# A NOTE ON NON LOWER SEMICONTINUOUS PERIMETER FUNCTIONALS ON PARTITIONS

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Abstract. We consider isotropic non lower semicontinuous weighted perimeter functionals defined on partitions of domains in $\mathbb{R}^n$. Besides identifying a condition on the structure of the domain which ensures the existence of minimizing configurations, we describe the structure of such minima, as well as their regularity.

## 1. Introduction

In this note we address the existence, the structure and the regularity properties of minimizing configurations for weighted non lower semicontinuous perimeter functionals of the form

$$\mathcal{F}_{\Omega,\mathrm{m}}(E_1, E_2, E_3) := \sum_{i<j\in\{1,2,3\}} \sigma_{ij}\mathcal{H}^{n-1}(\partial^* E_i \cap \partial^* E_j \cap \Omega)\,, \tag{1}$$

defined on partitions of a domain $\Omega \subset \mathbb{R}^n$ in three sets with prescribed Lebesgue measures $\mathrm{m} := (m_1, m_2, m_3)$, where $\sigma_{ij} > 0$. These functionals arise in the modelling of multicomponent systems interacting at the contact interfaces via isotropic energies. Their diffuse approximation as well as methods to study their diffuse gradient dynamic have been recently considered in [3] and [2]. We believe that non lower semicontinuous functionals as (1) can represent a good model to describe from a macroscopic point of view the effect that surface tension has in selecting equilibrium configurations of biological cell sorting phenomena due to differential adhesion. A rigorous microscopic mathematical analysis of these phenomena have been given amongst others in [9] (see also references therein), whose results are in accordance with the ones in our work. Identifying the sets $E_1$ and $E_2$ with the regions in the domain $\Omega$ which are occupied by the two cell types and denoting by $E_3$ the remaining environment in which the cells can move, we can find a stable condition (equation (4)) on the three surface tensions $\sigma_{ij}$ under which the minima of the functional exhibit separation between one of the cell types and the environment. For a particular class of domains, including the most significant ones for the biological interpretation (see Definition 3.1), we can also describe quite explicitly the shape which is taken by each region in correspondence of a minimum (Theorem 3.4). In particular we obtain an engulfment of the most adhesive cell type into the less adhesive one.
The first results on weighted perimeter functionals on partitions have been proven with different methods in [1], [10], where it is shown that (independently of the number of the sets in the partition) (1) is lower semicontinuous if and only if, under the assumption that

$$\begin{aligned} &\sigma_{ij} \leq \sigma_{ik} + \sigma_{jk} \quad \text{for all}\, i,j,k \in \{1,2,3\},\ \text{with}\, i \neq j \neq k\, \text{and}\, i \neq k\,,\\ &\sigma_{ij} = \sigma_{ji} \qquad\quad \text{for all } i \neq j \in \{1,2,3\}\,, \text{with}\, i \neq j\,. \end{aligned} \tag{2}$$

As for the regularity of the minimizers, the strict triangle inequalities in (2) are sufficient to prove that $\mathcal{H}^{n-1}-$almost every point of the minimizing interfaces belongs to the boundary

of just two elements of the partition (see [10] and [4]) and this allows to apply the standard regularity results for minimizing boundaries with prescribed volume.
In this paper we are interested in minimizers of (1) when the condition (2) is violated. Namely, we shall assume that

$$\sigma_{13} > \sigma_{12} + \sigma_{23} \,. \tag{3}$$

Some topological properties of such minimizers have been described in [6] (see also [10]).
The plan of the paper is the following. In Section 3 we introduce the class of domains $\Omega \subset \mathbb{R}^n$ which are foliated by isoperimetric sets (see Definition 3.1 below) and we prove that the functional (1) has always a minimum in these domains. In Section 4, relying on the elimination Theorem (Theorem 3.1 in [4]), we show that if (1) admits a minimizer $(F_1, F_2, F_3)$, the boundary of each set $F_i$ is regular out of a closed singular set of zero $\mathcal{H}^{n-1}$ measure. This result is slightly surprising since the functional, under assumption (3), is not lower semicontinous. We also point out that in our proof it is essential that the *strict* inequality holds in (3). Eventually, in Section 5 we give an example of a domain $\Omega \subset \mathbb{R}^2$ in which (1) has no minimizer.

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## 2. Notation

With $\mathcal{L}^n$ we denote the $n$-dimensional Lebesgue measure. Given a set $F \subset \mathbb{R}^n$ with finite perimeter, we denote its perimeter with $\mathrm{P}(F)$ and its relative perimeter with respect to an open set $\Omega \subset \mathbb{R}^n$ with $\mathrm{P}(F;\Omega)$. By $\chi_F$ we denote the characteristic function of $F$, by $\partial^* F$ its reduced boundary and by $|F| := \int_{\mathbb{R}^n} \chi_F(x)dx$ its volume. For $\mathcal{L}^n$-almost all points $x \in \mathbb{R}^n$ the density at $x$ with respect to the Lebesgue measure of a set $F \subset \mathbb{R}^n$ having finite perimeter is denoted by

$$\theta_E(x) := \lim_{r\downarrow 0} \frac{|B_r(x) \cap E|}{|B_r(x)|} \,,$$

where $B_r(x) \subset \mathbb{R}^n$ is the Euclidean ball with center $x$ and radius $r$.
In order to have notions of boundary, closure and interior for a set $F \subset \mathbb{R}^n$ with finite perimeter, which are invariant under $\mathcal{L}^n$ negligible changes, we define the measure theoretic boundary, closure and interior part respectively as

$$\partial F := \{x \in \mathbb{R}^n : \forall r > 0 \quad |F \cap B_r(x)| \notin \{0, |B_r(x)|\}\} \,,$$
$$\overline{F} := \{x \in \mathbb{R}^n : \forall r > 0 \quad |F \cap B_r(x)| \neq 0\} \,,$$
$$\mathring{F} := \{x \in \mathbb{R}^n : \exists r > 0 \quad |F \cap B_r(x)| = |B_r(x)|\} \,.$$

If $\partial^* F$ is sufficiently regular, $\mathrm{H}_{\partial^* F}(x)$ denotes the scalar mean curvature of $\partial^* F$ at $x \in \partial^* F$ (i.e. the sum of the principal curvatures of the surface at the point $x$).
Let $\Omega \subset \mathbb{R}^n$ be an open set and let $\mathrm{m} := (m_1, m_2)$, with $0 < m_1, m_2 < \infty$. Let also $E_1, E_2, E_3 \subset \Omega$ be three sets with finite perimeter. We say that $(E_1, E_2, E_3)$ belongs to $\mathcal{C}_{\Omega,\mathrm{m}}$ (the set of admissible test configurations) if $|E_i \cap E_j| = 0$ for all $i < j \in \{1, 2, 3\}$, $|E_1| = m_1$,

$|E_2| = m_2$ and $|\Omega \setminus (E_1 \cup E_2 \cup E_3)| = 0$.
For $i < j \in \{1, 2, 3\}$ consider $\sigma_{ij} > 0$ such that

$$\sigma_{13} > \sigma_{12} + \sigma_{23}\,. \tag{4}$$

On the set $\mathcal{C}_{\Omega,\mathrm{m}}$ we define the functional

$$\mathcal{F}_{\Omega,\mathrm{m}}(E_1, E_2, E_3) := \sum_{i<j\in\{1,2,3\}} \sigma_{ij}\mathcal{H}^{n-1}(\partial^* E_i \cap \partial^* E_j \cap \Omega)\,. \tag{5}$$

The set of triples $(F_1, F_2, F_3) \in \mathcal{C}_{\Omega,\mathrm{m}}$ minimizing $\mathcal{F}_{\Omega,\mathrm{m}}$ will be denoted by $\mathcal{M}(\mathcal{F}_{\Omega,\mathrm{m}})$.

We are interested at existence and regularity of minimizers of $\mathcal{F}_{\Omega,\mathrm{m}}$ on $\mathcal{C}_{\Omega,\mathrm{m}}$.

## 3. Existence of minimizers in foliated domains

We now define a class of domains on which the functional (5) admits a minimum for any choice of m.

**Definition 3.1.** Let $\Omega \subset \mathbb{R}^n$ be an open set with (possibly infinite) Lebesgue measure $|\Omega|$ and suppose that for any $0 < \rho < |\Omega|$ there exists a minimizer for the problem

$$\min\left\{\mathrm{P}(E;\Omega):\ E \subset \Omega,\ |E| = \rho\right\}. \tag{6}$$

We say that $\Omega$ is an *isoperimetrically foliated domain* if there exists a selection of minimizers $E^{(\rho)}$ of (6) such that

$$\rho_1 < \rho_2 \in (0, |\Omega|) \Longrightarrow \begin{cases} E^{(\rho_1)} \subset E^{(\rho_2)} \\ \mathcal{H}^{n-1}(\partial E^{(\rho_1)} \cap \partial E^{(\rho_2)} \cap \Omega) = 0 \end{cases} \tag{7}$$

*Remark* 3.2. By standard regularity theory (see [5]), $\partial E^{(\rho)}$ is $C^\infty$ away from a closed singular set of Hausdorff dimension at most $n - 8$.
We also observe that if $\Omega$ is an isoperimetrically foliated set as in Definition 3.1 and $|\Omega| < \infty$, then solutions to problem (6) which satisfy (7) foliate $\Omega$ with their boundaries, i.e.

$$|\Omega \setminus \bigcup_{0<\rho<|\Omega|} (\partial E^{(\rho)} \cap \Omega)| = 0\,. \tag{8}$$

To see this, consider a point $x \in \Omega$ which is not an element of $\partial E^{(\rho)} \cap \Omega$ for all $0 < \rho < |\Omega|$ and define $\rho_-(x) := \sup\{0 < \rho < |\Omega|, x \in \Omega \setminus E^{(\rho)}\}$, $\rho_+(x) := \inf\{0 < \rho < |\Omega|, x \in E^{(\rho)}\}$. Clearly it holds $\rho_-(x) \le \rho_+(x)$, but if $\rho_-(x) < \rho_+(x)$, we would have a contradiction. Thus we obtain $\rho_-(x) = \rho_+(x) =: \rho(x)$ and $x \in \partial^* E^{(\rho)}$. This is possible only if $\rho(x) = 0$ and consequently (8) follows.

*Remark* 3.3. Examples of isoperimetrically foliated domains are $\mathbb{R}^n$ itself, ellipses in $\mathbb{R}^2$ and $B^n$. Notice however that the unit square in $\mathbb{R}^2$ is not an isoperimetrically foliated domain, as for $\rho \in [0, \frac{1}{\pi}]$ the corresponding $E^{(\rho)}$ is a quarter of disk centred at any of the four vertices, while, for $\rho \in [\frac{1}{\pi}, 1 - \frac{1}{\pi}]$, $E^{(\rho)}$ is a vertical (or horizontal) stripe.

**Theorem 3.4.** *Let $\Omega \subset \mathbb{R}^n$ be an isoperimetrically foliated domain and let $m := (m_1, m_2)$, with $0 < m_1, m_2 < \infty$ and $m_1 + m_2 < |\Omega|$. Then $\mathcal{F}_{\Omega,m}$ attains its minimum in $\mathcal{C}_{\Omega,m}$.*

*Proof.* Let $(E_1, E_2, E_3) \in \mathcal{C}_{\Omega,\mathrm{m}}$. By means of (4) and (5), we obtain

$$\begin{aligned} \mathcal{F}_{\Omega,\mathrm{m}}(E_1,E_2,E_3) &\geq (\sigma_{12}+\sigma_{23})\mathcal{H}^{n-1}(\partial^* E_1 \cap \partial^* E_3 \cap \Omega) + \sigma_{12}\mathcal{H}^{n-1}(\partial^* E_1 \cap \partial^* E_2 \cap \Omega) \\ &\quad + \sigma_{23}\mathcal{H}^{n-1}(\partial^* E_2 \cap \partial^* E_3 \cap \Omega) \\ &= \sigma_{12}\mathrm{P}(E_1;\Omega) + \sigma_{23}\mathrm{P}(E_3;\Omega) \\ &\geq \sigma_{12}\mathrm{P}(E^{(m_1)};\Omega) + \sigma_{23}\mathrm{P}(E^{(m_1+m_2)};\Omega)\,. \end{aligned} \tag{9}$$

Since $\Omega$ is isoperimetrically foliated, the infimum of (5) is attained for the admissible choice $F_1 := E^{(m_1)}$, $F_2 := E^{(m_1+m_2)} \setminus E^{(m_1)}$ and $F_3 := \Omega \setminus E^{(m_1+m_2)}$. □

*Remark* 3.5. Thanks to the regularity of $\partial E^{(\rho)}$ (see Remark 3.2), the minimizing sets $F_1$, $F_2$, $F_3$ constructed in Theorem 3.4 have boundaries of class $C^\infty$ away from a closed singular set of Hausdorff dimension at most $n-8$.

*Remark* 3.6. It is easy to see that the existence of an isoperimetric foliation of $\Omega$ is sufficient but not necessary for the existence of minimizers of (5). Actually the existence of a minimizer is ensured if there exist minimizers $E^{(m_1)}$ and $E^{(m_1+m_2)}$ of (6) satisfying $E^{(m_1)} \subset E^{(m_1+m_2)}$ and $\mathcal{H}^{n-1}(\partial^* E^{(m_1)} \cap \partial^* E^{(m_1+m_2)}) = 0$. In view of Remark 3.3, if $\Omega = (0,1)\times(0,1) \subset \mathbb{R}^2$, $m_1 < \pi/16$, $m_3 < \pi/16$ and $m_2 = 1 - m_1 - m_3$, the minimum of (5) is attained for $F_1 = \{x^2+y^2 < 4m_1/\pi\} \cap \Omega$, $F_3 = \{(x-1)^2 + (y-1)^2 < 4m_2/\pi\} \cap \Omega$ and $F_2 = \Omega \setminus \overline{(F_1 \cup F_2)}$. Referring again to Remark 3.3, we notice that the same construction holds if $m_1, m_3 < 1/\pi$.

## 4. Regularity of minimizers in general domains

We state a result which can be found in [10] in a slightly different form, and whose proof is an easy modification of the one of Theorem 3.1 in [4].

**Theorem 4.1.** *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $m := (m_1, m_2)$, with $0 < m_1, m_2 < \infty$ and $m_1 + m_2 < |\Omega|$, let $(F_1, F_2, F_3) \in \mathcal{M}(\mathcal{F}_{\Omega,m})$ and let $R > 0$. Then there exist $\eta, r > 0$ such that, for all $x \in \Omega$ with $B_R(x) \subset \Omega$, $0 < \rho < r$ and $k \in \{1,3\}$ it holds*

$$|F_k \cap B_\rho(x)| < \eta\rho^n \quad \Rightarrow \quad |F_k \cap B_{\rho/2}(x)| = 0\,. \tag{10}$$

*Remark* 4.2. Theorem 4.1 have been proven in [4] under the hypothesis that the triangle inequality (2) holds in the strict sense. This is the reason for which Theorem 4.1 holds for $k \in \{1,3\}$, indeed in this cases (4) gives immediately $\sigma_{12} < \sigma_{23} + \sigma_{13}$ and $\sigma_{23} < \sigma_{13} + \sigma_{12}$.

We now prove a result on the structure of the minimizers of (5), which does not depend on the fact that the domain $\Omega$ is isoperimetrically foliated.

**Lemma 4.3.** *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $m := (m_1, m_2)$, with $0 < m_1, m_2 < \infty$ and $m_1 + m_2 < |\Omega|$. If $(F_1, F_2, F_3) \in \mathcal{M}(\mathcal{F}_{\Omega,m})$, then for every $x \in \partial^* F_2 \cap \partial^* F_1 \cap \Omega$ (resp. $x \in \partial^* F_2 \cap \partial^* F_3 \cap \Omega$) there exists $r > 0$ such that*

$$B_r(x) \cap F_3 = \emptyset \qquad (\textit{resp. } B_r(x) \cap F_1 = \emptyset)\,. \tag{11}$$

*Proof.* We consider $x \in \partial^* F_2 \cap \partial^* F_1 \cap \Omega$, since the other case follows by the same argument. By a well known property of sets of finite perimeter it follows that both $F_1$ and $F_2$ have density $1/2$ at $x$. Thus, for a sufficiently small $r_0 > 0$ it holds $|F_3 \cap B_{r_0}(x)| < \eta r_0^n$ and Theorem 4.1 ensures that $|F_3 \cap B_{r_0/2}(x)| = 0$. Consequently (11) holds with $0 < r \leq r_0/2$. □

**Proposition 4.4.** *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $m := (m_1, m_2)$, with $0 < m_1, m_2 < \infty$ and $m_1 + m_2 < |\Omega|$, let $(F_1, F_2, F_3) \in \mathcal{M}(\mathcal{F}_{\Omega,m})$, and let $R > 0$. Then there exist $\gamma \in (0,1)$ and $r > 0$ such that, for every $x \in \partial F_k$ with $B_R(x) \subset \Omega$, $0 < \rho < r$ and $k \in \{1,3\}$, it holds*

$$\gamma \leq \frac{|F_k \cap B_\rho(x)|}{\omega_n \rho^n} \leq 1 - \gamma \,. \tag{12}$$

*Proof.* Without loss of generality we can assume $k = 1$. The inequality on the left-hand side of (12) follows immediately from (10) applied to $\partial F_1$ (resp. $\partial F_3$) and it follows that $\eta \leq \gamma$. We claim that the inequality on the right-hand side of (12) holds for a $\gamma \geq \eta$ and we consider the case of $x \in \partial F_1$, being the other case identical. Suppose that there exists $x \in \partial F_1$, such that $\frac{|F_1 \cap B_r(x)|}{\omega_n r^n} > 1 - \eta$. This implies that $\frac{|F_3 \cap B_r(x)|}{\omega_n r^n} < \eta$ and consequently (by (10)) that $|F_3 \cap B_{r/2}(x)| = 0$. Thus we conclude that $x \in \partial F_1 \cap \partial F_2$ and the standard regularity results for minimizing boundaries with fixed volume apply to give the desired estimate. □

From (12) and the relative isoperimetric inequality in the ball $B_\rho(x)$, we obtain the following lower bound for the perimeter of minimizers.

**Proposition 4.5.** *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $m := (m_1, m_2)$, with $0 < m_1, m_2 < \infty$ and $m_1 + m_2 < |\Omega|$, let $(F_1, F_2, F_3) \in \mathcal{M}(\mathcal{F}_{\Omega,m})$, and let $R > 0$. Then, there exist $\theta, r > 0$ such that, for all $x \in \partial F_1$ (resp. $x \in \partial F_3$) with $B_R(x) \subset \Omega$ and $0 < \rho < r$, it holds*

$$P(F_1, B_\rho(x)) \geq \theta \rho^{n-1} \,. \tag{13}$$

**Corollary 4.6.** *If $(F_1, F_2, F_3) \in \mathcal{M}(\mathcal{F}_{\Omega,m})$, it holds*

$$\mathcal{H}^{n-1}((\partial F_1 \cap \Omega) \setminus (\partial^* F_1 \cap \Omega)) = 0 \qquad \textit{and} \qquad \mathcal{H}^{n-1}((\partial F_3 \cap \Omega) \setminus (\partial^* F_3 \cap \Omega)) = 0 \,. \tag{14}$$

*Proof.* We prove the claim just for $F_1$, since the proof for $F_3$ is identical. For any Borel set $B \subset \partial F_1$, by (13), we have

$$|\nabla \chi_{F_1}|(B) \geq \frac{\theta}{\omega_{n-1}} \mathcal{H}^{n-1}(B) \,,$$

where $|\nabla \chi_{F_1}|$ is the total variation measure associated to $\chi_{F_1}$. With the choice $B := \partial F_1 \setminus (\partial^* F_1 \cap \Omega)$, since $|\nabla \chi_{F_1}|$ is concentrated on $\partial^* F_1$, the thesis follows. □

**Proposition 4.7.** *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $m := (m_1, m_2)$, with $0 < m_1, m_2 < \infty$ and $m_1 + m_2 < |\Omega|$. For any $(F_1, F_2, F_3) \in \mathcal{M}(\mathcal{F}_{\Omega,m})$ it holds*

$$\mathcal{H}^{n-1}(\partial^* F_1 \cap \partial^* F_3 \cap \Omega) = 0 \,. \tag{15}$$

*Proof.* Suppose by contradiction that $\mathcal{H}^{n-1}(\partial^* F_1 \cap \partial^* F_3 \cap \Omega) > 0$. Thanks to (14), for any $\varepsilon > 0$ small enough, we can approximate $F_1$ from inside with smooth sets $F_1^\varepsilon \subset F_1$ as in [8], in such a way that $\mathcal{H}^{n-1}(\partial^* F_1 \cap \partial^* F_1^\varepsilon \cap \Omega) = 0$ and

$$|P(F_1, \Omega) - P(F_1^\varepsilon, \Omega)| < \varepsilon \,, \qquad |F_1| - |F_1^\varepsilon| < \varepsilon \,. \tag{16}$$

We now define $F_3^\varepsilon := F_3$, $F_2^\varepsilon := \Omega \setminus (\overline{F_1^\varepsilon} \cup \overline{F_3^\varepsilon})$. In order to restore the prescribed values for the volumes of $F_1^\varepsilon$ and $F_2^\varepsilon$, we consider a point $x \in \mathbb{R}^n$ with $\theta_{F_2}(x) = 1$ (which exists, since $|F_2| > 0$). By Theorem 4.1, there exists $r \in (0,1)$ such that $|B_r(x) \cap F_2| = |B_r(x)|$. We take $\varepsilon > 0$ small enough, so that $\varepsilon < |B_r(x)|$ and we define $\tilde{F}_1^\varepsilon := F_1^\varepsilon \cup B_{r'}(x)$ (with

$|B_{r'}(x)| = |F_1| - |F_1^\varepsilon|$), $\tilde F_2^\varepsilon := F_2^\varepsilon \setminus B_{r'}(x)$ and $\tilde F_3^\varepsilon := F_3^\varepsilon$. Notice that we have

$$\begin{aligned}
\mathcal{H}^{n-1}(\partial^* \tilde F_1^\varepsilon \cap \partial^* \tilde F_3^\varepsilon \cap \Omega) &= 0\,,\\
\mathcal{H}^{n-1}(\partial^* \tilde F_2^\varepsilon \cap \partial^* \tilde F_3^\varepsilon \cap \Omega) &= \mathcal{H}^{n-1}(\partial^* F_3 \cap \Omega)\\
&= \mathcal{H}^{n-1}(\partial^* F_1 \cap \partial^* F_3 \cap \Omega) + \mathcal{H}^{n-1}(\partial^* F_2 \cap \partial^* F_3 \cap \Omega)\,,\\
\mathcal{H}^{n-1}(\partial^* \tilde F_1^\varepsilon \cap \partial^* \tilde F_2^\varepsilon \cap \Omega) &= \mathcal{H}^{n-1}(\partial^* F_1^\varepsilon \cap \Omega) + P(B_{r'}(x))\\
&\le \mathcal{H}^{n-1}(\partial^* F_1 \cap \partial^* F_2 \cap \Omega) + \mathcal{H}^{n-1}(\partial^* F_1 \cap \partial^* F_3 \cap \Omega) + \varepsilon + n\omega_n^{\frac{1}{n}} \varepsilon^{\frac{n-1}{n}}\,.
\end{aligned}$$

Since we have assumed $\mathcal{H}^{n-1}(\partial^* F_1 \cap \partial^* F_3 \cap \Omega) > 0$, taking into account (16) and (4), for $\varepsilon$ small enough we conclude that

$$\begin{aligned}
\mathcal{F}_{\Omega,\mathrm{m}}(\tilde F_1^\varepsilon, \tilde F_2^\varepsilon, \tilde F_3^\varepsilon) &\le \big(\mathcal{H}^{n-1}(\partial^* F_1 \cap \partial^* F_3 \cap \Omega) + \mathcal{H}^{n-1}(\partial^* F_2 \cap \partial^* F_3 \cap \Omega)\big)\,\sigma_{23}\\
&+ \big(\mathcal{H}^{n-1}(\partial^* F_1 \cap \partial^* F_2 \cap \Omega) + \mathcal{H}^{n-1}(\partial^* F_1 \cap \partial^* F_3 \cap \Omega)\big)\,\sigma_{12}\\
&+ (\varepsilon + n\omega_n^{\frac{1}{n}} \varepsilon^{\frac{n-1}{n}})\sigma_{12}\\
&< \mathcal{F}_{\Omega,\mathrm{m}}(F_1, F_2, F_3)\,,
\end{aligned}$$

which contradicts the minimality of $(F_1, F_2, F_3)$. □

**Theorem 4.8.** *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $m := (m_1, m_2)$, with $0 < m_1, m_2 < \infty$ and $m_1 + m_2 < |\Omega|$. If $(F_1, F_2, F_3) \in \mathcal{M}(\mathcal{F}_{\Omega,m})$, then for any $i \in \{1, 2, 3\}$ the boundary $\partial F_i \cap \Omega$ is of class $C^\infty$ out of a closed singular set with zero $\mathcal{H}^{n-1}$ measure.*

*Proof.* By (14) and Proposition 4.7 we have that $\mathcal{H}^{n-1}-$almost every $x \in \partial F_1 \cap \Omega$ is an element of $\partial^* F_1 \cap \partial^* F_2 \cap \Omega$. Thus, using Lemma 4.3, there exist $r > 0$ such that $B_r(x) \cap F_3 = \emptyset$ (with $B_r(x) \subset\subset \Omega$) and, by standard regularity theory for minimizing boundaries with prescribed volume (see [5]), we can conclude that $B_r(x) \cap \partial F_1 = B_r(x) \cap \partial F_2$ is $C^\infty$ on the complement of a closed set of Hausdorff dimension smaller or equal to $n - 8$. In particular, $\partial F_1$ is $C^\infty$ on the complement of a closed set with zero $\mathcal{H}^{n-1}$ measure.
The same argument holds for the set $F_3$, and consequently $\partial F_3$ is $C^\infty$ on the complement of a closed set with zero $\mathcal{H}^{n-1}$ measure.
In particular, the sets $\partial F_2 \cap \partial F_1 \cap \Omega$ and $\partial F_2 \cap \partial F_3 \cap \Omega$ are $C^\infty$ on the complement of a closed set with zero $\mathcal{H}^{n-1}$ measure. This implies that also $\partial F_2 \cap \Omega$ is $C^\infty$ on the complement of a closed set with zero $\mathcal{H}^{n-1}$ measure. □

*Remark* 4.9. If we consider $\Omega := \mathbb{R}^2$ and $0 < m_1, m_2 < \infty$, it is easy to see that $\mathcal{M}(\mathcal{F}_{\Omega,\mathrm{m}})$ is the set of the triples $(F_1, F_2, F_3)$, where $F_1$ is a metric ball with $|F_1| = m_1$, $F_2 := B^{(m_1+m_2)} \setminus F_1$, where $B^{(m_1+m_2)}$ is a metric ball of mass $m_1 + m_2$ which contains $F_1$, and $F^3 := \mathbb{R}^2 \setminus B^{(m_1+m_2)}$. One of the possible minimizing configurations is realized when $F_1$ is tangent at a point (from the inside) to $F_2$. The point of contact between the two sets is not a point of $\partial^* F_2$ and this shows that even one dimensional minimizing boundaries for (5) are not necessarily everywhere regular.

## 5. Nonexistence of minimizers in general domains

In this final section we show that on domains $\Omega$ which do not satisfy Definition 3.1, there are choices of m for which the infimum of $\mathcal{F}_{\Omega,\mathrm{m}}$ is not attained.

**Proposition 5.1.** *Let* $\Omega = (0,1) \times (0,1) \subset \mathbb{R}^2$, $\sigma_{12} = \sigma_{23} = 1$, $\sigma_{13} > 2$, $m_1 = \frac{\pi(1+\varepsilon^2)}{16}$ *(for* $\varepsilon > 0$ *small enough),* $m_2 = 1/2 - \frac{\pi(1+\varepsilon^2)}{16}$ *and* $m_3 = 1/2$*. The functional* $\mathcal{F}_{\Omega,m}$ *has no minimum on* $\mathcal{C}_{\Omega,m}$*.*

*Proof.* We argue by contradiction. If $\mathcal{F}_{\Omega,\mathrm{m}}$ would attain its minimum at a triple $(F_1, F_2, F_3)$, by Proposition 4.7, we would have that $\mathcal{H}^{n-1}(\partial^* F_1 \cap \partial^* F_3 \cap \Omega) = 0$ and, by standard regularity, the interfaces $\partial^* F_1 \cap \partial^* F_2 \cap \Omega$ and $\partial^* F_2 \cap \partial^* F_3 \cap \Omega$ would be either straight segments or circular arcs meeting $\partial\Omega$ orthogonally.

If both $\partial^* F_1 \cap \partial^* F_2 \cap \Omega$ and $\partial^* F_2 \cap \partial^* F_3 \cap \Omega$ were segments, the value of the minimum of (5) would be 2. This would contradict the minimality of $(F_1, F_2, F_3)$, since

$$\mathcal{F}_{\Omega,\mathrm{m}}(F_1, F_2, F_3) = 2 > 1 + \frac{\sqrt{\pi(1+\varepsilon)}}{4} = \mathcal{F}_{\Omega,\mathrm{m}}(E_1, E_2, E_3)\,,$$

where

$$\begin{aligned} E_1 &= (\{x^2 + y^2 < 1/2\} \cup \{x^2 + (y-1)^2 = \varepsilon/2\}) \cap \Omega \\ E_3 &= \Omega \cap \{(x,y) \in \mathbb{R}^2, x > 1/2\} \\ E_2 &= \Omega \setminus \overline{(F_1 \cup F_3)}\,, \end{aligned}$$

for $\varepsilon > 0$ sufficiently small.

If $F_1$ and $F_3$ were quarter of disks centred at different consecutive corners of $\Omega$, one would easily see that $F_1$ and $F_3$ would overlap and consequently they could not be element of a triple in $\mathcal{C}_{\Omega,\mathrm{m}}$. On the other hand, if $F_1$ and $F_3$ would be quarter of disks with centers at opposite corners of $\Omega$ and one easily checks that the value of $\mathcal{F}_{\Omega,\mathrm{m}}$ taken at this configuration can not be a minimum. If we set $F_1 = \{x^2 + y^2 < \frac{(1+\varepsilon)^2}{4}\}$, $F_3 = \Omega \setminus \{x^2 + y^2 < \frac{2}{\pi}\}$ and $F_2 = \Omega \setminus \overline{(F_1 \cup F_3)}$, we still have $\mathcal{F}_{\Omega,\mathrm{m}}(F_1, F_2, F_3) = \sqrt{\frac{\pi}{2}} + \frac{\pi(1+\varepsilon)}{4} > 2$. As a consequence, $\mathcal{F}_{\Omega,\mathrm{m}}$ has no minimum on $\mathcal{C}_{\Omega,\mathrm{m}}$. □

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