A semidiscrete scheme for a one-dimensional Cahn-Hilliard equation

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Abstract

We analyze a semidiscrete scheme for the Cahn-Hilliard equation in one space dimension, when the interface length parameter is equal to zero. We prove convergence of the scheme for a suitable class of initial data, and we identify the limit equation. We also characterize the long-time behavior of the limit solutions.

keywords: Nonconvex functionals, forward-backward parabolic equations, finite element method.

1 Introduction

Motivated by several models in phase transitions, granular media and image processing, Cahn-Hilliard type equations have been extensively studied in recent years. In one space dimension, a typical example of such equation is

$$u_t = \frac{1}{2}\left(W'(u_x)\right)_x \quad \text{in} \quad [0,1]\times[0,T], \tag{1.1}$$

where u_x is the gradient of a continuous, one-periodic function $u:[0,1]\to\mathbb{R}$ and W is the nonconvex energy density $W(p)=\frac{1}{2}(p^2-1)^2$ (double well potential). Equation (1.1) is the formal L^2-gradient flow of the functional

$$E[u] := \frac{1}{2}\int_0^1 W(u_x)\,dx. \tag{1.2}$$

Notice that, by the change of variables $v=u_x$, equation (1.1) reduces to

$$v_t = \frac{1}{2}\left(W'(v)\right)_{xx} \quad \text{in} \quad [0,1]\times[0,T], \tag{1.3}$$

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which corresponds to the H^{-1}-gradient flow of (1.2). We point out that, due to the nonconvexity of W, equations (1.1) and (1.3) are not well-posed.

In this paper, we deal with the semidiscrete problem

$$\begin{aligned} \frac{du^h}{dt} &= D^+ W'(D^- u^h) && \text{in } [0,1]\times[0,T] \\ u^h(\cdot,0) &= \overline{u}^h && \text{on } [0,1]\times\{0\} \end{aligned} \tag{1.4}$$

where $h > 0$ is the grid size, D^+, D^- are the difference quotients defined in Definition 2.1, and $\overline{u}^h$ is the discretization of a piecewise-smooth function with nondifferentiable points $a_1^h, \ldots, a_m^h$. We consider (1.4) coupled with the periodic boundary conditions

$$\begin{aligned} u^h(0,t) &= u^h(1,t) && \text{on } \{0,1\}\times[0,T] \\ D^- u^h(0,t) &= D^- u^h(1,t) && \text{on } \{0,1\}\times[0,T] \end{aligned} \tag{1.5}$$

In Proposition 2.4 we show that if the initial datum satisfies

$$\frac{1}{\sqrt{3}} \leq \left| D_h^- \overline{u}^h \right| \leq \alpha := \frac{1}{2}\left(\sqrt{3} + \frac{1}{\sqrt{3}}\right) \qquad \text{in } \bigcup_{j=1}^{m-1} \left[a_j^h, a_{j+1}^h\right], \tag{1.6}$$

then this property holds for all times $t \geq 0$. This assumption is necessary to prevent the development of backward-parabolic zones in the stable region, see the discussion at the end of Section 2.2.

The main result of this paper, proved in Section 3, is the convergence of solutions to (1.4) and (1.5), as $h \to 0$, for initial data satisfying (1.6). Indeed, in Section 3.1 we show that the limit function u is a continuous, piecewise-smooth solution to (1.1) satisfying the boundary conditions $W'(u_x(a_j^+)) = W'(u_x(a_j^-))$ on the jump points of the derivative. We also prove existence of a unique asymptotic state of u, as $t \to \infty$, whose derivative assumes exactly two values.

There is no classical theory for solutions of forward-backward parabolic equations like (1.1) and (1.3), a part from some partial results (see for instance [10]). However, several notions of weak solution have been considered in the literature: in [5] the author defines an implicit variational scheme which produces a solution to (1.1) with W replaced by the convexified potential

$$W^{**} = \max\{f \leq W : \; f \text{ is convex}\}.$$

In [4] the following fourth-order regularization of (1.1) is proposed:

$$u_t = -\varepsilon u_{xxxx} + W'(u_x)_x, \tag{1.7}$$

and the author conjectures the existence of a pointwise limit as $\varepsilon \to 0$. The dynamics of this regularization for small ε, which is quite involved and has at least three relevant scales, was studied in [13, 3], where the asymptotic behavior as $t \to \infty$ is also discussed. In [12] the author considers instead the regularization

$$u_t = \varepsilon u_{txx} + W'(u_x)_x \,, \tag{1.8}$$

proving the convergence, as $\varepsilon \to 0$, to a measure valued solution to (1.1). In [8] further properties of such limit solutions are discussed, with particular emphasis on a hysteresis phenomenon which also appears in our analysis (see Section 3.1).

Our approach is different from the ones mentioned above: instead of studying continuous regularizations, we perform a spatial semidiscretisation using the standard finite element method. In Section 2, we discuss the properties of the Cauchy problem for the semidiscrete scheme (see (2.5)) and provide suitable assumptions on initial data under which solutions are stable. One expects convergence of the scheme to classical solutions of (1.1) at least when the gradient of the initial datum takes values in the forward parabolic region, and we confirm such expectation with the only restriction that the the gradient is not too big (see (1.6)). This is an advantage with respect to variational methods like the implicit scheme discussed in [5], which selects a local minimum of (1.2) and automatically forces all 1-Lipschitz functions not to move. Convergence of the scheme as the grid size h goes to zero is proved in Section 3, where we also identify the limit problem. We point out that our limit problem coincides with the limit of the continuous regularization (1.8), but not with the regularization (1.7). The reason is that our boundary conditions do not allow jump points of the gradient to vanish as $t \to +\infty$; an aspect that we discuss in the last section.

In order to keep the focus on the analytical aspects of the problem, we will not discuss the optimal convergence rate of the scheme, nor provide numerical simulations. We address the interested reader to [3, 6] for numerical simulations in the one-dimensional case, or to [9] for higher dimensions. A finite element discretisation of a simplified granular material model related to (1.1) was performed in [14] (see also [7]), where the authors study the limit profiles as $t \to +\infty$ of the discrete solutions.

2 Spatial semidiscretisation

Let $\mathrm{I} := [0, 1]$ and let $\{h, \dots, N\,h\}$ be a uniform grid on I with grid size $h = 1/N$, where $N \in \mathbb{N}$. Since we will work with 1-periodic functions, we identify the node 0 with the node N, hence $N + i$ with i. We denote by $PL(\mathrm{I})$ the N-dimensional vector subspace of $W^{1,\infty}(\mathrm{I})$, consisting of all continuous functions $u : \mathrm{I} \to \mathbb{R}$, with $u(0) = u(1)$, which are linear on the intervals $((i-1)h, ih)$, for all $i \in \{1, \dots, N\}$. We also let $PC(\mathrm{I})$ be the N-dimensional vector subspace of $L^2(\mathrm{I})$ of all right-continuous piecewise-constant functions on the grid. Letting $u_i := u(ih)$, we can identify $u \in PL(\mathrm{I})$ (resp. $u \in PC(\mathrm{I})$) with the vector $u^h := (u_1, u_2, \dots, u_N)$. Both $PL(\mathrm{I})$ and $PC(\mathrm{I})$ are endowed with the norms

$$\|u^h\|_{L^\infty(\mathrm{I})} := \max\{|u_i| : i = 1, \dots, N\} \qquad \|u^h\|^2_{L^2(\mathrm{I})} := h \sum_{i=1}^N u_i^2.$$

Definition 2.1. *We define the map $D^- : PL(\mathrm{I}) \to PC(\mathrm{I})$ and its adjoint $D^+ : PC(\mathrm{I}) \to PL(\mathrm{I})$ as*

$$(D^- u^h)_i = \frac{u_i - u_{i-1}}{h} \qquad (D^+ w)_i = \frac{w_{i+1} - w_1}{h} \qquad i \in 1, \dots, N. \tag{2.1}$$

With this notation, the space discretisation of (1.1) can be expressed by the following system of ODEs on $PL(\mathrm{I})$:

$$\frac{du_i}{dt} = -\frac{1}{h}\frac{\partial \mathcal{F}}{\partial u_i} = \frac{1}{h}\left(W'\left(\frac{u_{i+1} - u_i}{h}\right) - W'\left(\frac{u_i - u_{i-1}}{h}\right)\right) = (D^+ W'(D^- u))_i, \tag{2.2}$$

for all $i \in \{1, \dots, N\}$, with periodic boundary conditions.

We now introduce the class of initial data for (1.1) which we will consider in this paper.

Assumption 2.1. *Let $\{a_j\}_{j=1}^m \in (0,1)$, with $a_1 < a_2 < \ldots < a_m$. We shall consider initial $\overline{u} \in W^{1,\infty}(\mathrm{I}) \cap C^1(\mathrm{I} \setminus \{a_1, \dots, a_m\})$ such that $\overline{u}(0) = \overline{u}(1)$ and $\overline{u}_x(0) = \overline{u}_x(1)$.*

Remark 2.1. Notice that, if u solves

$$\begin{aligned} u_t &= W'(u_x)_x && \text{in } \mathrm{I} \times [0, +\infty) \\ u(0,t) &= u(1,t) && \text{on } \partial\mathrm{I} \times [0, +\infty) \\ u_x(0,t) &= u_x(1,t) && \text{on } \partial\mathrm{I} \times [0, +\infty), \end{aligned} \tag{2.3}$$

then $v = u_x$ solves

$$\begin{aligned} v_t &= W'(v)_{xx} && \text{in } \mathrm{I} \times [0, +\infty) \\ v(0) &= v(1) && \text{on } \partial\mathrm{I} \times [0, +\infty) \\ v_x(0,t) &= v_x(1,t) && \text{on } \partial\mathrm{I} \times [0, +\infty). \end{aligned} \tag{2.4}$$

Conversely, if $v = u_x$ solves (2.4) and $\int_\mathrm{I} v\, dx = 0$, then u solves (2.3). To get the full equivalence, i.e. for $\int_\mathrm{I} v\, dx = c$, it is enough to substitute the second line in (2.3) with $u(0,t) = u(1,t) + c$. For simplicity of the presentation, we restrict to the case $c = 0$.

Assumption 2.2. *Let $\overline{u}$ be as in Assumption 2.1. We denote by $a_1^h, \dots, a_m^h$ be the grid points corresponding to the nondifferentiable points of $\overline{u}$, that is, $a_i \in [a_i^h, a_i^h + h)$ for all $i \in \{1, \dots, N\}$. For the discrete initial data $\overline{u}^h \in PL(\mathrm{I})$ we require*

$$\|\overline{u}^h - \overline{u}\|_{L^\infty(\mathrm{I})} \underset{h \to 0}{\longrightarrow} 0\,; \quad \|D^- \overline{u}^h - \overline{u}_x\|_{L^1(\mathrm{I})} \underset{h \to 0}{\longrightarrow} 0\,; \quad \|D^- \overline{u}^h\|_{L^\infty(\mathrm{I})} \le C,$$

for some $C > 0$ independent of h.

The Cauchy problem corresponding to (2.2) is

$$\begin{aligned} \frac{du^h}{dt} &= D^+W'(D^-u^h) && \text{in } \mathrm{I}\times[0,T] \\ u^h(0,t) &= u^h(1,t) && \text{on } \partial\mathrm{I}\times[0,T] \\ D^-u^h(0,t) &= D^-u^h(1,t) && \text{on } \partial\mathrm{I}\times[0,T] \\ u^h(\cdot,0) &= \overline{u}^h && \text{on } \mathrm{I}\times\{0\}. \end{aligned} \tag{2.5}$$

where $\overline{u}^h \in PL(\mathrm{I})$ denotes the discrete initial datum with the properties listed in Assumption 2.2. Note that, due to the smoothness of W, the scheme (2.5) admits a unique solution $u^h \in C^\infty([0,t_0], PL(\mathrm{I}))$ for a suitable $t_0 > 0$. Moreover, by direct integration we get

$$\int_{\mathrm{I}} u^h(x,t)\,dx = \int_{\mathrm{I}} \overline{u}^h(x)\,dx. \tag{2.6}$$

In many cases, it will be useful to work with the system governing the evolution of the spatial derivative of $u^h(x,t)$.

Proposition 2.1. *Let $\overline{u}^h \in PL(\mathrm{I})$ be a discrete initial datum for (2.5) satisfying Assumption 2.2. If $u^h(x,t)$ is a solution to the Cauchy problem* (2.5)*, then $v^h := D^-u^h$ is a solution to the following system of ODEs:*

$$\begin{aligned} \frac{dv^h}{dt} &= D^-D^+W'(v^h)\,, && \text{in } \mathrm{I}\times[0,T] \\ v^h(0,t) &= v^h(1,t) && \text{on } \partial\mathrm{I}\times[0,T] \\ D^-v^h(0,t) &= D^-v^h(1,t) && \text{on } \partial\mathrm{I}\times[0,T] \\ v^h(\cdot,0) &= D^-\overline{u}^h && \text{on } \mathrm{I}\times\{0\}. \end{aligned} \tag{2.7}$$

2.1 A priori estimates

Figure 1: *At the left, the graph of the potential W; at the right, the graph of its derivative.*

We denote by $\alpha > 1$ the real number such that $W'(\alpha) = \alpha^3 - \alpha = W'\left(-\frac{1}{\sqrt{3}}\right)$, see Figure 2.1. Let us denote by $M(t) := \max_{i=1,\dots,N} v_i(t)$ and $m(t) := \min_{i=1,\dots,N} v_i(t)$ the maximum and minimum of the nodal values of v, respectively.

The following result will be needed in Proposition 2.2; the proof can be found in [1, Lemma 5.1 and 5.2].

Lemma 2.1. *Let $u_1, \dots, u_N$ be real continuous right-differentiable functions in an interval $[0, T)$. Define $M(t) := \max_{i=1,\dots,N} u_i(t)$. Then $M(t)$ is continuous, right-differentiable in $[0, T)$ and*

$$\frac{d}{dt^+} M(t) = \max_{i=1,\dots,N} \left\{ \frac{d}{dt^+} u_i(t) \ : \ u_i(t) = M(t) \right\}, \qquad t \in [0, T).$$

Proposition 2.2 (L^∞ estimate)**.** *Let $u^h(t)$ be solutions of the discrete Cauchy problem (2.5) with initial data $\overline{u}^h$ satisfying Assumption 2.2. Then*

$$\|v^h(t)\|_{L^\infty(\mathrm{I})} \le c \quad \forall\, t \in [0, \infty) \tag{2.8}$$

$$\|u^h(t)\|_{L^\infty(\mathrm{I})} \le c \quad \forall\, t \in [0, \infty) \tag{2.9}$$

where the constant $c > 0$ is independent of h.

Proof. At time $t = 0$, the statement follows directly from the assumptions on the initial data.

Step 1. Let us first prove (2.8). We will show that $\max_i |v_i(t)|$ is nonincreasing whenever it is greater than α. We distinguish two cases.

Case 1: $\max_i |v_i(t)| = M(t) \ge \alpha$. As $M(t)$ is a solution to (2.7), by Lemma 2.1 we have

$$\begin{aligned}\frac{d}{dt^+} M(t) &= \max_{i=1,\dots,N} \left\{ \frac{d}{dt^+} v_i(t) \ : \ v_i(t) = M(t) \right\} \ = \max_{i:v_i(t)=M(t)} D^- D^+ W'(v_i) \\ &= \frac{1}{h^2} \max_{i:v_i(t)=M(t)} \left(W'(v_{i+1}) - 2W'(M) + W'(v_{i-1}) \right).\end{aligned}$$

From $M \ge \alpha$ and $M \ge v_{i\pm1}$ we then get

$$W'(M(t)) \ge \max\{W'(v_{i-1}), W'(v_{i+1})\}.$$

Hence

$$\max_{i:v_i(t)=M(t)} \frac{dv_i}{dt^+} \le 0, \tag{2.10}$$

which gives the upper bound

$$\max_{i=1,\dots,N} v_i(t) \ \le \ \max_{i=1,\dots,N} \{\alpha, \max_i v_i(0)\}. \tag{2.11}$$

Case 2: $\max_i |v_i(t)| = -m(t) \ge \alpha$. Reasoning as above we obtain

$$\min_{i=1,\dots,N} v_i(t) \ \ge \ \min_{i=1,\dots,N} \{-\alpha, \min_i v_i(0)\}. \tag{2.12}$$

Putting together (2.11) and (2.12), we finally get

$$\|v^h(t)\|_{L^\infty(\mathrm{I})} \le \max\{\alpha, \|D^- \overline{u}^h\|_{L^\infty(\mathrm{I})}\} \qquad \forall t \in [0, \infty)$$

which is (2.8).

Step 2. Estimate (2.9) now follows directly from (2.6) and (2.8).

□

Theorem 2.1 (Global existence of the flow (2.5)). *Let $u^h(t)$ be a solution of the Cauchy problem (2.5). If the initial data $\overline{u}$ is in $PL(\mathrm{I})$ and one-periodic, then the solution $u^h(t)$ exists for all times t.*

Proof. As we noted before, $u^h \in C^\infty([0,t_0], PL(\mathrm{I}))$, for some $t_0 > 0$. Proposition 2.2 established L^∞ bounds on both $u^h(t)$ and its discrete derivative $v^h(t)$, which are uniform in time. As a consequence, the solution to the Cauchy problem can be extended for all times $t \in [0, +\infty)$. □

Proposition 2.3 (Energy decreasing property). *Let $u^h(x,t)$ be the solution of (2.5) with an initial datum $\overline{u}^h$ satisfying Assumption 2.2. Define the discrete energy*

$$E^h(t) := E[u^h(\cdot,t)] := h \sum_{i=1}^{N} W(D^- u_i(t)). \tag{2.13}$$

Then the following relation holds:

$$\frac{d}{dt} E[u^h(\cdot,t)] = -\|u_t^h\|^2_{L^2(\mathrm{I})} \le 0. \tag{2.14}$$

Proof. Keeping in mind the periodic boundary conditions, we compute

$$\begin{aligned} \frac{d}{dt} E[u^h(\cdot,t)] &= h \sum_{i=1}^{N} W'(D^- u_i)\partial_t(D^- u_i) + W'(D^- u_N)\partial_t u_N - W'(D^- u_0)\partial_t u_0 \\ &= -h \sum_{i=1}^{N} D^+ W'(D^- u^h)_i \, \partial_t u_i = -\|u_t^h(\cdot,t)\|^2_{L^2(\mathrm{I})} \le 0. \end{aligned}$$

□

As a consequence, we have

$$E^h(t) \le E^h(0) \le C \qquad \forall t \ge 0, \tag{2.15}$$

as the discrete initial datum is bounded in $W^{1,\infty}(\mathrm{I})$ uniformly in h, by Assumption 2.2.

Corollary 2.1 (Hölder continuity). *Let $\overline{u}^h$ be initial data satisfying Assumption 2.2. Then the solutions u^h of (2.5) are uniformly bounded in $C^{\frac{1}{2}}\big([0,T); L^2(\mathrm{I})\big)$.*

Proof. Let $0 \le t_1 < t_2 < +\infty$. We have to show $\|u^h(t_1) - u^h(t_2)\|_{L^2} \le c\,|t_2 - t_1|^{\frac{1}{2}}$, for some constant $c > 0$ independent of h. Using Hölder inequality and (2.15), we get

$$\|u^h(t_1) - u^h(t_2)\|_{L^2(\mathrm{I})} \le |t_2 - t_1|^{\frac{1}{2}} \Big(E^h(t_1) - E^h(t_2)\Big)^{\frac{1}{2}} \le \sqrt{E^h(0)}\,|t_2 - t_1|^{\frac{1}{2}}.$$

□

The following corollary will be an important ingredient in the convergence proof.

Corollary 2.2. *Let $u^h(x,t)$ be solutions to the Cauchy problem* (2.5) *with initial data $\overline{u}^h$ satisfying Assumption 2.2. Then $\frac{d}{dt}u^h \in L^2\big([0,\infty), L^2(\mathrm{I})\big)$, i.e.*

$$\int_0^\infty \left\| \frac{d}{dt} u^h(\cdot,t) \right\|^2_{L^2(\mathrm{I})} dt \le E^h(0) \le c, \tag{2.16}$$

where the constant c is independent of h.

Proof. Recalling (2.14) we have

$$E^h(0) - E^h(t) = \int_0^t \left\| u^h_t(\cdot,\tau) \right\|^2_{L^2(\mathrm{I})} d\tau \qquad \forall t \ge 0\,.$$

As $E^h(0) \le C$ by Assumption 2.2, the thesis follows letting $t \to +\infty$. □

2.2 The stability estimate

We shall make another assumption on initial data $\overline{u}^h$ which guarantees the stability of the solution to (2.5): we take initial data $\overline{u} \in W^{1,\infty}(\mathrm{I})$ as in Assumption 2.1 which further satisfy

$$\frac{1}{\sqrt{3}} \;\le\; (-1)^{j+1}\,\overline{u}_x(x) \;\le\; \alpha \qquad \forall\, x \in (a_{j-1}, a_j)\,, \quad j \in \{1,\dots,m\}\,. \tag{2.17}$$

Note that (2.17) implies in particular that m is even and $\overline{u}_x$ takes values only in the regions where the potential W is convex.

We point out that a similar assumption was made in [2] for the Perona-Malik equation. We now formulate the discrete analog of (2.17).

Assumption 2.3. *Let α as above and let $\overline{u}^h \in PL(\mathrm{I})$ be discrete initial data satisfying Assumption 2.2, with $a^h_1, \dots, a^h_m$ the grid points corresponding to the nondifferentiability points of $\overline{u}$. We require that $\overline{u}^h$ satisfies*

$$\frac{1}{\sqrt{3}} \;\le\; (-1)^{j+1}\,D^-\overline{u}_i \;\le\; \alpha \qquad \forall\, ih \in \left(a^h_{j-1}, a^h_j\right]\,, \quad j \in \{1,\dots,m\}\,. \tag{2.18}$$

Proposition 2.4 (Stability estimate)**.** *Let u^h be solutions to (2.5) with initial data $\overline{u}^h$ satisfying Assumptions 2.2 and 2.3. Then u^h satisfies*

$$\frac{1}{\sqrt{3}} \;\le\; (-1)^{j+1}\,D^-u_i(t) \;\le\; \alpha \qquad \forall\, ih \in \left(a^h_{j-1}, a^h_j\right]\,, \quad j \in \{1,\dots,m\}\,, \quad t \ge 0\,. \tag{2.19}$$

Proof. Fix $j \in \{1,\dots,m\}$. Without loss of generality we can assume that $\overline{u}^h$ is monotone increasing on $[a_j^h h, a_{j+1}^h h]$, that is $v_i(0) = D^-\overline{u}_i \in [1/\sqrt{3}, \alpha]$ in $[a_j^h h, a_{j+1}^h h]$. We let

$$m(t) := \min_{i=1,\dots,N} v_i(t) \qquad M(t) := \max_{i=1,\dots,N} v_i(t)\,,$$

and distinguish two cases:
Case 1: $M(t) = \alpha$ for some $t \geq 0$. By Lemma 2.1 and (2.7) we have

$$\begin{aligned} \frac{d}{dt^+}M(t) &= \max_{i:\, v_i(t)=M(t)} \frac{d}{dt^+}v_i(t) \;=\; \max_{i:\, v_i(t)=M(t)} D^-D^+W'(v_i) \\ &= \frac{1}{h^2}\max_{i:\, v_i(t)=M(t)} \big(W'(v_{i+1}) - 2W'(M) + W'(v_{i-1})\big) \leq 0, \end{aligned} \tag{2.20}$$

where we used the fact that $W'(\alpha) \geq W'(x)$ for all $x \leq \alpha$.
Case 2: $m(t) = 1/\sqrt{3}$ for some $t \geq 0$. As above, we have

$$\begin{aligned} \frac{d}{dt^+}m(t) &= \min_{i:\, v_i(t)=m(t)} \frac{d}{dt^+}v_i(t) \;=\; \min_{i:\, v_i(t)=m(t)} D^-D^+W'(v_i) \\ &= \frac{1}{h^2}\min_{i:\, v_i(t)=m(t)} \big(W'(v_{i+1}) - 2W'(M) + W'(v_{i-1})\big) \geq 0, \end{aligned} \tag{2.21}$$

where we used the fact that $W'(1/\sqrt{3}) \leq W'(x)$ for all $x \geq -\alpha$.

The thesis follows from (2.20) and (2.21). □

3 Convergence of the scheme

Proposition 3.1. *Let the initial data $\overline{u}^h$ satisfy Assumption 2.2. Then the solutions u^h converge, up to a subsequence as $h \to 0$, to a limit function $u \in C(\mathrm{I} \times [0,+\infty))$, uniformly on compact subset of* $\mathrm{I} \times [0,+\infty)$.

Proof. By Proposition 2.2 and Corollary 2.2 we know that the solutions u^h are uniformly bounded in $X_T := H^1\big([0,T], L^2(\mathrm{I})\big) \cap L^\infty\big([0,T], W^{1,\infty}(\mathrm{I})\big)$, for all $T > 0$. The thesis follows form the compact embedding of X_T into $C(\mathrm{I} \times [0,T])$ [2]. □

Recalling Proposition 2.4 and reasoning exaclty as in [11, Proposition 3.3], we obtain the following estimate.

Lemma 3.1. *Let $u^h(t)$ be a solution to the Cauchy problem (2.5) with initial data $\overline{u}^h$ satisfying Assumptions 2.2 and 2.3. Then, for every open set $\mathrm{I}_1 \subset\subset \mathrm{I} \setminus \{a_1, a_m\}$, there exists a constant $c = c(\mathrm{I}_1)$ such that for h small enough there holds*

$$\left\| \frac{d}{dt}u^h(t) \right\|^2_{L^2(\mathrm{I}_1)} \leq\; E^h(0)\; \left(\frac{1}{t} + c\right) \qquad \forall t > 0. \tag{3.1}$$

Proposition 3.2. *Let* $\overline{u}^h$ *be initial data satisfying Assumptions 2.2 and 2.3, and let* u^h *be the corresponding solutions to the Cauchy problem (2.5). Then, for any compact subset* K *of* $\mathrm{I} \setminus \{a_1, \dots, a_m\}$ *and for every* $t > 0$*, there exists a function* $\psi \in H^1(K)$ *such that*

$$W'\big(v^h\big) \longrightarrow \psi \quad \text{uniformly on } K \text{ (up to a subsequence).}$$

Proof. As $W'(v^h)$ is uniformly bounded in $L^\infty(\mathrm{I})$ by Assumption 2.3, up to a suitable subsequence we have

$$W'\big(D^- u^h\big) \longrightarrow \psi \quad \text{weakly}^* \text{ in } L^\infty(K).$$

Moreover, by Lemma 3.1 $\frac{d}{dt} u^h = D^+ W'\left(D^- u^h\right)$ is uniformly bounded in $L^2(\mathrm{I})$. The thesis then follows from the Arzelà-Ascoli Theorem. □

Proposition 3.2 allows us to obtain the strong convergence of $D^- u^h$, which is needed to pass to the limit in the nonlinear problem (2.5).

Proposition 3.3. *Let* $u^h(t)$ *be solutions to the Cauchy problem* (2.5) *with initial data* $\overline{u}^h$ *satisfying Assumptions 2.2 and 2.3. Then, up to a subsequence as* $h \to 0$*,*

$$D^- u^h \longrightarrow u_x \qquad \text{a.e. on } \mathrm{I} \times [0, +\infty) \tag{3.2}$$

and

$$W'\big(D^- u^h\big) \longrightarrow W'(u_x) \text{ in } L^2_{\mathrm{loc}}(\mathrm{I} \times [0, +\infty)). \tag{3.3}$$

Proof. By Propositions 2.2 and 3.1 we have

$$D^- u^h \longrightarrow u_x \quad \text{weakly}^* \text{ in } L^\infty(\mathrm{I}) \text{ for every } t \geq 0. \tag{3.4}$$

Let K be as in Proposition 3.2. As W' is invertible on $[-\alpha, -1/\sqrt{3}]$ and $[1/\sqrt{3}, \alpha]$, Proposition 3.2 implies

$$D^- u^h(t) = (W')^{-1}\Big(W'(D^- u^h(t))\Big) \longrightarrow u_x(t) \quad \text{uniformly on } K$$

for all $t > 0$, which gives (3.2). Claim (3.3) then follows from (3.2) and Lebesgue's Theorem. □

3.1 The limit problem

Theorem 3.1. *Let* $\overline{u} \in W^{1,\infty}(\mathrm{I})$ *be an initial datum satisfying Assumptions 2.1 and* (2.17)*. Let* $\overline{u}^h$ *be finite element discretizations of* $\overline{u}$ *satisfying Assumptions 2.2 and 2.3, let* u^h *be* *the corresponding solutions to (2.5), and let $u \in C(\mathrm{I} \times [0,+\infty))$ be the limit of u^h, as $h \to 0$, given by Proposition 3.1. Then u is the unique solution to the following PDE:*

$$\begin{array}{lll} (i) & u_t = W'(u_x)_x & \text{in } (\mathrm{I} \setminus \{a_1,\dots,a_m\}) \times [0,+\infty) \\ (ii) & W'(u_x^-) = W'(u_x^+) & \text{on } \{a_1,\dots,a_m\} \times [0,+\infty) \\ (iii) & u^- = u^+ & \text{on } \{a_1,\dots,a_m\} \times [0,+\infty) \\ (iv) & u(0) = \overline{u} & \text{at } \mathrm{I} \times \{0\}, \end{array} \tag{3.5}$$

where we set

$$u^\pm := \lim_{x \to a_j^\pm} u(x) \qquad u_x^\pm := \lim_{x \to a_j^\pm} u_x(x).$$

In particular $W'(u_x) \in C(\mathrm{I} \times [0,+\infty))$ and $u \in C^\infty((\mathrm{I} \setminus \{a_1,\dots,a_m\}) \times (0,+\infty))$.

Proof. Multiplying by $\varphi \in C_0^1(\mathrm{I} \times [0,+\infty))$ the first equation in (2.5), after an integration by parts we get

$$\int_0^\infty \int_{\mathrm{I}} u^h \varphi_t \, dx \, dt = \int_0^\infty \int_{\mathrm{I}} W'(D^- u^h) \, D^- \varphi \, dx \, dt. \tag{3.6}$$

As $u^h \to u$ locally uniformly on $\mathrm{I} \times [0,+\infty)$ by Proposition 3.1, and $W'(D^- u^h) D^- \varphi \to W'(u_x)\varphi_x$ in $L^2(\mathrm{I} \times [0,+\infty))$ by Proposition 3.3, we can pass to the limit in (3.6):

$$\int_0^\infty \int_{\mathrm{I}} u \, \varphi_t \, dx \, dt = \int_0^\infty \int_{\mathrm{I}} W'(u_x) \, \varphi_x \, dx \, dt. \tag{3.7}$$

Since $u_t \in L^2(\mathrm{I} \times [0,+\infty))$, (3.7) imples $W'(u_x) \in L^2\left([0,+\infty), H^1(\mathrm{I})\right)$, so that we can integrate by parts and obtain

$$\int_0^\infty \int_{\mathrm{I}} u_t \varphi \, dx dt = \int_0^\infty \int_{\mathrm{I}} W'(u_x)_x \, \varphi \, dx dt, \tag{3.8}$$

which proves statement (i).

Equalities (ii) and (iii) follow from the continuity of $W'(u_x)$ and u, respectively. $\square$

Remark 3.1. Problem (3.9) is equivalent to the limit problem derived in [12, 8] for the regularization (1.8). On the other hand, due to the numerical simulations performed in [3], it is expected to be different form the limit problem corresponding to the Cahn-Hilliard regularization (1.7) discussed in [4, 13].

Corollary 3.1. *If u satisfies (3.5), then $v = u_x = \lim_{h \to 0} v^h$ is the unique solution to the following PDE:*

$$\begin{array}{rcll} v_t &=& W'(v)_{xx} & \text{in } (\mathrm{I} \setminus \{a_1,\dots,a_m\}) \times [0,+\infty) \\ W'(v^-) &=& W'(v^+) & \text{on } \{a_1,\dots,a_m\} \times [0,+\infty) \\ W'(v)_x^- &=& W'(v)_x^+ & \text{on } \{a_1,\dots,a_m\} \times [0,+\infty) \\ v(0) &=& \overline{u}_x & \text{on } (\mathrm{I} \setminus \{a_1,\dots,a_m\}) \times \{0\}. \end{array} \tag{3.9}$$

Passing to the limit in (2.16) as $h \to 0$, we obtain an integral estimate on the time derivative of u.

Proposition 3.4. *Let u be as in Theorem 3.1. We have $u_t \in L^2\big(\mathrm{I} \times (0,\infty)\big)$ and*

$$\int_{\mathrm{I}\times(0,\infty)} \left(\frac{du}{dt}(x,t)\right)^2 dxdt = E[\overline{u}].$$

3.2 Long-time behavior

Theorem 3.2. *Let u be a solution of* (3.5)*. Then there exists a unique limit*

$$u_\infty(x) := \lim_{t\to+\infty} u(t,x) \qquad x \in \mathrm{I},$$

which is given by the piecewise-linear solution to

$$\begin{array}{lll} (i) & W'\left((u_\infty)_x\right)_x = 0 & \text{in } \mathrm{I} \setminus \{a_1,\dots,a_m\} \\ (ii) & W'\big((u_\infty)_x^-\big) = W'\big((u_\infty)_x^+\big) & \text{on } \{a_1,\dots,a_m\} \\ (iii) & u_\infty^- = u_\infty^+ & \text{on } \{a_1,\dots,a_m\}. \end{array} \tag{3.10}$$

Proof. We divide the proof into three steps.
Step 1 (Existence of u_∞). By Proposition 3.4, there exists a sequence of times $t_n \to +\infty$ such that

$$\int_{t_n}^{t_{n+1}} \|u_t\|^2_{L^2(\mathrm{I})}\, dt \;\longrightarrow\; 0. \tag{3.11}$$

We now define a sequence w^n of solutions to (3.5) in the following way:

$$w^n(x,t) \;:=\; u(x, t_n + t) \quad t \in [0,1].$$

From (3.11) we have

$$\int_0^1 \|w^n_t\|^2_{L^2(\mathrm{I})} dt \underset{n\to\infty}{\longrightarrow} 0, \tag{3.12}$$

whence $w^n \to w \in H^1\left([0,1], L^2(\mathrm{I})\right) \cap L^\infty([0,1], W^{1,\infty}(\mathrm{I}))$, with $w_t \equiv 0$, that is the limit function $w = u_\infty$ does not depend on t.
Step 2 (Limit equation). As every w^n solves (3.5), from (3.7) we get

$$\int_0^1 \int_{\mathrm{I}} W'(w^n_x)\varphi_x \, dxdt \;=\; 0,$$

for all test functions $\varphi \in C^1(\mathrm{I})$ independent of t. Passing to the limit as $n \to \infty$ and recalling (3.12), we get (i) and (ii), while (iii) follows from the Lipschitz continuity of w.

We now show that w_x is a piecewise-constant function which assumes exactly two values, p^- and p^+. Indeed, (3.10) (i) implies that, for all $j \in \{1, \ldots, m\}$, there exists $p_j \in [-\alpha, -1/\sqrt{3}] \cup [1/\sqrt{3}, \alpha]$ such that $W'(w_x) \equiv p_j$. Moreover, from condition (ii) we have that

$$W'(p_i) = W'(p_j) \qquad \forall\, i,\, j \in \{1, \ldots, m\}\,. \tag{3.13}$$

Since W' is monotone in the intervals $[-\alpha, -1/\sqrt{3}]$ and $[1/\sqrt{3}, \alpha]$, we get that for all $p \in [1/\sqrt{3}, \alpha]$ there exists only one value $\tilde{p} \in [-\alpha, -1/\sqrt{3}]$ such that

$$W'(p) \;=\; W'(\tilde{p}). \tag{3.14}$$

The claim then follows from (3.13) and (3.14).

Step 3 (Uniqueness). Once we know that w_x assumes precisely two values $p^- < p^+$, with $p^- \in [-\alpha, -1/\sqrt{3}]$ and $p^+ \in [1/\sqrt{3}, \alpha]$, the uniqueness of such values follows by direct integration. More precisely, assuming without loss of generality $w_x = p^+ > 0$ on $[0, a_1]$ and recalling (3.10) (iii), we have

$$0 = w(1) - w(0) = \sigma(p^+), \tag{3.15}$$

where

$$\sigma(p) \;:=\; p \sum_{\ell=0}^{\frac{m}{2}-1} (a_{2\ell+1} - a_{2\ell}) \;+\; \tilde{p} \sum_{k=1}^{\frac{m}{2}} (a_{2k} - a_{2k-1}) \qquad p \in \left[\frac{1}{\sqrt{3}}, \alpha\right].$$

Since σ is strictly increasing on $[1/\sqrt{3}, \alpha]$, equation (3.15) uniquely determines the value of p^+, and consequently of p^-.

□

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