A FUNCTIONAL CHARACTERIZATION OF NONSTANDARD MODELS

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Elementary approaches to nonstandard models

Many introductions to nonstandard methods are now available, which are "elementary" in the sense that they use only "every day" mathematics, without any appeal to notions and results from mathematical logic proper.

The survey [BDNF06] presents various such "elementary" introductions of the nonstandard methods, considered by the authors in the last decade, including different general characterizations of nonstandard models given by purely algebraic or topological means.

The topological approach of [DNF05], focusing on continuous extensions, makes it apparent that only in Hausdorff spaces, where functions have *unique* continuous extensions, the topology is really responsible of the nonstandard structure. In the general case, when uniqueness gets lost, it is rather the choice of a *distinguished continuous extension* made by the "$*$" operator that "induces" a topology on ${}^*X$.

These considerations suggest that "purely functional" conditions could characterize the nonstandard extensions of arbitrary sets, without any mention of topological or algebraic structure.

The paper [H87], where general conditions on extensions of n-ary relation characterize the nonstandard models of the real numbers, can be viewed as a first attempt in this direction.

THE FUNCTIONAL APPROACH

The main feature of all nonstandard models of Analysis is the existence of a canonical extension ${}^*f : {}^*\mathbb{R} \to {}^*\mathbb{R}$ of any (standard) function $f : \mathbb{R} \to \mathbb{R}$ (and also of any subset $A \subseteq \mathbb{R}$).

Given any set X, we consider an arbitrary superset ${}^*X$ of X, equipped with an operator $^* : X^X \to {}^*X^{{}^*X}$, which provides a *distinguished extension* of each function $f : X \to X$.

It turns out that *three simple, natural, purely functional* conditions isolated in [F??] are all what is needed for the strongest requirements of nonstandard models.

We assume that $0, 1 \in X$, so as to have at disposal the extensions of characteristic functions. These extensions being characteristic functions on ${}^*X$, we use them in extending subsets of X to subsets of ${}^*X$.

The *nonstandard models* are intended to preserve those properties of the standard structure which are currently being considered. The *Transfer* (*Leibniz*) *Principle* states that those properties that are expressible in a(n in)sufficiently expressive language are preserved.

In particular, an important property of nonstandard models of Analysis is that "disjoint functions have disjoint extensions". This property has a clear "analytic" flavour, and in fact it can be considered as the most characteristic feature of *nonstandard* extensions when compared with *continuous* extensions of functions in compactifications or topological completions, where equality may be reached only at limit points. So we should reasonably postulate this property, together with preservation of characteristic functions.

THE FUNCTIONAL EXTENSIONS

With this in mind, in the following definition we assume that "compositions and diagonals are preserved":

Definition. A superset ${}^*X$ of the set X is a *functional extension* of X if to every function $f : X \to X$ is associated a distinguished extension ${}^*f : {}^*X \to {}^*X$ in such a way that, for all $f, g : X \to X$ and all $\xi \in {}^*X$

(comp) $\quad {}^*g({}^*f(\xi)) = {}^*(g \circ f)(\xi)$,

(diag) $\quad {}^*(\chi \circ (f, g))(\xi) = \begin{cases} 1 & \text{if } {}^*f(\xi) = {}^*g(\xi) \\ 0 & \text{otherwise} \end{cases}$

where $\chi : X \times X \to \{0, 1\}$ is the characteristic function of the diagonal, i.e. $\chi(x, y) = 1 \iff x = y$.

PRESERVATION PROPERTIES

Theorem. *Let* ${}^*X$ *be a functional extension of* X. *Then*

- *if* $c_x : X \to X$ *is the constant function with value* $x \in X$, *then the extension* ${}^*c_x$ *is the constant with value* x *on* ${}^*X$;
- *if* $\chi_A : X \to X$ *is the characteristic function of* $A \subseteq X$, *then the extension* ${}^*\chi_A$ *is the characteristic function of a superset* ${}^*A$ *of* A *in* ${}^*X$.
- *The map* $* : A \mapsto {}^*A$ *commutes with binary union, intersection and complement. Moreover* ${}^*A \cap X = A$, *hence* $*$ *is a boolean isomorphism of* $\mathcal{P}(X)$ *onto a subfield* $St({}^*X)$ *of* $\mathcal{P}({}^*X)$.

Having defined $*$-extensions of sets, the property (diag) yields a sort of "preservation of equalizers", which corresponds to a generalization of the "analytic" property mentioned above, namely that "standard functions behave like germs":

Corollary. *Let ${}^*X$ be a functional extension of X. Then*

$$\{\xi \in {}^*X \mid {}^*f(\xi) = {}^*g(\xi)\} = {}^*\{x \in X \mid f(x) = g(x)\}$$

for all $f, g : X \to X$, or equivalently, for all $\xi \in {}^*X$,

$${}^*f(\xi) = {}^*g(\xi) \iff \exists A \subseteq X. (\xi \in {}^*A \;\; \& \; \forall x \in A \; . f(x) = g(x)\,).$$

THE IDENTITY MAP

The identity map may not be preserved by functional extensions.

If $\imath : X \to X$ is the identity of X, in the general case one only obtains

$$ {}^*\!f(\xi) = {}^*\!f({}^*\imath(\xi)) = {}^*\imath({}^*\!f(\xi)) \quad \text{for all } f : X \to X \text{ and all } \xi \in {}^*X. $$

Thus ${}^*\imath$ is the identity exactly on those points of ${}^*X$ which are reached by some function ${}^*\!f$, and all functions ${}^*\!f$ map each nonstandard point ξ to the same point as ${}^*\imath(\xi)$.

When ${}^*\imath$ is not the identity, the extension ${}^*X$ can be considered "redundant", in the sense that the extensions of all functions are completely determined by their restrictions to ${}^*\imath({}^*X)$, and the remaining elements of ${}^*X$ are not attained by any function ${}^*\!f$.

On the other hand ${}^*\imath({}^*X)$, equipped with the restrictions of all ${}^*\!f$s, becomes a functional extension where the identity is preserved.

IRREDUNDANT EXTENSIONS

Call *irredundant* a functional extension ${}^*X$ of X if

for all $\xi \in {}^*X$ there exist $f : X \to X$ and $\eta \in {}^*X$ s. t. ${}^*f(\eta) = \xi$.

Important and natural preservation properties, concerning *ranges, injectivity,* and *finite subsets* can be proved only for irredundant extensions:

Corollary. *Let ${}^*X$ be an irredundant functional extension of X. Then, for all $f : X \to X$ and all $A \subseteq X$*

- ${}^*f({}^*A) = {}^*(f(A))$ *(in particular ${}^*f$ is onto if f is onto);*
- *if $f : X \to X$ is one-one on A, then ${}^*f$ is one-one on ${}^*A$;*
- *extensions of finite sets are trivial, i.e. $A = {}^*A$ whenever A is finite.*

ACCESSIBLE EXTENSIONS

It is always assumed by nonstandard analysts that all and only infinite sets are indeed *extended*, i.e. $A = {}^*A$ if and only if A is finite. Although we do not need this assumption, the other properties of the Corollary are relevant, and valid in any non-standard model. So we have in mind essentially only irredundant extensions.

Irredundancy can be also viewed as saying that "any point is *accessible* from some other point". We have not postulated this condition from the beginning because it is still *too weak* to obtain complete elementary extensions. In fact we need a slightly(?) stronger property, namely that "every *couple* of points is accessible from some point".

POINTED vs ULTRAPOWER EXTENSIONS

Let ${}^*X$ be a functional extension of X. For $\alpha \in {}^*X$ put

$$ {}^*X_\alpha = \{{}^*f(\alpha) \mid f : X \to X\} \quad \text{and} \quad \mathcal{U}_\alpha = \{A \in \mathcal{P}(X) \mid \alpha \in {}^*A\}. $$

Define the function $\delta_\alpha : X^X \to {}^*X_\alpha$ by $\delta_\alpha(f) = {}^*f(\alpha)$.

- ${}^*X_\alpha$ is the least subset of ${}^*X$ closed under all functions ${}^*f$ and containing α (${}^*\imath(\alpha)$ if ${}^*X$ is redundant and $\alpha \notin {}^*\imath({}^*X)$). Hence ${}^*X_\alpha$ is an irredundant functional extension of X.

- $\mathcal{U}_\alpha$ is an ultrafilter over X and $\delta_\alpha(f) = \delta_\alpha(g)$ if and only if f and g agree on a set $U \in \mathcal{U}_\alpha$. Hence δ_α induces a bijection $\psi_\alpha : X^X/\mathcal{U}_\alpha \to {}^*X_\alpha$ such that ${}^*g \circ \psi_\alpha = \psi_\alpha \circ \overline{g}$ for all $g \in X^X$.
  ($\overline{g}$ is g modulo $\mathcal{U}_\alpha$)

We say that ${}^*X$ is a *pointed extension* if it is equal to ${}^*X_\alpha$ for some $\alpha \in {}^*X$, and we call any such α a *generator* of ${}^*X$.
(We shall always assume w.l.o.g. that $\alpha \in {}^*X_\alpha$.)

Theorem. *A functional extension ${}^*X$ of X is isomorphic to an ultrapower $X^X/\mathcal{U}$ of X modulo an ultrafilter $\mathcal{U}$ over X if and only if ${}^*X$ is pointed.*
In this case, there exists a unique $\alpha \in {}^*X$ such that $\mathcal{U} = \mathcal{U}_\alpha$. This element α is a generator of ${}^*X$, and the canonical map ψ_α is the unique isomorphism between $X^X/\mathcal{U}$ and ${}^*X$.

ACCESSIBILITY OF PAIRS

Definition. A functional extension ${}^*X$ of X is *accessible* if any two elements $\xi, \eta \in {}^*X$ belong to some pointed subextension ${}^*X_\zeta$, or equivalently for all $\xi, \eta \in {}^*X$ there exists $\zeta \in {}^*X$ such that

(acc) $\quad {}^*f(\zeta) = \xi \quad$ and $\quad {}^*g(\zeta) = \eta \quad$ for suitable $f, g : X \to X$

All properties considered up to now, in particular the defining properties (comp) and (diag), are instances of transfer. This seems *prima facie* not to apply to the condition (acc), whose direct formalization is *second-order*.
On the contrary, let $\delta : X \to X \times X$ be a bijective function, and let p_1, p_2 be the compositions of δ with the projections $\pi_i : X \times X \to X$. Then the following strong uniform version of accessibility is an instance of transfer

- *for all* $\xi, \eta \in {}^*X$ *there is a unique* $\zeta \in {}^*X$ *such that* ${}^*p_1(\zeta) = \xi,\ {}^*p_2(\zeta) = \eta$.

In order to get full transfer, you need accessibility of pairs!

LIMIT ULTRAPOWERS

By Keisler's Theorem, a structure $\mathfrak{Y}$ is a complete elementary extension of a structure $\mathfrak{X}$ if and only if $\mathfrak{Y}$ is isomorphic to a limit ultrapower of $\mathfrak{X}$.

Recall that the *limit ultrapower* $X^I/\mathcal{D}|\mathcal{E}$, where $\mathcal{D}$ is an ultrafilter over I and $\mathcal{E}$ is a filter of equivalences on I, is the subset of the ultrapower $X^I/\mathcal{D}$ containing the $\mathcal{D}$-equivalence classes of all functions $f : I \to X$ that induce on I an equivalence $Eq(f) = \{(x,y) \mid f(x) = f(y)\}$ belonging to $\mathcal{E}$.
(one can assume w.l.o.g. that each $E \in \mathcal{E}$ gives a partition of size not exceeding $|X|$.)

Given an accessible functional extension ${}^*X$ of X, we go to define a limit ultrapower $X^I/\mathcal{D}|\mathcal{E}$ isomorphic to ${}^*X$.

Let ${}^*X$ be an accessible functional extension of X, and for $\alpha \in {}^*X$

- put $\breve{\alpha} = \{\xi \in {}^*X \mid \alpha \in {}^*X_\xi\}$, and let $\mathcal{V}$ be an ultrafilter over ${}^*X$ including the family $\{\breve{\alpha} \mid \alpha \in {}^*X\}$ (which has the finite intersection property);

- define $\widehat{\alpha} : {}^*X \times X \to X$ by $\widehat{\alpha}(\xi, x) = \begin{cases} f_{\xi\alpha}(x) & \text{if } \xi \in \breve{\alpha} \\ x & \text{otherwise} \end{cases}$

  (where $f_{\xi\alpha} : X \to X$ is fixed so as to have ${}^*f_{\xi\alpha}(\xi) = \alpha$ for all $\xi \in \breve{\alpha}$);

- let $\mathcal{E}$ be the filter of equivalences on $I = {}^*X \times X$ generated by the equivalences $E_\alpha = \{(i,j) \in I \times I \mid \widehat{\alpha}(i) = \widehat{\alpha}(j)\}$ (NB: $\beta, \gamma \in {}^*X_\alpha \implies E_\alpha \subseteq E_\beta \cap E_\gamma$);

- let $\mathcal{D} = \sum_{\mathcal{V}} \mathcal{U}_\xi$ be the ultrafilter on I such that

$$D \in \mathcal{D} \iff \{\xi \in {}^*X \mid \{x \in X \mid (\xi, x) \in D\} \in \mathcal{U}_\xi\} \in \mathcal{V}$$

(where $\mathcal{U}_\xi$ is the ultrafilter on X generated by ξ)

Then the map $\widehat{\ }: {}^*X \to X^I/\mathcal{D}|\mathcal{E}$ is an isomorphism

(of structures for the language $\mathcal{L} = \{f \mid f \in X^X\}$).

EXTENDING n-ARY FUNCTIONS

A *unique unambiguous "parametric"* definition of the extension ${}^*\varphi$ of any n-ary function $\varphi : X^n \to X$ can be given without recurring to the machinery of any isomorphic limit ultrapower:

Theorem. *Let ${}^*X$ be an accessible functional extension of X. There is a unique way of assigning an extension ${}^*\varphi$ to every n-ary function $\varphi : X^n \to X$ so as to preserve all compositions (i.e. ${}^*\varphi \circ ({}^*\psi_1, \dots, {}^*\psi_n) = {}^*(\varphi \circ (\psi_1, \dots, \psi_n))$ for all all $m, n \geq 1$, all $\varphi : X^n \to X$, and all $\psi_1, \dots, \psi_n : X^m \to X$), namely:*

$${}^*\varphi(\xi_1, \dots, \xi_n) = {}^*(\varphi \circ (f_1, \dots, f_n))(\zeta),$$

where $f_i : X \to X$ and $\zeta \in {}^*X$ are such that ${}^*f_i(\zeta) = \xi_i$ for $i = 1, \dots, n$.

A "LOGICAL ARGUMENT"

By using the characteristic functions in n variables one can assign an extension ${}^*R$ to each n-ary relation R over X. In this way, given a first order structure $\mathfrak{X} = \langle X; R_i\,, i \in I; F_j\,, j \in J\rangle$ and an accessible functional extension ${}^*X$ of its universe X, one produces a superstructure ${}^*\mathfrak{X} = \langle {}^*X; {}^*R_i\,, i \in I; {}^*F_j\,, j \in J\rangle$.

Proceeding by induction on the complexity of the formula Φ, one obtains that

$$\forall x_1, \ldots, x_n \in X.\ {}^*\mathfrak{X} \models \Phi[x_1, \ldots, x_n] \iff \mathfrak{X} \models \Phi[x_1, \ldots, x_n],$$

thus providing a "logical proof" that $\mathfrak{X}$ is an *elementary substructure* of ${}^*\mathfrak{X}$.

TOPOLOGICAL EXTENSIONS

Definition. [DNF05] A T_1 topological space ${}^*X$ is a *topological extension* of X if X is a *dense* subspace of ${}^*X$, and to every function $f: X \to X$ is associated a distinguished *continuous extension* ${}^*f: {}^*X \to {}^*X$ in such a way that compositions and local identities are preserved, i.e.

(c) ${}^*g \circ {}^*f = {}^*(g \circ f)$ for all $f, g: X \to X$, and

(i) if $f(x) = x$ for all $x \in A \subseteq X$, then ${}^*f(\xi) = \xi$ for all $\xi \in \overline{A}$.

The topological extension ${}^*X$ is *analytic* if, for all $f, g: X \to X$,

(d) $f(x) \neq g(x)$ *for all* $x \in X \implies {}^*f(\xi) \neq {}^*g(\xi)$ *for all* $\xi \in {}^*X$.

The topological extension ${}^*X$ is *coherent* if

(f) for all $\xi, \eta \in {}^*X$ there exist functions $p, q: X \to X$ and a point $\zeta \in {}^*X$ such that ${}^*p(\zeta) = \xi$ and ${}^*q(\zeta) = \eta$.

THE STAR-TOPOLOGY

The properties (cidf) hold in any accessible functional extension. Actually, the properties (comp, diag, acc) have been suggested by the above definition. So it is natural to look for a topology that turns an accessible functional extension into a coherent analytic topological extension.

In fact the *Star-topology* of [DNF05], whose *closed sets* are the arbitrary intersections of sets of the form

$$E(\vec{f};\vec{\eta}) = E(f_1,\ldots,f_n;\eta_1,\ldots,\eta_n) = \{\xi \in {}^*X \mid {}^*f_i(\xi) = \eta_i,\ i = 1,\ldots,n\}$$

(for all n-tuples of functions $f_i : X \to X$, and of points $\eta_i \in {}^*X$) can be given to every nonstandard model ${}^*X$ so as to make all functions ${}^*f$ continuous.

Theorem. *Every accessible functional extension ${}^*X$ of X, when endowed with the Star-topology, becomes a coherent analytic topological extension of X.*

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