Quantifiers in Limits

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Robinson's limit definition

The nonstandard approach to calculus eliminates two quantifier blocks in the limit definition. The standard definition of

$$\lim_{z \to \infty} F(z) = \infty$$

requires three quantifier blocks,

$$\forall x \, \exists y \, \forall z \, [y \leq z \Rightarrow x \leq F(z)].$$

A. Robinson 1960: For standard functions F, this is equivalent to the universal sentence

$$\forall x \, [I(x) \Rightarrow I(F(x))]$$

where $I(x)$ means x is infinite.

Quantifier hierarchies

Fix an ordered structure $\mathcal{M} = (M, \leq, \ldots)$ with no greatest element.

Hierarchy of sentences in $L(\mathcal{M}) \cup \{F\}$:
Π_n: n quantifier blocks starting with $\forall$
Σ_n: n quantifier blocks starting with $\exists$
Δ_n: Both Π_n and Σ_n
B_n: Boolean in Π_n.

$$\Delta_1 \subset \Pi_1 \subset B_1 \subset \Delta_2 \subset \Pi_2 \subset B_2 \subset \Delta_3 \subset \Pi_3,$$

$$\Delta_1 \subset \Sigma_1 \subset B_1 \subset \Delta_2 \subset \Sigma_2 \subset B_2 \subset \Delta_3 \subset \Sigma_3.$$

Problem. *Given $\mathcal{M}$, locate*
$LIM = \{(\mathcal{M}, F) : \lim_{z\to\infty} F(z) = \infty\}$
in the quantifier hierarchy.

For each $\mathcal{M}$, it's Π_3 or lower.

Cases with low quantifier level

Theorem. *For every $\mathcal{M}$, LIM is not B_1.*

Theorem. *If $\mathcal{M}$ has universe $\mathbb{R}$ and a symbol for each function definable in $(\mathbb{R}, \leq, +, \cdot, \mathbb{N})$, LIM is Δ_2.*

Proof: $\mathcal{M}$ has a symbol for a function $G(x,n)$ such that $\mathbb{R}^{\mathbb{N}} = \{G(x,\cdot) : x \in \mathbb{R}\}$.
LIM is equivalent to
$\exists x \, \forall n \, \forall y [G(x,n) \leq y \Rightarrow n \leq F(y)]$.
The negation of LIM is equivalent to
$\exists m \, \exists x \, \forall n \, [n \leq G(x,n) \wedge F(G(x,n)) \leq m]$.

Cases with maximum quantifier level

Theorem. *If $\mathcal{M}$ is countable, then LIM is not Σ_3.*

Theorem. *If $\mathcal{M} = (\mathbb{R}, \leq, \mathcal{N})$ with $\mathcal{N} = (\mathbb{N}, \ldots)$, then LIM is not Σ_3.*

Theorem. *If $\mathcal{M}$ is saturated (or special), then LIM is not Σ_3.*

K. Sullivan, Ph.D. Thesis 1974, showed that LIM is not Π_2 and not Σ_2 when $\mathcal{M}$ saturated.

Theorem. *If $\mathcal{M} = (\mathcal{K}, I)$, $\mathcal{K}$ saturated and $I = \{x : x$ is infinite $\}$, LIM is not Σ_3.*

So Robinson's result for standard functions does not extend to arbitrary functions.

Infinitely long sentences

Given a set of sentences Q,
$\bigwedge Q = \{\bigwedge_n \theta_n : \theta_n \in Q\}$
$\bigvee Q = \{\bigvee_n \theta_n : \theta_n \in Q\}$.

If $\mathcal{M}$ has universe $\mathbb{R}$ and a constant for each $n \in \mathbb{N}$, then LIM is $\bigwedge\bigvee\Pi_1$,

$$\bigwedge_m \bigvee_n \forall z\,[n \leq z \Rightarrow m \leq F(z)],$$

and LIM is $\bigwedge\Sigma_2$,

$$\bigwedge_m \exists y \forall z\,[y \leq z \Rightarrow m \leq F(z)].$$

Theorem. *If $\mathcal{M}$ has universe $\mathbb{R}$, LIM is not $\bigvee\bigwedge B_1$.*

o-minimal structures

$\mathcal{M}$ is o-minimal if every set definable in $\mathcal{M}$ with parameters is a finite $\bigcup$ of intervals and points. (Van den Dries 1984, Pillay and Steinhorn 1986).

Examples of o-minimal structures:
$(\mathbb{R}, \leq, +, \cdot)$ (Tarski 1939).
$(\mathbb{R}, \leq, +, \cdot, \exp)$ (Wilkie 1991).
Above plus restricted analytic functions (van den Dries and C. Miller 1994).

Theorem. *If $\mathcal{M}$ is an o-minimal expansion of $(\mathbb{R}, \leq, +, \cdot)$, LIM is not $\bigwedge \Pi_2$ and not $\bigvee B_2$.*

Proof uses recent results of H. Friedman and C. Miller (2005) on fast sequences.

Conjecture. *For every o-minimal expansion $\mathcal{M}$ of $(\mathbb{R}, \leq, +, \cdot)$, LIM is not Σ_3.*

Summary

Quantifier Level of LIM Over $\mathcal{M}$

$$\Delta_1 \subset \Pi_1 \subset B_1 \subset \Delta_2 \subset \Pi_2 \subset B_2 \subset \Delta_3 \subset \Pi_3$$

$\leq \Pi_3$	always
$\leq \bigwedge \Sigma_2$	$(\mathbb{R}, \leq, 0, 1, 2, \ldots)$
$\leq \bigwedge \bigvee \Pi_1$	$(\mathbb{R}, \leq, 0, 1, 2, \ldots)$
$> \Delta_3$	countable
$> \Delta_3$	$(\mathcal{M}, I)$ with $\mathcal{M}$ saturated
$> \Delta_3$	$(\mathbb{R}, \leq, (\mathbb{N}, \ldots))$
$> \bigvee B_2$	o-minimal $(\mathbb{R}, \leq, +, \cdot, \ldots)$
$> \bigwedge \Pi_2$	o-minimal $(\mathbb{R}, \leq, +, \cdot, \ldots)$
$> \bigvee \bigwedge B_1$	$(\mathbb{R}, \leq, \ldots)$
Δ_2	$(\mathbb{R}, \leq, +, \cdot, \mathbb{N}, \text{definable})$
$> B_1$	always

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