

A SYMMETRY RESULT FOR THE ORNSTEIN-UHLENBECK OPERATOR

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ABSTRACT. In 1978 E. De Giorgi formulated a conjecture concerning the one-dimensional symmetry of bounded solutions to the elliptic equation $\Delta u = F'(u)$, which are monotone in some direction. In this paper we prove the analogous statement for the equation $\Delta u - \langle x, \nabla u \rangle u = F'(u)$, where the Laplacian is replaced by the Ornstein-Uhlenbeck operator. Our theorem holds without any restriction on the dimension of the ambient space, and this allows us to obtain a similar result in infinite dimensions by a limit procedure.

1. Introduction. A celebrated conjecture by De Giorgi [6] asks if bounded entire solutions to the equation

$$\Delta u = u^3 - u \tag{1}$$

which are strictly increasing in some direction are one-dimensional, in the sense that the level sets $\{u = \lambda\}$ are hyperplanes, at least if $n \leq 8$. This conjecture has been proved by Ghoussoub and Gui [14] in dimension $n = 2$, and by Ambrosio and Cabré [2] in dimension $n = 3$, and a counterexample has been given by del Pino, Kowalczyk and Wei in [7] for $n \geq 9$. While the conjecture is still open for $4 \leq n \leq 8$, a very nice proof has been presented by O. Savin [17] under the additional assumption that

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u connects -1 to 1 along the direction where it increases. See also [4] for another proof in dimension $n = 2$ and [12] for a review on the subject.

In this paper, we are interested in a variant of (1) where the Laplacian Δ is substituted by the Ornstein-Uhlenbeck operator $\Delta - \langle x, \nabla \rangle$. Namely, we consider the semilinear elliptic equation

$$\Delta u - \langle x, \nabla u \rangle + f(u) = 0 \quad (2)$$

and show the one-dimensional symmetry of bounded entire solutions which are monotone in some direction.

Let us state our main result.

Theorem 1.1. *Let $n \in \mathbb{N}$, $\alpha \in (0, 1)$. Let $u \in C^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ be a solution of*

$$\Delta u - \langle x, \nabla u \rangle + f(u) = 0 \quad \text{in } \mathbb{R}^n,$$

where $f : \mathbb{R} \rightarrow \mathbb{R}$ is a locally Lipschitz function. Assume that

$$\langle \nabla u(x), w \rangle > 0 \quad \text{for any } x \in \mathbb{R}^n \quad (3)$$

for some $w \in \mathbb{R}^n$. Then, u is one-dimensional, i.e. there exist $U : \mathbb{R} \rightarrow \mathbb{R}$ and $\omega \in \mathbb{R}^n$ such that

$$u(x) = U(\langle \omega, x \rangle)$$

for any $x \in \mathbb{R}^n$.

Notice that (2) can be regarded as the analog of (1) in the so-called Gauss space, that is, in \mathbb{R}^n endowed with the Gaussian instead of the Lebesgue measure. Indeed, while the Pde in (1) is the Euler-Lagrange equation of the Allen-Cahn Energy

$$\int_{\mathbb{R}^n} \left(\frac{|\nabla u|^2}{2} + \frac{(u^2 - 1)^2}{4} \right) dx, \quad (4)$$

the Pde in (2) is the Euler-Lagrange equation of the functional

$$\int_{\mathbb{R}^n} \left(\frac{|\nabla u|^2}{2} + F(u) \right) d\gamma(x), \quad (5)$$

where $F' = -f$ and

$$d\gamma(x) = \gamma(x)dx = \frac{e^{-|x|^2/2}}{(2\pi)^{n/2}} dx \quad (6)$$

is the standard Gaussian probability measure. It is interesting to remark that Theorem 1.1 holds for general type of nonlinearities, as it happens for the conjecture of De Giorgi when $n \leq 3$ (see [1], and this is a major difference with respect to the techniques in [17]).

As in the case of the Laplacian, Theorem 1.1 is closely related to the Bernstein problem in the Gauss space, which asks for flatness of entire minimal surfaces which are graphs in some direction. We point out that minimal surfaces in the Gauss space are interesting geometric objects, since they correspond to self-similar shrinkers of the mean curvature flow (see for instance [8]), and satisfy the equation

$$\kappa = \langle x, \nu \rangle \quad (7)$$

where κ is the mean curvature at x and ν is the normal vector. In this context, the analog of the Bernstein Theorem has been proved by Ecker and Huisken [8], under a polynomial growth assumption on the volume of the minimal surface, and more recently by Wang in [20] without any further assumption. We point out that, differently from the Euclidean case, the result holds without any restriction on

the dimension of the ambient space, and in fact there is no such restriction also in Theorem 1.1. This is due to the exponential decay of the Gaussian measure associated to the Ornstein-Uhlenbeck operator which allows for better estimates than the corresponding Euclidean ones.

Since Theorem 1.1 holds in any dimension and the Gauss space (\mathbb{R}^n, γ) formally converges to a Wiener space (X, H, γ) (see Section 2.1 for a precise definition) as $n \rightarrow \infty$, one may expect that an analogous result holds in such infinite dimensional setting. Indeed, in this paper we confirm this expectation and show the infinite dimensional extension of Theorem 1.1:

Theorem 1.2. *Let $u \in C^1(X) \cap L^\infty(X)$ satisfy*

$$\Delta_\gamma u = f(u) \tag{8}$$

where $f : \mathbb{R} \rightarrow \mathbb{R}$ is a locally Lipschitz function. Assume that

$$\partial_i \partial_j u \in C(X) \quad \text{for all } i, j \in \mathbb{N} \tag{9}$$

and

$$\inf_{x \in B_R} [\nabla u(x), w] > 0 \tag{10}$$

for all $x \in X$, for all $R > 0$ and for some $w \in H$. Then, u is one-dimensional, in the sense that there exist $U : \mathbb{R} \rightarrow \mathbb{R}$ and $\omega \in X^*$ such that

$$u(x) = U(\langle \omega, x \rangle) \quad \text{for all } x \in X. \tag{11}$$

Notice that Theorem 1.1 can be recovered as a corollary of Theorem 1.2, when the function u depends only on finitely many variables. As far as we know, Theorem 1.1 is the first result of De Giorgi conjecture type in an infinite dimensional setting. The proof that we perform exploits and generalizes some geometric ideas of [18, 19, 10, 11].

2. Notation. We denote by (\mathbb{R}^n, γ) the n -dimensional Gauss space, where γ is the standard Gaussian measure on \mathbb{R}^n defined in (6).

2.1. The Wiener space. An abstract Wiener space is defined as a triple (X, γ, H) where X is a separable Banach space, endowed with the norm $\|\cdot\|_X$, γ is a nondegenerate centered Gaussian measure, and H is the Cameron–Martin space associated to the measure γ , that is, H is a separable Hilbert space densely embedded in X , endowed with the inner product $[\cdot, \cdot]_H$ and with the norm $|\cdot|_H$. The requirement that γ is a centered Gaussian measure means that for any $x^* \in X^*$, the measure $x^*_\# \gamma$ is a centered Gaussian measure on the real line \mathbb{R} , that is, the Fourier transform of γ is given by

$$\hat{\gamma}(x^*) = \int_X e^{-i\langle x, x^* \rangle} d\gamma(x) = \exp\left(-\frac{\langle Qx^*, x^* \rangle}{2}\right), \quad \forall x^* \in X^*;$$

here the operator $Q \in \mathcal{L}(X^*, X)$ is the covariance operator and it is uniquely determined by formula

$$\langle Qx^*, y^* \rangle = \int_X \langle x, x^* \rangle \langle x, y^* \rangle d\gamma(x), \quad \forall x^*, y^* \in X^*.$$

The nondegeneracy of γ implies that Q is positive definite: the boundedness of Q follows by Fernique’s Theorem (see for instance [5, Theorem 2.8.5]), asserting that there exists a positive number $\beta > 0$ such that

$$\int_X e^{\beta \|x\|^2} d\gamma(x) < +\infty.$$

This implies also that the maps $x \mapsto \langle x, x^* \rangle$ belong to $L^p_\gamma(X)$ for any $x^* \in X^*$ and $p \in [1, +\infty)$, where $L^p_\gamma(X)$ denotes the space of all functions $f : X \rightarrow \mathbb{R}$ such that

$$\int_X |f(x)|^p d\gamma(x) < +\infty.$$

In particular, any element $x^* \in X^*$ can be seen as a map $x^* \in L^2_\gamma(X)$, and we denote by $R^* : X^* \rightarrow \mathcal{H}$ the identification map $R^*x^*(x) := \langle x, x^* \rangle$. The space \mathcal{H} given by the closure of R^*X^* in $L^2_\gamma(X)$ is called *reproducing kernel*. By considering the map $R : \mathcal{H} \rightarrow X$ defined as

$$R\hat{h} := \int_X \hat{h}(x)x d\gamma(x),$$

we obtain that R is an injective γ -Radonifying operator, which is Hilbert–Schmidt when X is Hilbert. We also have $Q = RR^* : X^* \rightarrow X$. The space $H := R\mathcal{H}$, equipped with the inner product $[\cdot, \cdot]_H$ and norm $|\cdot|_H$ induced by \mathcal{H} via R , is the Cameron–Martin space and is a dense subspace of X . The continuity of R implies that the embedding of H in X is continuous, that is, there exists $c > 0$ such that

$$\|h\|_X \leq c|h|_H, \quad \forall h \in H.$$

We have also that the measure γ is absolutely continuous with respect to translation along Cameron–Martin directions; in fact, for $h \in H$, $h = Qx^*$, the measure $\gamma_h(B) = \gamma(B - h)$ is absolutely continuous with respect to γ with density given by

$$d\gamma_h(x) = \exp\left(\langle x, x^* \rangle - \frac{1}{2}|h|_H^2\right) d\gamma(x). \quad (12)$$

2.2. Cylindrical functions and differential operators. For $j \in \mathbb{N}$ we choose $x_j^* \in X^*$ in such a way that $\hat{h}_j := R^*x_j^*$, or equivalently $h_j := R\hat{h}_j = Qx_j^*$, form an orthonormal basis of H . We order the vectors x_j^* in such a way that the numbers $\lambda_j := \|x_j^*\|_{X^*}^{-2}$ form a non-increasing sequence. Given $m \in \mathbb{N}$, we also let $H_m := \langle h_1, \dots, h_m \rangle \subseteq H$, and $\Pi_m : X \rightarrow H_m$ be the closure of the orthogonal projection from H to H_m

$$\Pi_m(x) := \sum_{j=1}^m \langle x, x_j^* \rangle h_j \quad x \in X.$$

The map Π_m induces the decomposition $X \simeq H_m \oplus X_m^\perp$, with $X_m^\perp := \ker(\Pi_m)$, and $\gamma = \gamma_m \otimes \gamma_m^\perp$, with γ_m and γ_m^\perp Gaussian measures on H_m and X_m^\perp respectively, having H_m and H_m^\perp as Cameron–Martin spaces. When no confusion is possible we identify H_m with \mathbb{R}^m ; with this identification the measure $\gamma_m = \Pi_{m\#}\gamma$ is the standard Gaussian measure on \mathbb{R}^m (see [5]). Given $x \in X$, we denote by $\underline{x}_m \in H_m$ the projection $\Pi_m(x)$, and by $\bar{x}_m \in X_m^\perp$ the infinite dimensional component of x , so that $x = \underline{x}_m + \bar{x}_m$. When we identify H_m with \mathbb{R}^m we shall rather write $x = (\underline{x}_m, \bar{x}_m) \in \mathbb{R}^m \oplus X_m^\perp$.

We say that $u : X \rightarrow \mathbb{R}$ is a *cylindrical function* if $u(x) = v(\Pi_m(x))$ for some $m \in \mathbb{N}$ and $v : \mathbb{R}^m \rightarrow \mathbb{R}$. We denote by $\mathcal{F}C_b^k(X)$, $k \in \mathbb{N}$, the space of all C_b^k cylindrical functions, that is, functions of the form $v(\Pi_m(x))$ with $v \in C^k(\mathbb{R}^m)$, with continuous and bounded derivatives up to the order k . We denote by $\mathcal{F}C_b^k(X, H)$ the space generated by all functions of the form uh , with $u \in \mathcal{F}C_b^k(X)$ and $h \in H$.

We let

$$\nabla_\gamma u := \sum_{j \in \mathbb{N}} \partial_j u h_j \quad \text{for } u \in \mathcal{F}C_b^1(X)$$

$$\operatorname{div}_\gamma \varphi := \sum_{j \geq 1} \partial_j^* [\varphi, h_j]_H \quad \text{for } \varphi \in \mathcal{F}C_b^1(X, H)$$

$$\Delta_\gamma u := \operatorname{div}_\gamma \nabla_\gamma u \quad \text{for } u \in \mathcal{F}C_b^2(X)$$

where $\partial_j := \partial_{h_j}$ and $\partial_j^* := \partial_j - \hat{h}_j$ is the adjoint operator of ∂_j . With this notation, the integration by parts formula holds:

$$\int_X u \operatorname{div}_\gamma \varphi \, d\gamma = - \int_X [\nabla_\gamma u, \varphi]_H \, d\gamma \quad \forall \varphi \in \mathcal{F}C_b^1(X, H). \quad (13)$$

In particular, thanks to (13), the operator ∇_γ is closable in $L_\gamma^p(X)$, and we denote by $W_\gamma^{1,p}(X)$ the domain of its closure. The Sobolev spaces $W_\gamma^{k,p}(X)$, with $k \in \mathbb{N}$ and $p \in [1, +\infty]$, can be defined analogously [5], and $\mathcal{F}C_b^k(X)$ is dense in $W_\gamma^{j,p}(X)$, for all $p < +\infty$ and $k, j \in \mathbb{N}$ with $k \geq j$.

Given a vector field $\varphi \in L_\gamma^p(X, H)$, $p \in (1, \infty]$, using (13) we can define $\operatorname{div}_\gamma \varphi$ in the distributional sense, taking test functions u in $W_\gamma^{1,q}(X)$ with $\frac{1}{p} + \frac{1}{q} = 1$. We say that $\operatorname{div}_\gamma \varphi \in L_\gamma^p(X)$ if this linear functional can be extended to all test functions $u \in L_\gamma^q(X)$. This is true in particular if $\varphi \in W_\gamma^{1,p}(X, H)$.

Let $u \in W_\gamma^{2,2}(X)$, $\psi \in \mathcal{F}C_b^1(X)$ and $i, j \in \mathbb{N}$. From (13), with $u = \partial_j u$ and $\varphi = \psi h_i$, we get

$$\int_X \partial_j u \partial_i \psi \, d\gamma = \int_X -\partial_j(\partial_i u) \psi + \partial_j u \psi \langle x_i^*, x \rangle \, d\gamma \quad (14)$$

Let now $\varphi \in \mathcal{F}C_b^1(X, H)$. If we apply (14) with $\psi = [\varphi, h_j] =: \varphi^j$, we obtain

$$\int_X \partial_j u \partial_i \varphi^j \, d\gamma = \int_X -\partial_j(\partial_i u) \varphi^j + \partial_j u \varphi^j \langle x_i^*, x \rangle \, d\gamma$$

which, summing up in j , gives

$$\int_X [\nabla_\gamma u, \partial_i \varphi] \, d\gamma = \int_X -[\nabla_\gamma(\partial_i u), \varphi] + [\nabla_\gamma u, \varphi] \langle x_i^*, x \rangle \, d\gamma \quad \forall \varphi \in \mathcal{F}C_b^1(X, H). \quad (15)$$

The operator $\Delta_\gamma : W_\gamma^{2,p}(X) \rightarrow L_\gamma^p(X)$ is usually called the *Ornstein-Uhlenbeck operator*. Notice that, if u is a cylindrical function, that is $u(x) = v(y)$ with $y = \Pi_m(x) \in \mathbb{R}^m$ and $m \in \mathbb{N}$, then

$$\Delta_\gamma u = \sum_{j=1}^m \partial_{j,j} u - \langle x_j^*, x \rangle \partial_j u = \Delta v - \langle y, \nabla v \rangle_{\mathbb{R}^m}. \quad (16)$$

We write $u \in C(X)$ if $u : X \rightarrow \mathbb{R}$ is continuous and $u \in C^1(X)$ if both $u : X \rightarrow \mathbb{R}$ and $\nabla_\gamma u : X \rightarrow H$ are continuous.

For simplicity of notation, from now on we will omit the explicit dependence on γ of operators and spaces. We also indicate by $[\cdot, \cdot]$ and $|\cdot|$ respectively the scalar product and the norm in H . When no confusion is possible, we shall also write u_i to indicate the derivative $\partial_i u$.

3. Proof of Theorem 1.2. Recalling the integration by parts formula (13), equation (8) can be written in a weak form as

$$\int_X [\nabla u, \nabla \varphi] - f(u)\varphi \, d\gamma = 0 \quad \text{for any } \varphi \in W^{1,2}(X) \quad (17)$$

which is meaningful for $u \in W^{1,2}(X)$. Notice that, as $\mathcal{FC}_b^1(X)$ is dense in $W^{1,2}(X)$, it is enough to require (17) for all $\varphi \in \mathcal{FC}_b^1(X)$.

Remark 3.1. Since $L^\infty(X) \subset L^2(X)$, by [16, Th. 4.1] we have that a bounded weak solution of (8) belongs to $W^{2,2}(X)$.

3.1. The linearized equation. We now consider the equation solved by the derivatives of the solution u .

Lemma 3.2. *Let $u \in W^{2,2}(X)$ satisfy (8). For any $i \in \mathbb{N}$ let $u_i = \partial_i u \in W^{1,2}(X)$, then*

$$\int_X [\nabla u_i, \nabla \varphi] - f'(u)u_i\varphi + u_i\varphi \, d\gamma = 0 \quad \text{for any } \varphi \in W^{1,2}(X). \quad (18)$$

Proof. Notice first that it is enough to prove (18) for all $\varphi \in \mathcal{FC}_b^2(X)$. Letting $\varphi \in \mathcal{FC}_b^2(X)$, we multiply (8) by φ_i and recall (15), to get

$$\begin{aligned} 0 &= \int_X [\nabla u, \nabla \varphi_i] - f(u)\varphi_i \, d\gamma \\ &= \int_X -[\nabla u_i, \nabla \varphi] + \langle x_i^*, x \rangle [\nabla u, \nabla \varphi] - f(u)\varphi_i \, d\gamma \\ &= \int_X -[\nabla u_i, \nabla \varphi] + \langle x_i^*, x \rangle [\nabla u, \nabla \varphi] + f'(u)u_i\varphi - \langle x_i^*, x \rangle x_i f(u)\varphi \, d\gamma \\ &= \int_X -[\nabla u_i, \nabla \varphi] + [\nabla u, \nabla (\langle x_i^*, x \rangle \varphi) - \varphi \nabla \langle x_i^*, x \rangle] + f'(u)u_i\varphi \\ &\quad - \langle x_i^*, x \rangle x_i f(u)\varphi \, d\gamma \\ &= \int_X -[\nabla u_i, \nabla \varphi] - \varphi [\nabla u, \nabla \langle x_i^*, x \rangle] + f'(u)u_i\varphi \, d\gamma, \end{aligned}$$

where the last inequality follows from (17), with φ replaced by $\langle x_i^*, x \rangle \varphi$. \square

3.2. A variational inequality implied by the monotonicity. The next result shows that monotone solutions of (8) satisfy a variational inequality. In the Euclidean case, this fact boils down to the classical stability condition (namely, the second derivative of the energy functional being nonnegative). Differently from this, in our case, a negative eigenvalue appears in the inequality.

Lemma 3.3. *Let $u \in W^{2,2}(X)$ satisfy (8) and (10). Then, for any $\varphi \in W^{1,2}(X)$ it holds*

$$\int_X |\nabla \varphi|^2 - f'(u)\varphi^2 \, d\gamma \geq - \int_X \varphi^2 \, d\gamma. \quad (19)$$

Proof. The proof is a variation of a classical technique (see, e.g., [1, 11]).

Without loss of generality we may assume $w = h_1$, and we let $\varphi \in W^{1,2}(X)$ be such that $\varphi^2/u_1 \in W^{1,2}(X)$. Notice that, thanks to (10), the space of such functions is dense in $W^{1,2}(X)$. We use (18), with $i = 1$ and test function φ^2/u_1 ,

and we obtain

$$\begin{aligned}
& \int_X f'(u)\varphi^2 - \varphi^2 d\gamma \\
&= \int_X [\nabla u_1, \nabla(\varphi^2/u_1)] d\gamma \\
&= \int_X 2(\varphi/u_1)[\nabla u_1, \nabla\varphi] - (\varphi/u_1)^2 |\nabla u_1|^2 d\gamma \\
&= \int_X |\nabla\varphi|^2 - \left| (\varphi/u_1)\nabla u_1 - \nabla\varphi \right|^2 d\gamma \\
&\leq \int_X |\nabla\varphi|^2 d\gamma. \quad \square
\end{aligned}$$

3.3. A geometric Poincaré inequality. Now we show that a suitable geometric Poincaré inequality stems naturally from solutions of (8) satisfying (19). In the Euclidean case, it boils down to the inequality discovered in [18, 19].

Lemma 3.4. *Let $u \in W^{2,2}(X)$ satisfy (8) and (19). For any $\varphi \in W^{1,\infty}(X)$ we have*

$$\int_X (|\nabla^2 u|^2 - |\nabla|\nabla u||^2)\varphi^2 d\gamma \leq \int_X |\nabla u|^2 |\nabla\varphi|^2 d\gamma \quad (20)$$

where

$$|\nabla^2 u|^2 := \sum_{i,j} u_{ij}^2.$$

Proof. We use (19) with test function $|\nabla u|\varphi$, and we see that

$$\begin{aligned}
& \int_X (f'(u) - 1)|\nabla u|^2 \varphi^2 d\gamma \\
&\leq \int_X |\nabla(|\nabla u|\varphi)|^2 d\gamma \\
&= \int_X \varphi^2 |\nabla|\nabla u||^2 + |\nabla u|^2 |\nabla\varphi|^2 + 2[\nabla|\nabla u|, \nabla\varphi] |\nabla u|\varphi d\gamma \\
&= \int_X \varphi^2 |\nabla|\nabla u||^2 + |\nabla u|^2 |\nabla\varphi|^2 + \frac{1}{2}[\nabla|\nabla u|^2, \nabla\varphi^2] d\gamma.
\end{aligned} \quad (21)$$

We now exploit (18) with test function $u_i\varphi^2$ and we get

$$\begin{aligned}
& \int_X (f'(u) - 1)u_i^2 \varphi^2 d\gamma \\
&= \int_X [\nabla u_i, \nabla(u_i\varphi^2)] d\gamma \\
&= \int_X |\nabla u_i|^2 \varphi^2 + u_i[\nabla u_i, \nabla\varphi^2] d\gamma \\
&= \int_X |\nabla u_i|^2 \varphi^2 + \frac{1}{2}[\nabla u_i^2, \nabla\varphi^2] d\gamma.
\end{aligned}$$

Summing over $i \in \mathbb{N}$, we conclude that

$$\begin{aligned}
& \int_X (f'(u) - 1)|\nabla u|^2 \varphi^2 d\gamma \\
&= \int_X |\nabla^2 u|^2 \varphi^2 + \frac{1}{2}[\nabla|\nabla u|^2, \nabla\varphi^2] d\gamma.
\end{aligned} \quad (22)$$

From (21) and (22), we conclude that

$$\begin{aligned} & \int_X |\nabla^2 u|^2 \varphi^2 + \frac{1}{2} [\nabla |\nabla u|^2, \nabla \varphi^2] d\gamma \\ & \leq \int_X \varphi^2 |\nabla |\nabla u|^2|^2 + |\nabla u|^2 |\nabla \varphi|^2 + \frac{1}{2} [\nabla |\nabla u|^2, \nabla \varphi^2] d\gamma \end{aligned}$$

which gives (20). \square

Let $u \in C^1(X) \cap L^\infty(X)$ satisfying (9), let $N \in \mathbb{N}$ and $\bar{x}_N \in X_N^\perp$. We consider the map $\psi_{N, \bar{x}_N} : \mathbb{R}^N \rightarrow \mathbb{R}$ defined as $\psi_{N, \bar{x}_N}(\underline{x}_N) := u(\underline{x}_N, \bar{x}_N)$, and let

$$\begin{aligned} \mathcal{N}_N(\bar{x}_N) & := \{ \underline{x}_N \in \mathbb{R}^N : \nabla \psi_{N, \bar{x}_N}(\underline{x}_N) \neq 0 \} \\ & = \{ \underline{x}_N \in \mathbb{R}^N : \exists i \in \{1, \dots, N\} \text{ such that } u_i(\underline{x}_N, \bar{x}_N) \neq 0 \} \end{aligned}$$

be its noncritical set. By the Implicit Function Theorem, the level set of ψ_{N, \bar{x}_N} in $\mathcal{N}_N(\bar{x}_N)$ are $(N-1)$ -dimensional hypersurfaces of class C^2 . Thus we can consider the principal curvatures of these hypersurfaces, that we denote by $\kappa_{1,N}, \dots, \kappa_{N-1,N}$, and the tangential gradient of ψ_{N, \bar{x}_N} ¹, that we denote by $\nabla_{T,N}$. We also set

$$\nabla_N u := \Pi_N \nabla u = \nabla \psi_{N, \bar{x}_N}, \quad \nabla_N^2 u := \nabla_N (\nabla_N u) = \nabla^2 \psi_{N, \bar{x}_N},$$

$$\mathcal{K}_N := \sqrt{\sum_{i=1}^{N-1} \kappa_{i,N}^2}$$

and

$$\mathcal{N}_N := \{ x = (\underline{x}_N, \bar{x}_N) \in X : \underline{x}_N \in \mathcal{N}_N(\bar{x}_N) \} = \{ x \in X : \nabla_N u(x) \neq 0 \}.$$

With this notation, we have the following

Lemma 3.5. *Let $u \in C^1(X) \cap L^\infty(X)$ satisfy (8), (9) and (19), and fix $N \in \mathbb{N}$. For any $\varphi \in W^{1,\infty}(X)$ we have*

$$\begin{aligned} & \int_{\mathcal{N}_N} \left(|\nabla_N u|^2 \mathcal{K}_N^2 + |\nabla_{T,N} |\nabla_N u|^2|^2 \right) \varphi^2 d\gamma \\ & \leq \int_X \left(|\nabla^2 u|^2 - |\nabla |\nabla u|^2|^2 \right) \varphi^2 d\gamma \\ & \leq \int_X |\nabla u|^2 |\nabla \varphi|^2 d\gamma. \end{aligned} \tag{23}$$

Proof. Let

$$\begin{aligned} \mathcal{D}_N & := |\nabla_N^2 u|^2 - |\nabla_N |\nabla_N u|^2|^2 \\ & = \sum_{1 \leq i, j \leq N} u_{ij}^2 - \sum_{1 \leq i \leq N} \left[\frac{\nabla_N u}{|\nabla_N u|}, \nabla_N u_i \right]^2 \\ & = \sum_{1 \leq i, j \leq N} \left(u_{ij}^2 - \left(\frac{u_j u_{ij}}{|\nabla_N u|} \right)^2 \right). \end{aligned}$$

Since $|\nabla_{N-1} u| \leq |\nabla_N u|$ and

$$\left| \frac{u_j u_{ij}}{|\nabla_N u|} \right| \leq \frac{|u_j|}{|\nabla_N u|} |u_{ij}| \leq |u_{ij}|$$

¹The tangential gradient of a function g along a hypersurface with normal ν is $\nabla g - (\nabla g \cdot \nu)\nu$, that is, the tangential component of the full gradient

for any $i, j \leq N$, it follows that

$$\mathcal{D}_N - \mathcal{D}_{N-1} \geq \sum_{\substack{N-1 \leq i, j \leq N \\ \max(i, j) = N}} \left(u_{ij}^2 - \left(\frac{u_j u_{ij}}{|\nabla_N u|} \right)^2 \right) \geq 0$$

so that \mathcal{D}_N is non-decreasing in N . Accordingly,

$$|\nabla^2 u|^2 - |\nabla|\nabla u||^2 = \lim_{M \rightarrow +\infty} \mathcal{D}_M \geq \mathcal{D}_N \quad (24)$$

for any $N \in \mathbb{N}$. Moreover, by Stampacchia's Theorem we have that $\nabla_N |\nabla_N u| = 0$ for almost any $\underline{x}_N \in \mathbb{R}^N \setminus \mathcal{N}_N(\bar{x}_N)$, and similarly $u_{ij} = 0$ for almost any $\underline{x}_N \in \mathbb{R}^N \setminus \mathcal{N}_N(\bar{x}_N)$. Therefore

$$\mathcal{D}_N = |\nabla_N^2 u|^2 - |\nabla_N |\nabla_N u||^2 = 0 \text{ for almost any } \underline{x}_N \in \mathbb{R}^N \setminus \mathcal{N}_N(\bar{x}_N). \quad (25)$$

On the other hand, by [19, Formula (2.1)],

$$\mathcal{D}_N = |\nabla_N u|^2 \mathcal{K}_N^2 + |\nabla_{T,N} |\nabla_N u||^2 \quad \text{when } \underline{x}_N \in \mathcal{N}_N(\bar{x}_N).$$

From this, (24) and (25), we obtain

$$\begin{aligned} & \int_X \left(|\nabla^2 u|^2 - |\nabla|\nabla u||^2 \right) \varphi^2 d\gamma \\ & \geq \int_X \mathcal{D}_N \varphi^2 d\gamma \\ & = \int_{\mathcal{N}_N} \mathcal{D}_N \varphi^2 d\gamma \\ & = \int_{\mathcal{N}_N} \left(|\nabla_N u|^2 \mathcal{K}_N^2 + |\nabla_{T,N} |\nabla_N u||^2 \right) \varphi^2 d\gamma, \end{aligned}$$

which, recalling (20), implies (23). \square

3.4. A symmetry result. We now use the previous material to obtain a one-dimensional symmetry result for the N -dimensional projection of the solution. The idea of using geometric Poincaré inequalities as the ones in [18, 19] in order to obtain symmetry properties goes back to [10] and it was widely used in [11] in the finite dimensional Euclidean setting. The result we present here is the following:

Proposition 3.6. *Fix $N \in \mathbb{N}$ and $\bar{x}_N \in X_N^{\perp}$. Let $u \in C^1(X) \cap L^\infty(X)$ satisfy (8), (9) and (19). Then, the map ψ_{N, \bar{x}_N} is one-dimensional, i.e. there exists $U_{N, \bar{x}_N} : \mathbb{R} \rightarrow \mathbb{R}$ and $\omega_{N, \bar{x}_N} \in \mathbb{R}^N$, with $|\omega_{N, \bar{x}_N}| = 1$, such that*

$$u(\underline{x}_N, \bar{x}_N) = U_{N, \bar{x}_N}(\langle \omega_{N, \bar{x}_N}, \underline{x}_N \rangle) \quad (26)$$

for any $\underline{x}_N \in \mathbb{R}^N$.

Proof. We fix $R > 1$, to be taken arbitrarily large in what follows, and let $\Lambda = \max_i \lambda_i$. Let $\Phi \in C^\infty(\mathbb{R})$ be such that $\Phi(t) = 1$ if $t \leq R$, $\Phi(t) = 0$ if $t \geq R + 1$ and $|\Phi'(t)| \leq 3$ for any $t \in [R, R + 1]$. We take $\varphi(x) := \Phi(|x|)$. Then $|\nabla \varphi(x)| \leq \sqrt{\Lambda} |\Phi'(|x|)| \leq 3\sqrt{\Lambda}$, and (23) yields

$$\int_{\mathcal{N}_N \cap \{|x| \leq R\}} |\nabla_N u|^2 \mathcal{K}_N^2 + |\nabla_{T,N} |\nabla_N u||^2 d\gamma \leq 9\Lambda \int_{\{R \leq |x| \leq R+1\}} |\nabla u|^2 d\gamma. \quad (27)$$

Also, due to our assumptions on u ,

$$\int_X |\nabla u|^2 d\gamma < +\infty.$$

Therefore, by sending $R \rightarrow +\infty$ in (27), we conclude that

$$|\nabla_N u|^2 \mathcal{K}_N^2 + |\nabla_{T,N} |\nabla_N u||^2 = 0$$

for any $x \in \mathcal{N}_N$. From this and [11, Lemma 2.11] we get (26). \square

From the finite dimensional symmetry result of Proposition 3.6, one can take the limit as $N \rightarrow +\infty$ and obtain:

Corollary 3.7. *Let $u \in C^1(X) \cap L^\infty(X)$ satisfy (8), (9) and (19). Then, u is necessarily one-dimensional, i.e. there exists $U : \mathbb{R} \rightarrow \mathbb{R}$ and $\omega \in X^*$ such that*

$$u(x) = U(\langle \omega, x \rangle)$$

for any $x \in X$.

Proof. We first show that there exists $h \in H$ such that

$$\frac{\nabla u}{|\nabla u|} = h \quad \text{in } \mathcal{N} := \{x \in X : \nabla u(x) \neq 0\} = \bigcup_{N \in \mathbb{N}} \mathcal{N}_N. \quad (28)$$

Let $V \subset X$ be defined as $V = \bigcup_N H_N$. Since V is a dense subset of X , it is enough to show that (28) holds in $\mathcal{N} \cap V = \bigcup_N V_N$, where $V_N := \mathcal{N}_N \cap H_N$.

However, from Proposition 3.6 we know that

$$\frac{\nabla_N u}{|\nabla_N u|} = \omega_{N,0} \quad \text{in } V_N, \quad (29)$$

which implies that

$$\frac{\nabla u}{|\nabla u|} = \lim_{N \rightarrow \infty} \frac{\nabla_N u}{|\nabla_N u|} = \lim_{N \rightarrow \infty} \omega_{N,0} =: h \quad \text{in } V.$$

From (28) it follows that there exists a function $U : \mathbb{R} \rightarrow \mathbb{R}$ such that $U(t) = u(th)$ for all $t \in \mathbb{R}$, and

$$u(x) = U(\hat{h}(x)) \quad x \in X. \quad (30)$$

Moreover, U is a bounded non-decreasing solution to the ODE

$$U'' - tU' + f(U) = 0 \quad t \in \mathbb{R}.$$

Being u continuous, if U is nonconstant (otherwise the thesis follows immediately) then the function \hat{h} is also continuous, so that $h \in QX^*$ and $\hat{h}(x) = \langle \omega, x \rangle$ for some $\omega \in X^*$, which implies the thesis. \square

3.5. Proof of Theorem 1.2. The proof of Theorem 1.2 follows directly from Lemma 3.3 and Corollary 3.7. \square

Remark 3.8. We observe that, in the infinite dimensional case, there may exist weak solutions to (8), satisfying (10), which are not continuous. Indeed, given $U : \mathbb{R} \rightarrow \mathbb{R}$ satisfying (31) and (32) below, the function $u(x) = U(\hat{h}(x))$ in (30) is a solution to (8), monotone in the direction given by h , for any $h \in H$. However, such a solution is continuous only if $h \in QX^*$. As a possible generalization of Theorem 1.2, one could ask if any bounded weak solution to (8), satisfying $[\nabla u, w] > 0$ for some $w \in H$, is of this form.

4. Heteroclinic solutions. The results in Theorems 1.1 and 1.2 may be seen either as classification results (when one knows explicitly the solutions of the associated one-dimensional problem) or as nonexistence result (when the associated one-dimensional problem does not admit any solution). For this, we now give some simple conditions on the nonlinearity f ensuring existence or nonexistence of bounded solutions to the ODE

$$U'' - tU' + f(U) = 0 \quad t \in \mathbb{R} \quad (31)$$

satisfying

$$U' > 0 \quad t \in \mathbb{R}. \quad (32)$$

Notice that, from (32) it follows that there exist $U^\pm \in \mathbb{R}$, with $U^- < U^+$, such that

$$\lim_{t \rightarrow \pm\infty} U(t) = U^\pm. \quad (33)$$

Notice also that

$$\int_0^{+\infty} U'(\tau) d\tau = U^+ - U(0) < +\infty.$$

Thanks to this and (32) we can apply Lemma A.4 with $v := U'$ and obtain the existence of a sequence $\theta_n^+ \rightarrow +\infty$ such that

$$\lim_{n \rightarrow +\infty} \theta_n^+ U'(\theta_n^+) = 0 \quad \text{and} \quad \lim_{n \rightarrow +\infty} U''(\theta_n^+) = 0.$$

Similarly there is a sequence $\theta_n^- \rightarrow -\infty$ such that

$$\lim_{n \rightarrow +\infty} \theta_n^- U'(\theta_n^-) = 0 \quad \text{and} \quad \lim_{n \rightarrow +\infty} U''(\theta_n^-) = 0.$$

Consequently, from (33) and (32) we can deduce that

$$f(U^-) = f(U^+) = 0. \quad (34)$$

Now we state a nonexistence result.

Proposition 4.1. *Assume that there exists $U_0 \in (U^-, U^+)$ such that*

$$f \geq 0 \text{ in } [U^-, U_0] \quad \text{or} \quad f \leq 0 \text{ in } [U_0, U^+].$$

Then, there are no solutions to (31) satisfying (32).

Proof. Let us assume that $f \leq 0$ in $[U_0, U^+]$, since the argument is analogous in the other case, and assume by contradiction that we are given a solution U of (31), (32).

Letting $t_0 > 0$ be such that $u(t_0) \in [U_0, U^+]$, we have that U satisfies the differential inequality

$$U'' \geq tU' \quad \text{for all } t \in [t_0, +\infty),$$

which implies, by direct integration,

$$U'(t) \geq U'(t_0)e^{(t^2-t_0^2)/2} \geq U'(t_0) > 0 \quad \text{for all } t \in [t_0, +\infty),$$

contradicting (33). \square

We consider the potential $F : \mathbb{R} \rightarrow \mathbb{R}$, defined as

$$F(t) = - \int_0^t f(s) ds + k.$$

where $k \in \mathbb{R}$. Notice that, if F is convex or concave, from (34) it follows that $f \equiv 0$ in $[U^-, U^+]$, so that by Proposition 4.1 there are no solutions to (31) satisfying (32).

Given $U : (0, +\infty) \rightarrow \mathbb{R}$, we let

$$G(U) := \int_0^{+\infty} \left(\frac{(U'(t))^2}{2} + F(U(t)) \right) d\gamma(t) \quad (35)$$

where $d\gamma(t) = e^{-t^2/2} dt$. Notice that (31) is the Euler-Lagrange equation of G .

As a counterpart of the nonexistence result in Proposition 4.1, we now give an existence result for monotone solutions to (31).

Proposition 4.2. *Assume that F satisfies the following properties:*

$$\begin{aligned} F(c) = F(-c) = 0 & \quad \text{for some } c > 0 \\ F(r) > 0 & \quad \text{for any } r \notin \{c, -c\} \\ F(r) = F(-r) & \quad \text{for any } r \in [0, +\infty). \\ f(r) = 0 & \quad \text{iff } r \in \{c, -c, 0\}. \end{aligned} \quad (36)$$

Assume also that there exists $U \in W_\gamma^{1,2}((0, +\infty))$ such that $U(0) = 0$ and

$$G(U) < G(0) = \sqrt{\frac{\pi}{2}} F(0). \quad (37)$$

Then, there exists a monotone solution to (31), connecting $-c$ to c .

Proof. Let \bar{U} be a solution to the minimum problem

$$\min \{ G(U) : U \in W_\gamma^{1,2}((0, +\infty)), U(0) = 0 \}. \quad (38)$$

Note that (36) implies

$$G(\min(|\bar{U}|, c)) \leq G(\bar{U}),$$

so that we may assume $\bar{U}(t) \in [0, c]$ for all $t \in (0, +\infty)$.

Let now \bar{U}^* be the Ehrhard rearrangement of \bar{U} [9], which is defined in such a way that \bar{U}^* is non-decreasing on $(0, +\infty)$, and

$$\gamma(\{t : \bar{U}^*(t) > r\}) = \gamma(\{t : \bar{U}(t) > r\}) \quad \text{for all } r \in (0, c).$$

Notice that $\bar{U}^*(0) = 0$ and $\bar{U}(t) \in [0, c]$ for all $t \in (0, +\infty)$. By [9] (see also [13, Prop. 3.12]), we have $\bar{U}^* \in W_\gamma^{1,2}((0, +\infty))$ and

$$\begin{aligned} \int_0^{+\infty} \frac{(U^{*\prime}(t))^2}{2} d\gamma(t) & \leq \int_0^{+\infty} \frac{(U'(t))^2}{2} d\gamma(t) \\ \int_0^{+\infty} F(U^*(t)) d\gamma(t) & = \int_0^{+\infty} F(U(t)) d\gamma(t), \end{aligned}$$

so that

$$G(\bar{U}^*) \leq G(\bar{U}).$$

In particular, we may assume that $\bar{U} = \bar{U}^*$, i.e. that \bar{U} is non-decreasing on $(0, +\infty)$.

As $U = c$ and $U = 0$ are solutions to (31), which is the Euler-Lagrange equation of G , we get that either $\bar{U} = 0$ or $\bar{U} = c$ or

$$\bar{U}(t) \in (0, c) \quad \text{for all } t \in (0, +\infty). \quad (39)$$

On the other hand, thanks to (37) and the fact that $\bar{U}(0) = 0$, we can exclude the first two possibilities, so that (39) holds. Moreover, since \bar{U} is non-decreasing and $f(r) \neq 0$ for all $r \in (0, c)$, it follows that $\bar{U}'(t) > 0$ for all $t \in (0, +\infty)$ and

$$\lim_{t \rightarrow +\infty} \bar{U}(t) = c.$$

Since by (36) the function $t \rightarrow -\bar{U}(-t)$ is a monotone solution to (31) on $(-\infty, 0)$, we get that the odd extension of \bar{U} on \mathbb{R} is a solution to (31) on the whole of \mathbb{R} which satisfies (32) and connects $-c$ to c . \square

Remark 4.3. Notice that for all $U \in W_\gamma^{1,2}((0, +\infty))$ we have

$$G(U) < \tilde{G}(U) := \int_0^{+\infty} \left(\frac{(U'(t))^2}{2} + F(U(t)) \right) dt.$$

If we let \bar{U} be the unique solution to

$$\begin{aligned} U''(t) + f(U(t)) &= 0 & t \in (0, +\infty) \\ U(0) &= 0 \\ \lim_{t \rightarrow +\infty} U(t) &= c, \end{aligned}$$

we have

$$G(\bar{U}) < \tilde{G}(\bar{U}) = \int_0^c \sqrt{2F(r)} dr.$$

In particular, condition (37) is verified whenever

$$\int_0^c \sqrt{2F(r)} dr \leq \sqrt{\frac{\pi}{2}} F(0)$$

which is satisfied, for instance, by the standard double-well potential $F(t) = (1 - t^2)^2/4$.

Appendix A. Asymptotics of functions of one variable. The purpose of this appendix is to collect some observations that are elementary in nature but useful for the proof of (34).

A.1. Asymptotics of monotone functions. We point out a simple result on the limit of monotone functions:

Lemma A.1. *Let $t_o \in \mathbb{R}$ and $w : [t_o, +\infty) \rightarrow [0, +\infty)$ be non-increasing. Suppose that*

$$\int_{t_o}^{+\infty} w(t) dt < +\infty.$$

Then

$$\lim_{t \rightarrow +\infty} tw(t) = 0.$$

Proof. Suppose not. Then there exists $c > 0$ and a diverging sequence t_n for which $t_n w(t_n) = |t_n w(t_n)| \geq c$. Up to a subsequence we may assume that $t_n > 2t_{n-1}$, that is $t_n - t_{n-1} > t_n/2$. Also, by monotonicity, $w(t) \geq w(t_n)$ for all $t \in (t_{n-1}, t_n)$. We obtain that

$$\begin{aligned} \int_{t_o}^{+\infty} w(t) dt &\geq \sum_{n=2}^{+\infty} \int_{t_{n-1}}^{t_n} w(t) dt \geq \sum_{n=2}^{+\infty} w(t_n)(t_n - t_{n-1}) \\ &\geq \sum_{n=2}^{+\infty} w(t_n)t_n/2 \geq \sum_{n=2}^{+\infty} c/2 = +\infty, \end{aligned}$$

that is in contradiction with our assumptions. \square

A.2. The non-increasing lower envelope of a function. Now we introduce the non-increasing lower envelop of a function v and we use it to deduce some asymptotic properties. In this section, we take $t_o \in \mathbb{R}$ and $v \in C([t_o, +\infty))$, and we define its non-increasing lower envelope by

$$V(t) := \min_{[t_o, t]} v.$$

By construction $V(t) \leq v(t)$ for any $t \in [t_o, +\infty)$ and V is non-increasing. Also,

$$\text{if } v(t) > 0 \text{ for all } t \geq t_o, \text{ then } V(t) > 0 \text{ for any } t \geq t_o. \quad (40)$$

Lemma A.2. $V \in C([t_o, +\infty))$.

Proof. Fix $\epsilon > 0$ and $a \geq t_o$. By the continuity of v , there exists $\delta > 0$ such that for any $b \geq t_o$ with $|a - b| \leq \delta$ we have that $|v(a) - v(b)| \leq \epsilon$. Let $A := \min\{a, b\}$ and $B := \max\{a, b\}$. Notice that $\{A, B\} = \{a, b\}$ and $|A - B| = |a - b| \leq \delta$, therefore

$$\min_{[A, B]} v \geq v(A) - \epsilon.$$

Then, using the monotonicity of V ,

$$\begin{aligned} |V(a) - V(b)| &= |V(A) - V(B)| = V(A) - V(B) = \min_{[t_o, A]} v - \min_{[t_o, B]} v \\ &= \min_{[t_o, A]} v - \min \left\{ \min_{[t_o, A]} v, \min_{[A, B]} v \right\} \\ &\leq \min_{[t_o, A]} v - \min \left\{ \min_{[t_o, A]} v, v(A) - \epsilon \right\} \\ &\leq \min_{[t_o, A]} v - \min \left\{ \min_{[t_o, A]} v, v(A) \right\} + \epsilon \\ &= \min_{[t_o, A]} v - \min_{[t_o, A]} v + \epsilon = \epsilon. \quad \square \end{aligned}$$

Lemma A.3. *Suppose that $v(t) > 0$ for all $t \geq t_o$ and that*

$$\int_{t_o}^{+\infty} v(t) dt < +\infty.$$

Then there exists a diverging sequence t_n for which $V(t_n) = v(t_n)$.

Proof. Suppose not. Then there exists $t_\star > t_o$ such that $V(t) < v(t)$ for all $t \geq t_\star$ and so

$$\min_{[t_\star, t]} v > \min_{[t_\star, t]} V \quad \text{for all } t \geq t_\star.$$

Since V is non-increasing, we obtain that the latter quantity equals to $V(t)$, hence

$$\min_{[t_\star, t]} v > V(t) = \min_{[t_o, t]} v \quad \text{for all } t \geq t_\star.$$

Consequently, for all $t \geq t_\star$, the minimum of v in $[t_o, t]$ is in fact attained in $[t_o, t_\star]$. Therefore

$$v(t) > V(t) = \min_{[t_o, t]} v = \min_{[t_o, t_\star]} v = V(t_\star)$$

for all $t \geq t_\star$, and the latter quantity is strictly positive, thanks to (40). It follows that

$$\int_{t_o}^{+\infty} v(t) dt \geq \int_{t_\star}^{+\infty} v(t) dt \geq \int_{t_o}^{+\infty} V(t_\star) dt = +\infty,$$

that is in contradiction with our assumptions. \square

Lemma A.4. *Suppose that $v \in C^1((t_o, +\infty))$, $v(t) > 0$ for all $t \geq t_o$ and that*

$$\int_{t_o}^{+\infty} v(t) dt < +\infty.$$

Then there exists a diverging sequence θ_n for which

$$\lim_{n \rightarrow +\infty} \theta_n v(\theta_n) = 0 \quad \text{and} \quad \lim_{n \rightarrow +\infty} v'(\theta_n) = 0. \quad (41)$$

Proof. First of all, we observe that

$$\int_{t_o}^{+\infty} V(t) dt \leq \int_{t_o}^{+\infty} v(t) dt < +\infty,$$

so by Lemma A.1

$$\lim_{t \rightarrow +\infty} tV(t) = 0. \quad (42)$$

Now we take t_n as in Lemma A.3: then we have that the set

$$I_n := \left\{ T \geq t_n \text{ s.t. } V(t) = v(t) \text{ for all } t \in [t_n, T] \right\}$$

contains t_n and so it is non-empty. Hence we can define

$$T_n := \sup I_n \in [t_n, +\infty]. \quad (43)$$

We distinguish two cases: either $T_{n_*} = +\infty$ for some n_* or T_n is always finite.

In the first case, we have that $V(t) = v(t)$ for any $t \geq t_{n_*}$ and so v is non-increasing and, from (42),

$$\lim_{t \rightarrow +\infty} tv(t) = 0. \quad (44)$$

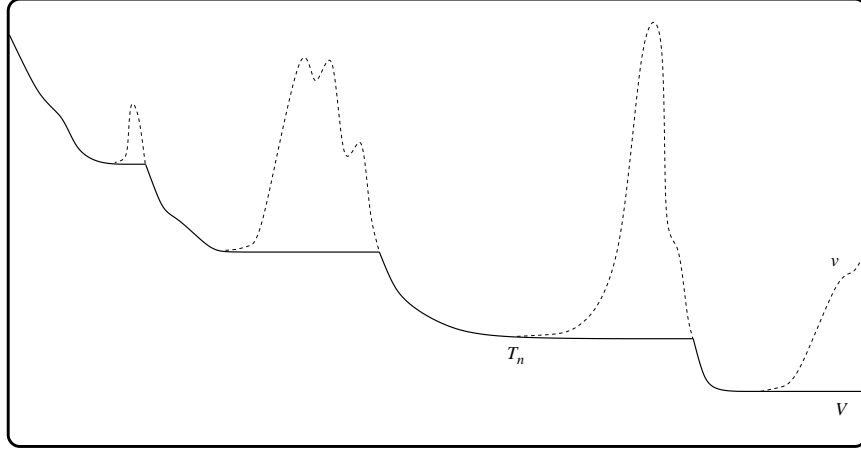
We show that $v'(\sigma_n)$ is infinitesimal for an appropriate sequence σ_n . If this were not true, then there exists $\tilde{t} > t_o$ and $c > 0$ such that $-v'(t) = |v'(t)| \geq c$ for any $t \geq \tilde{t}$. As a consequence

$$v(t) = v(\tilde{t}) + \int_{\tilde{t}}^t v'(\tau) d\tau \leq v(\tilde{t}) - c(t - \tilde{t}) < 0$$

for t large, and this is against the positivity of v . This shows the existence of a sequence for which $v'(\sigma_n)$ is infinitesimal, that, together with (44), proves (41) in the first case (here $\theta_n := \sigma_n$).

It remains to prove (41) in the second case, that is we suppose that T_n is always finite. Then $T_n \in I_n$ due to the continuity of V (recall Lemma A.2), that is $V(T_n) = v(T_n)$. We show that

$$v'(T_n) = 0. \quad (45)$$



Suppose indeed that $v'(T_n) > 0$. Then, if we take $t < T_n$ with t close enough to T_n we obtain

$$V(T_n) = v(T_n) > v(t) \geq V(t),$$

which is a contradiction since V is non-increasing. Similarly, if we suppose that $v'(T_n) < 0$, we would have that v' is strictly decreasing near T_n hence there would be $\tilde{T}_n > T_n$ such that for any $t \in [T_n, \tilde{T}_n]$

$$\min_{[T_n, t]} v < v(T_n) = V(T_n) = \min_{[t_o, T_n]} v$$

and so the minimum of v in $[t_o, t]$ is attained in $[T_n, t]$, for any $t \in [T_n, \tilde{T}_n]$. Therefore

$$v(t) = \min_{[T_n, t]} v = \min_{[t_o, t]} v = V(t)$$

for any $t \in [T_n, \tilde{T}_n]$. This would give that $\tilde{T}_n \in I_n$ and this contradicts (43). So (45) is proved.

Also, by (42), we have

$$\lim_{n \rightarrow +\infty} T_n v(T_n) = \lim_{n \rightarrow +\infty} T_n V(T_n) = 0. \quad (46)$$

Putting together (45) and (46) we complete the proof of (41) in the second case (here we take $\theta_n := T_n$). \square

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