ON THE STRUCTURE OF FIXED-POINT SETS OF NONEXPANSIVE MAPPINGS

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Topic #4: *Nonstandard Methods in Functional Analysis.*

The so-called metric fixed point theory is now a well-developed branch of fixed point theory with its own methods and problems but also with links to other fields such as geometry of Banach spaces, integral and differential equations, multivalued analysis and others.

Let C be a nonempty, bounded, closed and convex subset of a Banach space X. A mapping $T : C \to C$ is said to be nonexpansive if

$$\|Tx - Ty\| \le \|x - y\|$$

for all $x, y \in C$. It follows from the Banach contraction principle that there exists a sequence (x_n) of elements in C such that

$$\lim_{n\to\infty} \|Tx_n - x_n\| = 0.$$

A natural question arises whether, for a given two commuting nonexpansive mappings $T, S : C \to C$, there exists a sequence (x_n) in C such that

$$\lim_{n\to\infty} \|Tx_n - x_n\| = \lim_{n\to\infty} \|Sx_n - x_n\| = 0.$$

This is one of the long-standing problems in the theory.

Notice that the above problem has an intrinsic interpretation if we consider the nonstandard hull of a Banach space X:

$$\text{Does there exist } \hat{x} \in \hat{C} \text{ such that } \hat{T}\hat{x} = \hat{S}\hat{x} = \hat{x} \text{ ?}$$

One can prove that the answer is affirmative if there exists a nonexpansive mapping $r : \hat{C} \to \mathrm{Fix}\ \hat{T}$ such that $rx = x$ for $x \in \mathrm{Fix}\ \hat{T}$, where $\mathrm{Fix}\ \hat{T}$ denotes the set of fixed points of $\hat{T}$ (which is always nonempty).

In the present talk we show that for any countable set $A \subset \mathrm{Fix}\,\widetilde{T}$ there exists a nonexpansive mapping $r : \hat{C} \to \mathrm{Fix}\ \hat{T}$ such that $rx = x$ for $x \in A$. The strength of techniques from nonstandard analysis is compared with techniques based on the Banach space ultrapower construction.

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