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A SYMMETRY RESULT FOR DEGENERATE ELLIPTIC EQUATIONS ON THE WIENER SPACE WITH NONLINEAR BOUNDARY CONDITIONS AND APPLICATIONS

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Abstract. The purpose of this paper is to study a boundary reaction problem on the space $X\times\mathbb{R}$, where X is an abstract Wiener space. We prove that smooth bounded solutions enjoy a symmetry property, i.e., are one-dimensional in a suitable sense. As a corollary of our result, we obtain a symmetry property for some solutions of the following equation

$$(-\Delta_\gamma)^s u = f(u),$$

with $s \in (0,1)$, where $(-\Delta_\gamma)^s$ denotes a fractional power of the Ornstein-Uhlenbeck operator, and we prove that for any $s \in (0,1)$ monotone solutions are one-dimensional.

Contents

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1. **Introduction.** The main purpose of the present work is to investigate the following boundary reaction problem

$$\begin{cases} \operatorname{div}_{\gamma,y}(\mu(y)\nabla_{\gamma,y} v(x,y)) = 0 & \text{on } (x,y) \in X \times \mathbb{R}^+ \\ -\lim\limits_{y\to 0^+} \mu(y)\partial_y v(x,y) = f(v) & \text{on } X \end{cases} \tag{1}$$

where X is an abstract Wiener space and $\mu : \mathbb{R}^+ := (0,+\infty) \to \mathbb{R}^+$ is a degenerate weight. In the previous equation, $\operatorname{div}_{\gamma,y}$ and $\nabla_{\gamma,y}$ stand for the divergence and gradient operators in $X \times \mathbb{R}^+$ (see below).

The degeneracy is given in terms of A_2 classes (see [17]) on $\mathbb{R}^+$, i.e., the function μ satisfies the inequality: there exists $\kappa > 0$ such that for any $a, b > 0$

$$\int_a^b \mu(y)\,dy \int_a^b \mu^{-1}(y)\,dy \le \kappa(b-a)^2.$$

We investigate symmetry properties of bounded, smooth solutions of (1) satisfying a monotonicity assumption.

As a corollary of our main result, we get a symmetry property for some solutions of the equation

$$(-\Delta_\gamma)^s u = f(u) \qquad \text{on } X. \tag{2}$$

Here $(-\Delta_\gamma)^s$ denotes a fractional power of the laplacian $-\Delta_\gamma$ in the infinite-dimensional Wiener space (X,γ,H), $s \in (0,1)$ and in this application $\mu(y) = y^{1-2s}$. In the local case $s = 1$, such an equation has been investigated in [10]. In the present work, we investigate the non local case $s \in (0,1)$. Properties of entire smooth solutions of (2) will be investigated thanks to problem (1), realising the operator $(-\Delta_\gamma)^s$ as the boundary operator of a suitable differential extension in $X \times \mathbb{R}^+$.

Owing to the well-known relation between the Bernstein problem and the symmetry properties of solutions of Allen-Cahn type equation, we prove the one-dimensional symmetry of monotone solutions to (2). This is in the spirit of other symmetry results obtained in connection to a conjecture by De Giorgi on the flatness of level sets of entire solutions of the Allen-Cahn equation in the Euclidean space [11], which has motivated among others the works [1, 2, 15, 6, 12, 16, 19, 9].

Our main result is the following theorem. We refer to section 2 for the notation.

Theorem 1.1. *Let $v \in C^1(X \times \mathbb{R}^+) \cap L^\infty(X \times \mathbb{R}^+)$ satisfy (1), where $f : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz function. Assume that*

$$\partial_i \partial_j v \in C(X \times \mathbb{R}^+) \qquad \text{for all } i, j \in \mathbb{N}$$

and for any $y > 0$

$$\inf_{x \in B_R} [\nabla_\gamma v(x,y), w]_H > 0$$

for all $R > 0$ and for some $w \in H$.

Then, v is one-dimensional in x, in the sense that there exist $V : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ and $\omega \in X^*$ such that

$$v(x,y) = V(\langle x, \omega \rangle, y) \qquad \textit{for all } x \in X, \quad y > 0.$$

The regularity assumptions on v are necessary to justify the computations. In the finite-dimensional case, they are satisfied assuming only that the weight μ is uniformly positive away from $y = 0$. However, in the infinite-dimensional setting we do not know if these conditions are met in general. It would require to develop as a first step a theory of degenerate elliptic operators like in [13], which is far from being understood at the moment.

We now specify Theorem 1.1 in the setting of the Gauss space $\mathbb{R}^N$ endowed with the standard Gaussian measure γ_N whenever the weight μ is uniformly positive away from $y = 0$. This is actually the case we are interested in, since this is the one occuring for the fractional laplacian. In the finite dimensional setting, one can sharpen the regularity assumptions and one can prove the following result.

Theorem 1.2. *Consider the following equation on $\mathbb{R}^N \times \mathbb{R}^+$*

$$\begin{cases} \operatorname{div}(\gamma_N(x)\mu(y)\nabla v(x,y)) = 0 & \text{on } (x,y) \in \mathbb{R}^N \times \mathbb{R}^+ \\ \\ -\lim_{y\to 0^+} \mu(y)\partial_y v(x,y) = f(v) & \text{on } \mathbb{R}^N \end{cases} \tag{3}$$

Let $v \in L^\infty(\mathbb{R}^N \times \mathbb{R}^+)$ satisfy (3), where $f : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz function. Assume that

$$[\nabla_x v(x,y), w]_{\mathbb{R}^N} > 0$$

for all $x \in \mathbb{R}^N$, $y > 0$ and for some $w \in \mathbb{S}^{N-1}$ and that the function μ is A_2 in $\mathbb{R}^N \times \mathbb{R}^+$, bounded below from 0 on any interval $[a, +\infty)$ for $a > 0$.

Then, v is one-dimensional in x, in the sense that there exist $V : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ and $\omega \in \mathbb{S}^{N-1}$ such that

$$v(x,y) = V(\langle x, \omega\rangle, y) \qquad \textit{for all } x \in X, \quad y > 0.$$

Remark 1. If we do not assume that the function μ is uniformly positive in half-spaces but that it is just A_2, the results in [13] just provide that v is locally Hölder continuous in $\mathbb{R}^N \times \mathbb{R}^+$. However, any derivative v_i of v satisfies a non divergence-form PDE with A_2 weights and the regularity theory for these PDEs is not known.

Theorem 1.1 also admits the following consequence.

Theorem 1.3. *Let $u \in L^\infty(X) \cap C(X)$ be a weak solution of (2), where $f : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz function. Assume that*

$$\inf_{x \in B_R} [\nabla_\gamma u(x), w]_H > 0$$

for all $R > 0$ and for some $w \in H$. Furthermore, assume that its extension v to $X \times \mathbb{R}^+$ defined by

$$v = \inf\Big\{ \int_{X\times\mathbb{R}^+} y^{1-2s} |\nabla_{\gamma,y} w|^2 d\gamma\, dy, w \in H^1(X \times \mathbb{R}^+, \gamma \otimes y^{1-2s}\, dy),\ w(x,0) = u \Big\}$$

satisfies the assumptions of Theorem 1.1. Then, u is one-dimensional , in the sense that there exist $U : \mathbb{R} \to \mathbb{R}$ and $\omega \in X^*$ such that

$$u(x) = U(\langle x, \omega\rangle) \qquad \textit{for all } x \in X.$$

By the construction of the extension v of u (see below), the function v satisfies (1) weakly. The proof of Theorem 1.3 indeed is based on the extension technique introduced in [8], later extended in [24] in a rather general abstract context. An analogue of Theorem 1.3 has been obtained in [21] in the classical Euclidean case in dimension 2. In dimension 3, the only available result is in [7] for $s \in [1/2, 1)$.

In the last section of the paper, we give a sufficient condition for the existence of a nontrivial monotone solution of (2), which is one-dimensional by Theorem 1.3.

2. **Notation and preliminary results.**

We denote by $(\mathbb{R}^N, \gamma_N)$ the N-dimensional Gauss space, where γ_N is the standard gaussian measure on $\mathbb{R}^N$ (with a slight abuse of notation we denote by γ_N both the density and the measure) defined as $d\gamma_N(x) = (2\pi)^{-N/2} \exp\{-|x|^2/2\}\, dx$.

2.1. **The Wiener space.** An abstract Wiener space is defined as a triple (X, γ, H) where X is a separable Banach space, endowed with the norm $\|\cdot\|_X$, γ is a nondegenerate centred Gaussian measure, and H is the Cameron–Martin space associated with the measure γ, that is, H is a separable Hilbert space densely embedded in X, endowed with the inner product $[\cdot,\cdot]_H$ and with the norm $|\cdot|_H$. The requirement that γ is a centred Gaussian measure means that for any $x^* \in X^*$, the measure $x^*_\#\gamma$ is a centred Gaussian measure on the real line $\mathbb{R}$, that is, the Fourier transform of γ is given by

$$\hat{\gamma}(x^*) = \int_X e^{-i\langle x, x^*\rangle}\, d\gamma(x) = \exp\left(-\frac{\langle Qx^*, x^*\rangle}{2}\right), \qquad \forall x^* \in X^*;$$

here the operator $Q \in \mathcal{L}(X^*, X)$ is the covariance operator and it is uniquely determined by the formula

$$\langle Qx^*, y^*\rangle = \int_X \langle x, x^*\rangle\langle x, y^*\rangle d\gamma(x), \qquad \forall x^*, y^* \in X^*.$$

The nondegeneracy of γ implies that Q is positive definite: the boundedness of Q follows by Fernique's Theorem (see for instance [4, Theorem 2.8.5]), asserting that there exists a positive number $\beta > 0$ such that

$$\int_X e^{\beta\|x\|^2} d\gamma(x) < +\infty.$$

This implies also that the maps $x \mapsto \langle x, x^*\rangle$ belong to $L^p_\gamma(X)$ for any $x^* \in X^*$ and $p \in [1, +\infty)$, where $L^p_\gamma(X)$ denotes the space of all γ-measurable functions $f : X \to \mathbb{R}$ such that

$$\int_X |f(x)|^p d\gamma(x) < +\infty.$$

In particular, any element $x^* \in X^*$ can be seen as a map $x^* \in L^2_\gamma(X)$, and we denote by $R^* : X^* \to \mathcal{H}$ the identification map $R^*x^*(x) := \langle x, x^*\rangle$. The space $\mathcal{H}$ given by the closure of R^*X^* in $L^2_\gamma(X)$ is usually called reproducing kernel. By considering the map $R : \mathcal{H} \to X$ defined as

$$R\hat{h} := \int_X \hat{h}(x) x\, d\gamma(x),$$

we obtain that R is an injective γ–Radonifying operator, which is Hilbert–Schmidt when X is Hilbert. We also have $Q = RR^* : X^* \to X$. The space $H := R\mathcal{H}$, equipped with the inner product $[\cdot,\cdot]_H$ and norm $|\cdot|_H$ induced by $\mathcal{H}$ via R, is the Cameron-Martin space and is a dense subspace of X. The continuity of R implies that the embedding of H in X is continuous, that is, there exists $c > 0$ such that

$$\|h\|_X \le c|h|_H, \qquad \forall h \in H.$$

We have also that the measure γ is absolutely continuous with respect to translation along Cameron–Martin directions; in fact, for $h \in H$, $h = Qx^*$, the measure $\gamma_h(B) = \gamma(B - h)$ is absolutely continuous with respect to γ with density given by

$$d\gamma_h(x) = \exp\left(\langle x, x^*\rangle - \frac{1}{2}|h|^2_H\right) d\gamma(x).$$

2.2. **Cylindrical functions and differential operators.** For $j \in \mathbb{N}$ we choose $x_j^* \in X^*$ in such a way that $\hat{h}_j := R^* x_j^*$, or equivalently $h_j := R\hat{h}_j = Qx_j^*$, form an orthonormal basis of H. We order the vectors x_j^* in such a way that the numbers $\lambda_j := \|x_j^*\|_{X^*}^{-2}$ form a non-increasing sequence. Given $m \in \mathbb{N}$, we also let $H_m := \langle h_1, \dots, h_m \rangle \subseteq H$, and $\Pi_m : X \to H_m$ be the closure of the orthogonal projection from H to H_m

$$\Pi_m(x) := \sum_{j=1}^{m} \left\langle x, x_j^* \right\rangle h_j \qquad x \in X.$$

The map Π_m induces the decomposition $X \simeq H_m \oplus X_m^\perp$, with $X_m^\perp := \ker(\Pi_m)$, and $\gamma = \gamma_m \otimes \gamma_m^\perp$, with γ_m and $\gamma_m^\perp$ Gaussian measures on H_m and $X_m^\perp$ respectively, having H_m and $H_m^\perp$ as Cameron–Martin spaces. When no confusion is possible we identify H_m with $\mathbb{R}^m$; with this identification the measure $\gamma_m = \Pi_{m\#}\gamma$ is the standard Gaussian measure on $\mathbb{R}^m$ (see [4]). Given $x \in X$, we denote by $\underline{x}_m \in H_m$ the projection $\Pi_m(x)$, and by $\overline{x}_m \in X_m^\perp$ the infinite dimensional component of x, so that $x = \underline{x}_m + \overline{x}_m$. When we identify H_m with $\mathbb{R}^m$ we rather write $x = (\underline{x}_m, \overline{x}_m) \in \mathbb{R}^m \times X_m^\perp$.

We say that $u : X \to \mathbb{R}$ is a *cylindrical function* if $u(x) = v(\Pi_m(x))$ for some $m \in \mathbb{N}$ and $v : \mathbb{R}^m \to \mathbb{R}$. We denote by $\mathcal{F}C_b^k(X)$, $k \in \mathbb{N}$, the space of all C_b^k cylindrical functions, that is, functions of the form $v(\Pi_m(x))$ with $v \in C^k(\mathbb{R}^n)$, with continuous and bounded derivatives up to the order k. We denote by $\mathcal{F}C_b^k(X, H)$ the space generated by all functions of the form uh, with $u \in \mathcal{F}C_b^k(X)$ and $h \in H$.

Given $u \in L^2_\gamma(X)$, we consider the canonical cylindrical approximation $\mathbb{E}_m$ given by

$$\mathbb{E}_m u(x) = \int_{X_m^\perp} u(\Pi_m(x), y)\, d\gamma_m^\perp(y). \tag{4}$$

Notice that $\mathbb{E}_m u$ depends only on the first m variables and $\mathbb{E}_m u$ converges to u in $L^p_\gamma(X)$ for all $1 \le p < \infty$.

We let

$$\nabla_\gamma u := \sum_{j\in\mathbb{N}} \partial_j u \, h_j \qquad \text{for } u \in \mathcal{F}C_b^1(X)$$

$$\mathrm{div}_\gamma \varphi := \sum_{j\ge 1} \partial_j^* [\varphi, h_j]_H \quad \text{for } \varphi \in \mathcal{F}C_b^1(X, H)$$

$$\Delta_\gamma u := \mathrm{div}_\gamma \nabla_\gamma u \qquad \text{for } u \in \mathcal{F}C_b^2(X)$$

where $\partial_j := \partial_{h_j}$ and $\partial_j^* := \partial_j - \hat{h}_j$ is the adjoint operator of ∂_j. With this notation, the following integration by parts formula holds:

$$\int_X u \,\mathrm{div}_\gamma \varphi \, d\gamma = -\int_X [\nabla_\gamma u, \varphi]_H \, d\gamma \qquad \forall \varphi \in \mathcal{F}C_b^1(X, H). \tag{5}$$

In particular, thanks to (5), the operator ∇_γ is closable in $L^p_\gamma(X)$, and we denote by $W^{1,p}_\gamma(X)$ the domain of its closure. The Sobolev spaces $W^{k,p}_\gamma(X)$, with $k \in \mathbb{N}$ and $p \in [1, +\infty]$, can be defined analogously [4], and $\mathcal{F}C_b^k(X)$ is dense in $W^{j,p}_\gamma(X)$, for all $p < +\infty$ and $k, j \in \mathbb{N}$ with $k \ge j$.

Given a vector field $\varphi \in L^p_\gamma(X; H)$, $p \in (1, \infty]$, using (5) we can define $\mathrm{div}_\gamma \varphi$ in the distributional sense, taking test functions u in $W^{1,q}_\gamma(X)$ with $\frac{1}{p} + \frac{1}{q} = 1$. We say that $\operatorname{div}_\gamma \varphi \in L^p_\gamma(X)$ if this linear functional can be extended to all test functions $u \in L^q_\gamma(X)$. This is true in particular if $\varphi \in W^{1,p}_\gamma(X; H)$.

Let $u \in W^{2,2}_\gamma(X)$, $\psi \in \mathcal{F}C^1_b(X)$ and $i, j \in \mathbb{N}$. From (5), with $u = \partial_j u$ and $\varphi = \psi h_i$, we get

$$\int_X \partial_j u\, \partial_i \psi \, d\gamma = \int_X -\partial_i(\partial_j u)\, \psi + \partial_j u\, \psi \langle x, x_i^* \rangle d\gamma \tag{6}$$

Let now $\varphi \in \mathcal{F}C^1_b(X, H)$. If we apply (6) with $\psi = [\varphi, h_j]_H =: \varphi^j$, we obtain

$$\int_X \partial_j u\, \partial_i \varphi^j \, d\gamma = \int_X -\partial_j(\partial_i u)\, \varphi^j + \partial_j u\, \varphi^j \langle x, x_i^* \rangle d\gamma$$

which, summing up in j, gives

$$\int_X [\nabla_\gamma u, \partial_i \varphi]_H \, d\gamma = \int_X -[\nabla_\gamma(\partial_i u), \varphi]_H + [\nabla_\gamma u, \varphi]_H \langle x, x_i^* \rangle d\gamma$$

for all $\varphi \in \mathcal{F}C^1_b(X, H)$.

The operator $\Delta_\gamma : W^{2,p}_\gamma(X) \to L^p_\gamma(X)$ is usually called the Ornstein-Uhlenbeck operator on X. Notice that, if u is a cylindrical function, that is $u(x) = v(y)$ with $y = \Pi_m(x) \in \mathbb{R}^m$ and $m \in \mathbb{N}$, then

$$\Delta_\gamma u = \sum_{j=1}^m \partial_{jj} u - \langle x, x_j^* \rangle \partial_j u = \Delta v - \langle \nabla v, y \rangle_{\mathbb{R}^m} .$$

We write $u \in C(X)$ if $u : X \to \mathbb{R}$ is continuous and $u \in C^1(X)$ if both $u : X \to \mathbb{R}$ and $\nabla_\gamma u : X \to H$ are continuous.

For simplicity of notation, from now on we omit the explicit dependence on γ of operators and spaces. We also indicate by $[\cdot, \cdot]$ and $|\cdot|$ respectively the inner product and the norm in H.

2.3. **Fractional Sobolev spaces and extension properties.** Since the operator $-\Delta_\gamma$ is positive and self-adjoint in $L^2_\gamma(X)$, one can define its fractional powers by means of the standard formula in spectral theory (see e.g. [25, §IX.11])

$$(-\Delta_\gamma)^s = \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{t\Delta_\gamma} - \mathrm{Id}\right) \frac{dt}{t^{1+s}},$$

where $s \in (0, 1)$ and $e^{t\Delta_\gamma}$ denotes the Ornstein-Uhlenbeck semigroup on X.

It is by now classical from non local PDEs involving the fractional laplacian to use the so-called Caffarelli-Silvestre extension (see [8]) to deal with these operators. Here we use a general formulation, due to Stinga and Torrea (see [24]). More precisely, a consequence of their main result is the following:

Theorem 2.1. *Let $u \in dom((-\Delta_\gamma)^s)$. A solution of the extension problem*

$$\begin{cases} \Delta_\gamma v + \dfrac{1-2s}{y} \partial_y v + \partial_y^2 v = 0 & \text{on } X \times \mathbb{R}^+ \\[2ex] v(x, 0) = u & \text{on } X, \end{cases}$$

is given by

$$v(x, y) = \frac{1}{\Gamma(s)} \int_0^\infty e^{t\Delta_\gamma}((-\Delta_\gamma)^s u)(x) e^{-y^2/4t} \frac{dt}{t^{1-s}}$$

and furthermore, one has at least in the distributional sense

$$- \lim_{y\to 0^+} y^{1-2s}\partial_y v(x,y) = \frac{2s\Gamma(-s)}{4^s\Gamma(s)}(-\Delta_\gamma)^s u(x). \tag{7}$$

Proof. We just sketch the proof since it is basically contained in [24]. Let $\{\phi_k\}_k$ be a basis of $L^2_\gamma(X)$ given by eigenfunctions of the Ornstein-Uhlenbeck operator (see e.g. [20]). Then any u in $L^2_\gamma(X)$ writes

$$u = \sum_k u_k\phi_k$$

and one has the spectral representation for any u in the domain of $(-\Delta_\gamma)^s$

$$(-\Delta_\gamma)^s u = \sum_k \lambda_k^s u_k\phi_k$$

where $\lambda_k \downarrow 0$ is the sequence of eigenvalues of $-\Delta_\gamma$. Consequently, we have

$$-\Delta_\gamma\phi_k = \lambda_k\phi_k$$

and the heat kernel writes

$$e^{t\Delta_\gamma}u = \sum_k e^{-\lambda_k t}u_k\phi_k.$$

One then checks easily that the formula for v makes sense. With this at hand, the same computations as in [24, Theorem 1.1] work and one gets the desired result. □

By means of Theorem 2.1, one can reformulate equation (2) into the following boundary value problem

$$\begin{cases} \Delta_\gamma v + \dfrac{1-2s}{y}\partial_y v + \partial_y^2 v = 0 & \text{on } X\times\mathbb{R}^+ \\ -\lim\limits_{y\to 0^+} y^{1-2s}\partial_y v(x,y) = c_s f(u) & \text{on } X, \\ v(x,0) = u & \text{on } X, \end{cases} \tag{8}$$

where $c_s > 0$ is the constant appearing in (7). To simplify the formulas, we drop the constant c_s. The trace term $v(x,0) = u$ has to be understood in the L^2 sense.

Notice that the first equation in (8) can be also written as

$$\operatorname{div}_{\gamma,y}(y^{1-2s}\nabla_{\gamma,y}v) = 0, \tag{9}$$

where we set

$$\nabla_{\gamma,y} := (\nabla_\gamma, \partial_y) \qquad \text{and} \qquad \operatorname{div}_{\gamma,y}F := \operatorname{div}_\gamma F_H + \partial_y F_{\mathbb{R}},$$

for a vector field $F = (F_H, F_{\mathbb{R}}) : X\times\mathbb{R}^+ \to H\times\mathbb{R}$. This equation involves an A_2 weight (see [17]) and such types of operators have been investigated in [13]. Recalling the integration by parts formula (5), equation (9) can then be written in a weak form as:

$$\int_{X\times\mathbb{R}^+} y^{1-2s}([\nabla_\gamma v, \nabla_\gamma\varphi] + \partial_y v\partial_y\varphi)d\gamma dy - \int_X f(u)\varphi\, d\gamma = 0\,, \tag{10}$$

for any $\varphi \in H^1(X\times\mathbb{R}^+, \gamma\otimes y^{1-2s}dy)$. Notice that, as $\mathcal{F}C_b^1(X\times\mathbb{R}^+)$ is dense in $H^1(X\times\mathbb{R}^+, \gamma\otimes y^{1-2s}dy)$, it is enough to require (10) to hold for all $\varphi \in \mathcal{F}C_b^1(X\times\mathbb{R}^+)$.

After defining the fractional laplacian, let us introduce the fractional Sobolev space

$$H^s_\gamma(X) = \Big\{ u \in L^2_\gamma(X) : \ [u]_{H^s_\gamma} < \infty \Big\}$$

where

$$[u]^2_{H^s_\gamma} = \inf\Big\{ \int_{X\times\mathbb{R}^+} |\nabla_{\gamma,y} v|^2 y^{1-2s} d\gamma(x) dy : v \in H^1_{\rm loc}(X\times\mathbb{R}^+), v(\cdot,0)=u(\cdot) \Big\}. \quad (11)$$

The space H^s_γ is endowed with the obvious Hilbert norm

$$\|u\|^2_{H^s_\gamma} = \|u\|^2_{L^2} + [u]^2_{H^s_\gamma}.$$

Remark 2. Let us define the space

$$H^1(X\times\mathbb{R}^+, \gamma\otimes y^{1-2s}dy) = \Big\{ v \in H^1_{\rm loc}(X\times\mathbb{R}^+) : \int_{X\times\mathbb{R}^+} (|v|^2 + |\nabla_{\gamma,y} v|^2) y^{1-2s} d\gamma(x) dy < \infty \Big\}.$$

A function $u \in L^2_\gamma(X)$ belongs to H^s_γ if and only if there is $v_u \in H^1(X\times\mathbb{R}^+, \gamma\otimes y^{1-2s}dy)$ such that the infimum in (11) is attained by v_u. We may therefore define the inner product

$$\begin{aligned}\langle u, w\rangle_{\dot H^s_\gamma} &= \int_{X\times\mathbb{R}^+} [\nabla_{\gamma,y} v_u, \nabla_{\gamma,y} v_w]_{H\times\mathbb{R}^+} y^{1-2s} d\gamma(x) dy \\ &= \int_{X\times\mathbb{R}^+} \Big([\nabla_\gamma v_u, \nabla_\gamma v_w]_H + \partial_y v_u \partial_y v_w \Big) y^{1-2s} d\gamma(x) dy, \quad u, w \in H^s_\gamma(X).\end{aligned}$$

3. Proof of Theorem 1.1.

The proof of Theorem 1.1 combines several techniques borrowed from [21] and [10]. Following the approach in [10], we first prove flatness of the level sets for cylindrical N-dimensional functions and, being the estimates independent of the dimension N, we then get the result for a general function on X by passing to the limit as $N \to +\infty$.

We analyse then geometric properties of the solutions of the problem:

$$\begin{cases} \mathrm{div}_{\gamma,y}(\mu(y)\nabla_{\gamma,y} v) = 0 & \text{on } X\times\mathbb{R}^+ \\ -\lim_{y\to 0^+} \mu(y)\partial_y v(x,y) = f(v) & \text{on } X. \end{cases} \quad (12)$$

3.1. Regularity properties for solutions of (12).

We first recall that equation (12) has a weak form and study the regularity properties of weak solutions. The weak form of (12) is

$$\int_{X\times\mathbb{R}^+} \mu(y)([\nabla_\gamma v, \nabla_\gamma \varphi] + \partial_y v \partial_y \varphi) d\gamma dy - \int_X f(v)\varphi\, d\gamma = 0\,, \quad (13)$$

for any $\varphi \in H^1(X\times\mathbb{R}^+, \gamma\otimes\mu(y)dy)$.

The following lemma is a direct consequence of a standard Caccioppoli estimate and the boundedness of $f(v)$ on X. Henceforth, we set

$$B_R := \{(x,y) \in X\times\mathbb{R} : \|x\|^2_X + |y|^2 < R^2\}, \qquad B^+_R := B_R \cap X\times\mathbb{R}^+.$$

Lemma 3.1. *Let v be a bounded weak solution of (13).*

Then, for any $R > 0$ there exists C, possibly depending on R, in such a way that

$$\|\mu(y)|\nabla_{\gamma,y} v|^2\|_{L^1(B^+_R)} \le C.$$

for any ball $B^+_R \subset X\times\mathbb{R}^+$.

Proof. We test (13) with $\varphi := v\tau^2$ where τ is a cutoff function such that $0 \le \tau \in C_0^\infty(B_{2R}^+)$, with $\tau = 1$ in B_R^+ and $|\nabla_{y,\gamma}\tau| \le 8/R$, with $R \ge 1$.

One then gets that

$$\int_{X\times\mathbb{R}^+} \mu(y)\left(|\nabla_{y,\gamma} v|^2\tau^2 + 2\tau\nabla_{y,\gamma} u \cdot \nabla_{y,\gamma}\tau\right) = \int_X f(v)v\tau^2.$$

Thus, by Cauchy-Schwarz inequality,

$$\begin{aligned}\int_{X\times\mathbb{R}^+} \mu(y)\,|\nabla_{\gamma,y} v|^2\tau^2 \le& \frac{1}{2}\int_{X\times\mathbb{R}^+} \mu(y)\,|\nabla_{\gamma,y} v|^2\tau^2 \\ &+ C_*\Big(\int_{X\times\mathbb{R}^+} \mu(y)|\nabla_{\gamma,y}\tau|^2 + \int_X |f(v)|\,|v|\,\tau^2\Big)\end{aligned}$$

for a suitable constant $C_* > 0$. The result follows from the fact that $\mu \in L^1_{\mathrm{loc}}(\mathbb{R}^+)$ by properties of A_2 functions and $f(v) \in L^\infty(X)$. □

Lemma 3.2. *Let v be a bounded $C^2_{\mathrm{loc}}(X\times\mathbb{R}^+)$ weak solution of (12). For any $i \in \mathbb{N}$ let $v_i = \partial_i v$; then*

$$\int_{X\times\mathbb{R}^+} \mu(y)([\nabla_\gamma v_i, \nabla_\gamma\varphi] + \partial_y v_i\partial_y\varphi + v_i\varphi)d\gamma\,dy - \int_X f'(v)v_i\varphi\,d\gamma = 0 \tag{14}$$

for any $\varphi \in H^1(X\times\mathbb{R}^+, \gamma\otimes\mu(y)dy)$.

Proof. By density it is enough to prove (14) for all $\varphi \in \mathcal{F}C_b^2(X\times\mathbb{R}^+)$. We test the weak formulation (10) with $\varphi_i := \partial_i\varphi$ and integrate by parts to get

$$\begin{aligned}0 =& \int_{X\times\mathbb{R}^+} \mu(y)\Big([\nabla_\gamma v, \nabla_\gamma\varphi_i] + \partial_y v\partial_y\varphi_i\Big)d\gamma\,dy - \int_X f(v)\varphi_i\,d\gamma \\ =& -\int_{X\times\mathbb{R}^+} \mu(y)\Big([\nabla_\gamma v_i, \nabla_\gamma\varphi] + \partial_y v_i\partial_y\varphi - \langle x, x_i^*\rangle([\nabla_\gamma v, \nabla_\gamma\varphi] + \partial_y v\partial_y\varphi)\Big)\,d\gamma\,dy \\ &+ \int_X (f'(v)v_i\varphi - \langle x, x_i^*\rangle f(v)\varphi)\,d\gamma \\ =& -\int_{X\times\mathbb{R}^+} \mu(y)\Big([\nabla_\gamma v_i, \nabla_\gamma\varphi] + \partial_y v_i\partial_y\varphi - \partial_y v\partial_y\varphi\langle x, x_i^*\rangle\Big)d\gamma\,dy \\ &+ \int_{X\times\mathbb{R}^+} [\nabla_\gamma v, \nabla_\gamma(\langle x, x_i^*\rangle\varphi) - \varphi\nabla_\gamma\langle x, x_i^*\rangle]d\gamma\,dy \\ &+ \int_X (f'(v)v_i\varphi - \langle x, x_i^*\rangle f(v)\varphi)\,d\gamma.\end{aligned}$$

Hence, using (10) with φ replaced by $\langle x, x_i^*\rangle\varphi$ and the fact that $[\nabla_\gamma v, \nabla_\gamma\langle x, x_i^*\rangle] = v_i$, one gets the desired result. □

One gets the following lemma using the fact that $\mu \in L^1_{\mathrm{loc}}(\mathbb{R}^+)$ and the regularity assumptions on v.

Lemma 3.3. *Let v be a bounded weak solution of (13). Assume furthermore that*

$$\partial_i\partial_j v \in C(X\times\mathbb{R}^+),\ \ \forall\, i, j \in \mathbb{N}.$$

Then,

$$\mu(y)|\nabla_{\gamma,y} v_j|^2 \in L^1(B_R^+)$$

for every $R>0$ and any $j\in\mathbb{N}$. Furthermore,

$$\text{for almost any } y>0, \text{ the map } X\ni x\mapsto \nabla_\gamma v(x,y) \text{ is in } W^{1,1}_{\mathrm{loc}}(X,H) \tag{15}$$

and

$$\begin{array}{l}\text{the map } X\times\mathbb{R}^+\ni(x,y)\mapsto \mu(y)\sum_{j=1}^\infty\left(|\nabla_{\gamma,y}v_j|^2+|v_j|^2\right)\\ \text{is in } L^1(B_R^+), \text{ for any } R>0.\end{array} \tag{16}$$

$$\begin{array}{l}\text{the map } X\times\mathbb{R}^+\ni(x,y)\mapsto \mu(y)\big(|\nabla_{\gamma,y}|\nabla_\gamma v||^2+|\nabla_\gamma v|^2\big)\\ \text{is in } L^1(B_R^+), \text{ for any } R>0.\end{array} \tag{17}$$

Proof. Since v is C^2 in $X\times\mathbb{R}^+$, for any $y>\varepsilon>0$ and any $R>0$

$$\int_{\{\|x\|_X<R}|\nabla_\gamma v(x,y)|+\sum_{j=1}^\infty|\nabla_\gamma v_{x_j}(x,y)|\,d\gamma\le C$$

for a suitable $C>0$, possibly depending on ε and R, which proves (15).

From the fact that $\partial_i\partial_j v\in C(X\times\mathbb{R}^+)$, we test the linearised equation (14), with $\varphi=v_i^2\eta$ with η a cut-off function in the ball B_R^+. As in the proof of Lemma 3.1, one gets easily, using the fact that $\mu\in L^1_{\mathrm{loc}}$, the property (16) (recall that f is locally Lipschitz).

To prove (17), we now perform the following standard approximation argument. Define $\Gamma:=\nabla_\gamma v$, and let $r,\rho>0$ and $P\in X\times\mathbb{R}^+$ be such that $B_{r+\rho}(P)\subset X\times\mathbb{R}^+$. Fix also $i\in\mathbb{N}$.

Then, for any $\varepsilon>0$,

$$\frac{\sum_{j=1}^\infty\Gamma_j\,\partial_i\Gamma_j}{\sqrt{\varepsilon^2+\sum_{j=1}^\infty\Gamma_j^2}}\le\frac{2\,|\Gamma|\,|\partial_i\Gamma|}{\varepsilon+|\Gamma|}\le 2|\partial_i\Gamma|\in L^1(B_r(P))$$

$$\lim_{\varepsilon\to0^+}\frac{\sum_{j=1}^n\Gamma_j\,\partial_i\Gamma_j}{\sqrt{\varepsilon^2+\sum_{j=1}^\infty\Gamma_j^2}}=\chi_{\{\Gamma\neq0\}}\frac{\sum_{j=1}^\infty\Gamma_j\,\partial_i\Gamma_j}{|\Gamma|}$$

$$\sqrt{\varepsilon^2+\sum_{j=1}^\infty\Gamma_j^2}\le\varepsilon+|\Gamma|\in L^1(B_r(P))$$

and

$$\lim_{\varepsilon\to0^+}\sqrt{\varepsilon^2+\sum_{j=1}^\infty\Gamma_j^2}=|\Gamma|,$$

thanks to (16).

Therefore, by Dominated Convergence Theorem,

$$\begin{aligned}\int_{X\times\mathbb{R}^+}\psi\chi_{\{\Gamma\neq0\}}\frac{\sum_{j=1}^\infty\Gamma_j\,\partial_i\Gamma_j}{|\Gamma|}&=\lim_{\varepsilon\to0^+}\int_{X\times\mathbb{R}^+}\psi\frac{\sum_{j=1}^\infty\Gamma_j\,\partial_i\Gamma_j}{\sqrt{\varepsilon^2+\sum_{j=1}^\infty\Gamma_j^2}}\\&=\lim_{\varepsilon\to0^+}\int_{X\times\mathbb{R}^+}\psi\,\partial_i\left(\sqrt{\varepsilon^2+\sum_{j=1}^\infty\Gamma_j^2}\right)\\&=-\lim_{\varepsilon\to0^+}\int_{X\times\mathbb{R}^+}(\partial_i\psi)\sqrt{\varepsilon^2+\sum_{j=1}^\infty\Gamma_j^2}\end{aligned}$$

$$= -\int_{X\times\mathbb{R}^+} (\partial_i\psi)|\Gamma|.$$

for any $\psi \in C_0^\infty(B_r(P))$.

Thus, since P, r and ρ can be arbitrarily chosen, we have that

$$\partial_i|\Gamma| = \chi_{\{\Gamma\neq 0\}}\frac{\sum_{j=1}^\infty \Gamma_j\partial_i\Gamma_j}{|\Gamma|}$$

weakly and almost everywhere in $X\times\mathbb{R}^+$.

Accordingly, we have

$$|\nabla_{\gamma,y}|\nabla_\gamma v||^2 = |\nabla_{\gamma,y}|\Gamma||^2 \le \sum_{j=1}^\infty |\nabla_{\gamma,y}v_{y_j}|^2.$$

Then, (16) implies (17). □

3.2. **Preliminary results.** We put ourselves under the hypothesis of the previous section on the regularity properties of weak solutions of (13).

The following lemma shows that monotone solutions satisfy a suitable stability inequality. We omit the proof, which is an obvious modification of the one of [10, Lemma 3.3].

Lemma 3.4. *Let v be a bounded weak solution of (12). Suppose that v satisfies the monotonicity condition*

$$\inf_{x\in B_R}[\nabla_\gamma v(x,y),w] > 0 \tag{18}$$

for all $y>0$, $R>0$ and for some $w\in H$. Then the inequality

$$\int_{X\times\mathbb{R}^+}\mu(y)(|\nabla_\gamma\varphi|^2+|\partial_y\varphi|^2)d\gamma\,dy - \int_X f'(v)\varphi^2\,d\gamma \ge -\int_{X\times\mathbb{R}^+}\varphi^2\mu(y)\,d\gamma\,dy \tag{19}$$

holds for any $\varphi\in H^1(X\times\mathbb{R}^+,\gamma\otimes\mu(y)dy)$.

3.3. **A geometric Poincaré inequality.** We now prove a geometric Poincaré inequality for solutions of (12) satisfying (19), in the spirit of [10, Lemma 3.4] (see also [22, 23, 14]).

Lemma 3.5. *Let v be a bounded weak solution of (12) and (19). For any $\varphi\in W^{1,\infty}(X\times\mathbb{R}^+)$ we have*

$$\int_{X\times\mathbb{R}^+}\mu(y)\big(|\nabla_\gamma^2 v|^2 - \big|\nabla_\gamma|\nabla_\gamma v|\big|^2\big)\varphi^2 d\gamma\,dy \le \int_{X\times\mathbb{R}^+}\mu(y)|\nabla_\gamma v|^2|\nabla_{\gamma,y}\varphi|^2\,d\gamma\,dy$$

where

$$|\nabla_\gamma^2 v|^2 := \sum_{i,j}(\partial_i\partial_j v)^2\,.$$

Proof. We use (19) with test function $|\nabla_\gamma v|\,\varphi$, and we see that

$$\begin{aligned}
&\int_X f'(v)|\nabla_\gamma v|^2\,\varphi^2 d\gamma - \int_{X\times\mathbb{R}^+}\mu(y)|\nabla_\gamma v|^2\,\varphi^2 d\gamma\\
&\le \int_{X\times\mathbb{R}+}\mu(y)(|\nabla_\gamma(\,|\nabla_\gamma v|\varphi)|^2 + |\partial_y(|\nabla_\gamma v|\varphi)|^2\,d\gamma\,dy\\
&= \int_{X\times\mathbb{R}^+}\mu(y)\Big(\varphi^2\big|\nabla_\gamma|\nabla_\gamma v|\big|^2 + |\nabla_\gamma v|^2|\nabla_\gamma\varphi|^2 + \frac{1}{2}[\nabla_\gamma|\nabla_\gamma v|^2,\nabla_\gamma\varphi^2]
\end{aligned}$$

$$+|\partial_y|\nabla_\gamma v||^2\varphi^2+\frac{1}{2}\partial_y|\nabla_\gamma v|^2\partial_y\varphi^2+|\nabla_\gamma v|^2|\partial_y\varphi|^2\Big)\,d\gamma\,dy.$$

Using (14) with the test function $v_i\varphi^2$ gives

$$\int_{X\times\mathbb{R}^+}\mu(y)([\nabla_\gamma v_i,\nabla_\gamma(v_i\varphi^2)]+\partial_y v_i\partial(v_i\varphi^2)+v_i^2\varphi^2)\,d\gamma\,dy-\int_X f'(v)v_i^2\varphi^2 d\gamma=0.$$

Hence

$$\begin{aligned}&\int_X f'(v)v_i^2\varphi^2 d\gamma-\int_{X\times\mathbb{R}^+}\mu(y)v_i^2\varphi^2 d\gamma\\&=\int_{X\times\mathbb{R}^+}\mu(y)([\nabla_\gamma v_i,\nabla_\gamma(v_i\varphi^2)]+(v_i)_y(v_i\varphi^2)_y)d\gamma\,dy\\&=\int_{X\times\mathbb{R}^+}\mu(y)(|\nabla_\gamma v_i|^2\varphi^2+(\partial_y v_i)^2\varphi^2+\frac{1}{2}\partial_y(v_i)^2\partial_y\varphi^2+\frac{1}{2}[\nabla_\gamma v_i^2,\nabla_\gamma\varphi^2])d\gamma\,dy.\end{aligned}$$

Summing up in i gives

$$\begin{aligned}&\int_X f'(v)|\nabla_\gamma v|^2\varphi^2 d\gamma-\int_{X\times\mathbb{R}^+}\mu(y)|\nabla_\gamma v|^2\varphi^2 d\gamma\,dy\\&=\int_{X\times\mathbb{R}^+}\mu(y)(|\nabla_\gamma^2 v|^2\varphi^2+\sum_i(\partial_y v_i)^2\varphi^2+\frac{1}{2}\partial_y|\nabla_\gamma v|^2\partial_y\varphi^2+\frac{1}{2}[\nabla_\gamma|\nabla_\gamma v|^2,\nabla_\gamma\varphi^2])d\gamma\,dy.\end{aligned}$$

Hence we have

$$\begin{aligned}&\int_{X\times\mathbb{R}^+}\mu(y)(|\nabla_\gamma^2 v|^2\varphi^2+\sum_i(\partial_y v_i)^2\varphi^2+\frac{1}{2}\partial_y|\nabla_\gamma v|^2\partial_y\varphi^2+\frac{1}{2}[\nabla_\gamma|\nabla_\gamma v|^2,\nabla_\gamma\varphi^2])d\gamma\,dy\\&\le\int_{X\times\mathbb{R}^+}\mu(y)\Big(\varphi^2\big|\nabla_\gamma|\nabla_\gamma v|\big|^2+|\nabla_\gamma v|^2|\nabla_\gamma\varphi|^2+\frac{1}{2}\left[\nabla_\gamma|\nabla_\gamma v|^2,\nabla_\gamma\varphi^2\right]\\&+|\partial_y|\nabla_\gamma v||^2\varphi^2+\frac{1}{2}\partial_y|\nabla_\gamma v|^2\partial_y\varphi^2+|\nabla_\gamma v|^2|\partial_y\varphi|^2\Big)d\gamma\,dy.\end{aligned}$$

Collecting terms, one gets

$$\begin{aligned}&\int_{X\times\mathbb{R}^+}\mu(y)(|\nabla_\gamma^2 v|^2-\big|\nabla_\gamma|\nabla_\gamma v|\big|^2)\varphi^2+\sum_i(\partial_y v_i)^2\varphi^2\,d\gamma\,dy\\&\le\int_{X\times\mathbb{R}^+}\mu(y)\Big(|\nabla_\gamma v|^2|\nabla_{\gamma,y}\varphi|^2+|\partial_y|\nabla_\gamma v||^2\varphi^2\Big)\,d\gamma\,dy.\end{aligned}$$

Now we claim that

$$\sum_i(\partial_y v_i)^2-|\partial_y|\nabla_\gamma v||^2\ge 0,$$

and this leads to the desired result. The claim follows directly by the Cauchy-Schwarz inequality:

$$|\partial_y|\nabla_\gamma v||^2=\Big(\frac{[\nabla_\gamma v,\nabla_\gamma v_y]}{|\nabla_\gamma v|}\Big)^2\le|\nabla_\gamma v_y|^2=\sum_i(\partial_y v_i)^2.$$

□

Following [21], we now introduce level sets parametrised by $y>0$ for cylindrical functions. Let $v\in L^\infty(X\times\mathbb{R}^+)$ satisfy $\partial_i v\in C(X\times\mathbb{R}^+)$ and

$$\partial_i\partial_j v\in C(X\times\mathbb{R}^+)\qquad\text{for all } i,j\in\mathbb{N}.\tag{20}$$

Let $N \in \mathbb{N}$ and $\overline{x}_N \in X_N^\perp$. We consider the map $\psi_{N,\overline{x}_N} : \mathbb{R}^N \times \mathbb{R}^+ \to \mathbb{R}$ defined as $\psi_{N,\overline{x}_N}(\underline{x}_N, y) := v(\underline{x}_N, \overline{x}_N, y)$, and let for $y > 0$

$$\begin{aligned}\mathcal{N}_N^y(\overline{x}_N) &:= \big\{\underline{x}_N \in \mathbb{R}^N : \nabla_x \psi_{N,\overline{x}_N}(\underline{x}_N, y) \neq 0\big\}\\ &= \big\{\underline{x}_N \in \mathbb{R}^N : \exists\, i \in \{1,\dots,N\} \text{ such that } v_i(\underline{x}_N, \overline{x}_N, y) \neq 0\big\}\end{aligned}$$

be its noncritical set. By the Implicit Function Theorem, the level set of $\psi_{N,\overline{x}_N}$ in $\mathcal{N}_N^y(\overline{x}_N)$ are $(N-1)$-dimensional hypersurfaces of class C^2. Thus we can consider the principal curvatures of these hypersurfaces, that we denote by $\kappa_{1,N}^y, \dots, \kappa_{N-1,N}^y$, and the tangential gradient of $\psi_{N,\overline{x}_N}$ [1], that we denote by $\nabla_{T,N}$. We also set

$$\begin{aligned}&\nabla_{N,\gamma}\, v := \Pi_N \nabla_\gamma v = \nabla_\gamma \psi_{N,\overline{x}_N},\\ &\nabla^2_{N,\gamma}\, v := \nabla_{N,\gamma}\big(\nabla_{N,\gamma} v\big) = \nabla^2 \psi_{N,\overline{x}_N},\\ &\mathcal{K}_N^y := \sqrt{\sum_{i=1}^{N-1} (\kappa_{i,N}^y)^2}\\ &\mathcal{N}_N^y := \Big\{(\underline{x}_N, \overline{x}_N, y) \in X \times \mathbb{R}^+ : \underline{x}_N \in \mathcal{N}_N^y(\overline{x}_N)\Big\} = \Big\{x \in X : \nabla_{N,\gamma} v(x,y) \neq 0\Big\}.\end{aligned}$$

With this notation, we have the following (see [10, Lemma 3.5] for the proof which is identical).

Lemma 3.6. *Let $v \in L^\infty(X \times \mathbb{R}^+)$ such that $\partial_i v \in C(X \times \mathbb{R}^+)$ satisfy (12), (19) and (20), and fix $N \in \mathbb{N}$. For any $\varphi \in W^{1,\infty}(X \times \mathbb{R}^+)$ we have*

$$\begin{aligned}&\int_{\mathcal{N}_N^y} \mu(y)\Big(|\nabla_{N,\gamma} v|^2 (\mathcal{K}_N^y)^2 + \big|\nabla_{T,N}|\nabla_{N,\gamma} v|\big|^2\Big)\varphi^2\, d\gamma\, dy\\ &\leq \int_{X\times\mathbb{R}^+} \mu(y) |\nabla_\gamma v|^2 |\nabla \varphi|^2\, d\gamma\, dy.\end{aligned} \tag{21}$$

We are now in the position to prove a symmetry results for cylindrical solution of (12) satisfying (19).

Proposition 1. *Fix $N \in \mathbb{N}$ and $\overline{x}_N \in X_N^\perp$. Let $v \in L^\infty(X \times \mathbb{R}^+)$ such that $\partial_i v \in C(X \times \mathbb{R}^+)$ satisfy (12), (19) and (20). Then, there exists a map $V_{N,\overline{x}_N} : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ and $\omega_{N,\overline{x}_N} \in \mathbb{R}^N$, with $|\omega_{N,\overline{x}_N}| = 1$, such that*

$$v(\underline{x}_N, \overline{x}_N, y) = V_{N,\overline{x}_N}\big(\langle \underline{x}_N, \omega_{N,\overline{x}_N}\rangle, y\big) \tag{22}$$

for any $\underline{x}_N \in \mathbb{R}^N$.

Proof. We fix $R > 1$, to be taken arbitrarily large in what follows, and let $\Lambda = \max_i \lambda_i$, where the λ_i are defined in Subsection 2.2. Let $\Phi \in C^\infty(\mathbb{R})$ be such that $\Phi(t) = 1$ if $t \leq R$, $\Phi(t) = 0$ if $t \geq R+1$ and $|\Phi'(t)| \leq 3$ for any $t \in [R, R+1]$. We take $\varphi(x,y) := \Phi(|(x,y)|)$. Then $|\nabla_{\gamma,y}\varphi(x,y)| \leq \sqrt{\Lambda}\,|\Phi'(|(x,y)|)| \leq 3\sqrt{\Lambda}$, and (21) yields

$$\begin{aligned}&\int\limits_{\mathcal{N}_N^y \cap \{|(x,y)| \leq R\}} \mu(y)\Big(|\nabla_{N,\gamma} v|^2 (\mathcal{K}_N^y)^2 + \big|\nabla_{T,N}|\nabla_{N,\gamma} v|\big|^2\Big)\varphi^2\, d\gamma\, dy\\ &\leq C \int\limits_{\{R \leq |(x,y)| \leq R+1\}} \mu(y) |\nabla_{\gamma,y} v|^2\, d\gamma\, dy.\end{aligned} \tag{23}$$

[1]The tangential gradient of a function g along a hypersurface with normal ν is $\nabla g - (\nabla g \cdot \nu)\nu$, that is, the tangential component of the full gradient.

On the other hand, since by Lemma 3.1

$$\mu(y)|\nabla_{\gamma,y}v|^2 \in L^1_{\rm loc}(X \times \mathbb{R}^+),$$

sending $R \to +\infty$ in (23) we conclude that

$$|\nabla_{N,\gamma}v|^2(\mathcal{K}_N^y)^2 + \big|\nabla_{T,N}|\nabla_{N,\gamma}v|\big|^2 = 0$$

for any $x \in \mathcal{N}_N^y$. From this and [14, Lemma 2.11] we get (22). □

From the finite dimensional symmetry result in Proposition 1, one can take the limit as $N \to +\infty$ and obtain, following verbatim the proof in [10, Corollary 3.7], the corollary

Corollary 1. *Let $v \in C^1(X \times \mathbb{R}^+) \cap L^\infty(X \times \mathbb{R}^+)$ satisfy (19), (20) and (12). Then, v is one-dimensional, in the sense that there exists $V : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ and $\omega \in X^*$ such that*

$$v(x, y) = V(\langle x, \omega\rangle, y)$$

for any $x \in X$ and $y > 0$.

From Corollary 1 and Lemma 3.4 we immediately deduce Theorem 1.1.

□

4. **Proof of Theorem 1.2.** The proof of this theorem follows directly from the proof of Theorem 1.1 as soon as one checks the desired regularity assumptions. First, it is easily checked that, defining the operator

$$Lv = \mathrm{div}(\gamma_N(x)\mu(y)\nabla v),$$

the weak form of equation $Lv = 0$ in $H^1(\mathbb{R}^N \times \mathbb{R}^+, dxdy)$ is equivalent to $\mathrm{div}_{\gamma,y}v = 0$ in $H^1(\mathbb{R}^N \times \mathbb{R}^+, \gamma \otimes \mu dy)$. We may therefore apply the known regularity results on weak solutions of elliptic equations with A_2 weights. Indeed, the Gaussian measure satisfies locally for any $x \in B_R$

$$0 < C_R^1 \le \gamma_N \le C_R^2,$$

hence the weight $w(x, y) = \gamma_N(x)\mu(y)$ is an A_2 weight in $\mathbb{R}^{N+1}$ in the sense that

$$\sup_{B_R \subset \mathbb{R}^{N+1}} \Big(\frac{1}{|B|}\int_{B_R} w(x, y)\, dxdy\Big)\Big(\frac{1}{|B|}\int_{B_R} w(x, y)^{-1}\, dxdy\Big) \le C$$

for some constant $C > 0$. Furthermore, since the weight is assumed to be uniformly positive, one can invoke classical regularity theory to deduce that $\partial_i\partial_j v$ is continuous in $\mathbb{R}^N \times \mathbb{R}^+$. This gives the desired result. □

5. **Proof of Theorem 1.3.** Recalling Theorem 2.1, Theorem 1.3 is a direct consequence of Theorem 1.1. Consider

$$\mu(y) = y^{1-2s}$$

for $s \in (0, 1)$. Clearly, this weight is an A_2 weight. By the construction in [24], we have that the trace of v on X, denoted u, is satisfied in the L^2 sense and satisfies equation (2). By construction, v satisfies

$$v(x, y) = \frac{1}{\Gamma(s)}\int_0^\infty e^{t\Delta_\gamma} f(u(x))e^{-y^2/4t}\frac{dt}{t^{1-s}}$$

and the Poisson formula (see [24])

$$v(x, y) = \frac{1}{\Gamma(s)}\int_0^\infty e^{\frac{y^2}{r}\Delta_\gamma} u(x)e^{-r}\frac{dr}{r^{1-s}}.$$

We now recall the following well-known expression of the Ornstein-Uhlenbeck semigroup

$$e^{t\Delta_\gamma}u(x) = \int_X u(e^{-t}x + \sqrt{1-e^{-2t}}y)\, d\gamma(y).$$

From the previous formula we obtain that for fixed $t > 0$, $e^{t\Delta_\gamma}$ maps $L^\infty(X)$ into itself with the bound

$$\|e^{t\Delta_\gamma}u\|_{L^\infty(X)} \le \|u\|_{L^\infty(X)}.$$

Since

$$\|v\|_\infty \le \frac{\|u\|_\infty}{\Gamma(s)} \int_0^\infty e^{-r} r^{s-1}\, dr = \|u\|_\infty,$$

we deduce that $v \in L^\infty(X \times \mathbb{R}^+)$. Furthermore, we have

$$\nabla_\gamma v(x,y) = \frac{1}{\Gamma(s)} \int_0^\infty e^{\frac{y^2}{r}\Delta_\gamma} \nabla_\gamma u(x) e^{-r} \frac{dr}{r^{1-s}},$$

and since $e^{\frac{y^2}{r}\Delta_\gamma}$ is order-preserving by the Ornstein-Uhlenbeck formula, the monotonicity assumption on v is satisfied. Then Theorem 1.1 holds and this leads to the desired result by taking $y \to 0$ in the L^2 sense as computed in [24].

□

6. **Existence of one-dimensional monotone solutions.** Given $F \in C^1(\mathbb{R})$, we introduce the energy

$$G(u) := \frac{1}{2}[u]^2_{H^s_{\gamma_1}(\mathbb{R})} + \int_{\mathbb{R}} F(u)\, d\gamma_1 \qquad u \in H^s_{\gamma_1}(\mathbb{R}),$$

where $[u]_{H^s_{\gamma_1}(\mathbb{R})}$ is defined in (11).Notice that a critical point of G satisfies the Euler-Lagrange equation

$$(-\Delta_{\gamma_1})^s u + F'(u) = 0. \tag{24}$$

The goal of this section is to prove existence of monotone solutions of (24).

Theorem 6.1. *Assume that F satisfies the following properties:*

$$F(\pm 1) = 0 \tag{25}$$
$$F(u) > 0 \qquad \text{for all } u \ne \pm 1 \tag{26}$$
$$F(u) = F(-u) \qquad \text{for all } u \in \mathbb{R} \tag{27}$$
$$1 + \sqrt{\frac{2}{\pi}} \max_{[-1,1]} F < F(0). \tag{28}$$

Then there exists a global minimiser $U^\star$ of G in $\mathcal{M}$, such that $U^\star$ is odd, monotonically increasing and strictly positive on $\mathbb{R}^+$. Moreover $U^\star \in C^2(\mathbb{R})$ and solves (24).

Proof. Observe that $\inf_{\mathcal{M}} G < +\infty$. Indeed, if we let

$$\tilde{u}(t) = \max(-1, \min(t,1)) \in \mathcal{M}$$

and

$$\tilde{v}(t,y) = \begin{cases} \max(\tilde{u}(t) - y, 0) & \text{if } t \ge 0 \\ \min(\tilde{u}(t) + y, 0) & \text{if } t < 0, \end{cases}$$

we have

$$[\tilde{u}]^2_{H^s_{\gamma_1}(\mathbb{R})} \le \int_{\mathbb{R}\times\mathbb{R}^+} \Big(|\partial_t \tilde{v}|^2 + |\partial_y \tilde{v}|^2\Big) y^{1-2s} d\gamma_1(t) dy \le 2,$$

which gives, recalling (28),

$$G(\tilde{u}) \leq 1 + \int_{\mathbb{R}} F(\tilde{u}(t))\, d\gamma_1(t) \leq 1 + \sqrt{\frac{2}{\pi}} \max_{[-1,1]} F < G(0). \tag{29}$$

Let now $U^\star$ be a minimiser of G among the functions u which are odd. Such a minimiser exists since G is lower semicontinuous in $H^s(\mathbb{R}, \gamma_1)$ and $\tilde{u}$ is odd with $G(\tilde{u}) < +\infty$. Moreover, (28) and (29) imply that $U^\star$ is not identically zero and, by [18, Corollary 3.4] $U^\star$ is monotonically increasing, and a simple truncation argument shows that $|U^\star| \leq 1$. Finally, by elliptic regularity, $U^\star$ is of class C^2 and solves (24). □

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