frac{y-x}{|y-x|}\cdot\nu(x)\right|^2
\end{aligned}
$$

where we used the equality $rf'(r) = \frac{1}{r^{d-1}} - (d-1)f(r) = \frac{1-(d-1)\log r}{r^{d-1}}$.
If we substitute this expression in (21) we get

$$
\begin{aligned}
&\omega_{d-1}\log\delta\mathrm{P}(E) = \int_{\partial E}\int_{\partial E\setminus B_\delta(y)} \frac{H(E,x)\nu(x)\cdot(y-x)}{|x-y|^{d-1}}\log|x-y| d\mathcal{H}^{d-1}(x) d\mathcal{H}^{d-1}(y) \\
&+ \frac{1}{d-1}\int_{\partial E}\int_{\partial E\setminus B_\delta(y)} \frac{1}{|x-y|^{d-1}} d\mathcal{H}^{d-1}(x) d\mathcal{H}^{d-1}(y) \\
&- \frac{1}{d-1}\int_{\partial E}\int_{\partial E\setminus B_\delta(y)} \frac{1-(d-1)\log|x-y|}{|x-y|^{d-1})}\left|\frac{y-x}{|y-x|}\cdot\nu(x)\right|^2 d\mathcal{H}^{d-1}(x) d\mathcal{H}^{d-1}(y).
\end{aligned}
$$

The conclusion then follows by substituting $\omega_{d-1}\log\delta\mathrm{P}(E)$ with the previous expression in the representation formula obtained in (i), and observing that $1-\nu(x)\nu(y) = (\nu(x)-\nu(y))^2/2$.

$\square$

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