# Second Order Properties of Models of First Order Arithmetic

Roman Kossak
City University of New York

RK, James H. Schmerl

*The Structure of Models of Peano Arithmetic*,
Oxford Logic Guides, 2006

- $M \restriction <$

  Friedman's 14th Problem: Let $M \models$ PA and let $T$ be a completion of PA. Is there $N \models T$ such that $M \restriction < \cong N \restriction <$?

  Pabion's Theorem: For each uncountable cardinal $\kappa$, $M \restriction <$ is $\kappa$-saturated iff $M$ is $\kappa$-saturated.

  Bovykin, Kaye 02: Various partial results.

- $M{\restriction}+$, $M{\restriction}\times$

  Tennenbaum's Theorem: If $M$ is nonstandard, then $+^M$ and $\times^M$ are not computable.

  For countable $M$, $N$, $M \restriction + \cong N \restriction +$ iff $M{\restriction}\times \cong N{\restriction}\times$.

  Each $M \restriction +$ has $2^{\aleph_0}$ nonisomorphic expansions to models of PA.

  Theorem (RK, Nadel, Schmerl): There are $M, N$ such that $M{\restriction}+ \cong N{\restriction}+$ and $M{\restriction}\times \ncong N{\restriction}\times$.

- $\mathsf{SSy}(M) = \{X \cap \mathbb{N} : X \in \mathsf{Def}(M)\}$

  For every $M \models \mathsf{PA}$, $(\mathbb{N}, \mathfrak{X}) \models \mathsf{WKL}_0$.

  Scott Set Problem: Let $(\mathbb{N}, \mathfrak{X}) \models \mathsf{WKL}_0$. Is there $M \models \mathsf{PA}$ such that $\mathsf{SSy}(M) = \mathfrak{X}$?

  Kanovei's Question: Is there a Borel model $M$ such that $\mathsf{SSy}(M) = \mathcal{P}(\mathbb{N})$?

- $\mathsf{Lt}(M) = (\{K : K \prec M\}, \prec)$

  Mills' Theorem: For every distributive lattice $L$ (satisfying certain immediate necessary conditions) there is $M \models \mathsf{PA}$ such that $\mathsf{Lt}(M) \cong L$.

  Question: Is there a finite lattice which cannot be represented as $\mathsf{Lt}(M)$?

- $\{\mathsf{Th}(M, \mathsf{Cod}(M/I)) : I \subseteq_{\mathsf{end}} M\}$

  For $I \subseteq_{\mathsf{end}} M$,
  $\mathsf{Cod}(M/I) = \{X \cap I : X \in \mathsf{Def}(M)\}$

  $I \subseteq_{\mathsf{end}} M$ is *strong* iff $(M, \mathsf{Cod}(M/I)) \models \mathsf{ACA}_0$

  A countable recursively saturated $M$ is *arithmetically saturated* iff $\mathbb{N}$ is strong in $M$

  (RK, Schmerl 95): Let $T$ be a completion of PA. If $M$, $N$ are countable arithmetically saturated models of $T$, then t.f.a.e:

  (1) $M \cong N$

  (2) $\mathsf{Lt}(M) \cong \mathsf{Lt}(N)$

  (3) $\mathsf{Aut}(M) \cong \mathsf{Aut}(N)$

  Key: If $M$ is arithmetically saturated, then $\mathsf{Aut}(M)$ and $\mathsf{Lt}(M)$ know $\mathsf{SSy}(M)$.

- Aut($M$)

  Schmerl's Theorem: Let $\mathfrak{A}$ be a linearly ordered structure. There is $M \models$ PA such that Aut($M$) $\cong$ Aut($\mathfrak{A}$).

- If $M \models$ PA is countable and recursively saturated, and $\mathfrak{A}$ is a countable linearly ordered structure, then there is $K \prec_{\text{end}} M$ such that Aut($K$, Cod($M/K$)) $\cong$ Aut($\mathfrak{A}$).

- Th(Aut($M$)) is undecidable.

- It all works for PA$^*$

- Nonstandard satisfaction classes

  $S \subseteq M$ is a *truth extension* iff for all $\varphi(x)$

  $$(M, S) \models \forall x[\langle \ulcorner \varphi \urcorner, x\rangle \in S \longleftrightarrow \varphi(x)].$$

- Let $M \models$ PA be countable. Then, $M$ is recursively saturated iff $M$ has a truth extension such that $(M, S) \models$ PA$^*$.

- "Kossak's conjecture"

  (model theory of countable *recursively saturated* models of PA)= (model theory of $(M, S) \models$ PA$^*$, where $S$ is a truth extension for $M$)

- Definable sets, inductive sets, classes

$\mathsf{Ind}(M) = \{X \subseteq M : (M, X) \models \mathsf{PA}^*\}$

$\mathsf{Class}(M) = \{X \subseteq M : \forall a \in M \ a \cap X \in \mathsf{Def}(M)\}$

**Proposition.** *For every model $M$ of* $\mathsf{PA}^*$,

$$\mathsf{Def}(M) \subseteq \mathsf{Ind}(M) \subseteq \mathsf{Class}(M).$$

**Proposition.** *If $M$ is countable, then*

$$\mathsf{Def}(M) \subset \mathsf{Ind}(M) \subset \mathsf{Class}(M).$$

- Undefinable inductive sets

**Theorem. (Simpson 74)** *Let* $M \models \mathsf{PA}^*$ *be countable. There is* $X \in \mathsf{Ind}(M)$ *such that every element of* $M$ *is definable in* $(M, X)$. *(Cohen forcing in arithmetic)*

**Theorem. (Enayat 88)** *There are nonstandard models* $M \models \mathsf{PA}$ *such that for every* $X \in \mathsf{Class}(M) \setminus \mathsf{Def}(M)$, *every element of* $M$ *is definable in* $(M, X)$.

**Theorem. (Schmerl 05)** *Let* $\{A_n\}_{n<\omega}$ *be a collection of inductive subsets of a countable model* $M$. *Then, there is* $X \in \mathsf{Ind}(M)$ *such that* $A_n \in \mathsf{Def}(M, X)$, *for each* $n$. *(Forcing with perfect trees)*

- A digression

**Definition.** *A subset of $X$ a model $M$ is large if every element of $M$ is definable in $(M, a)_{a \in X}$.*

**Proposition.** *All unbounded definable sets are large.*

**Lemma. (Schmerl)** *For every unbounded $X \in$ Def$(M)$ and every $a \in M$ there are an unbounded definable $Y \subseteq X$ and a Skolem term $t(x)$ such that for all $x \in Y$, $t(x) = a$.*

**Proposition.** *Every countable recursively saturated model of PA has an unbounded inductive subset which is not large.*

- Classes and reals

Keisler, Schmerl 91:

$M \longrightarrow \mathbb{Q}(M) \longrightarrow \mathbb{R}^M$

$\mathbb{R}^M = \{D \subseteq_{\mathsf{end}} \mathbb{Q}(M) : D \in \mathrm{Def}(M)\}$

$\mathbb{R}^M \longrightarrow \widehat{\mathbb{R}^M}$ Scott completion

A cut $I$ of an ordered field $F$ is *Dedekindean* if for each positive $\delta \in F$ there is $x \in I$ such that $x + \delta > I$.

A field $F$ is *Scott complete* is every Dedekindean cut of $F$ has a supremum in $F$.

(D. Scott, 69) Every ordered field field $F$ has a unique extension $\hat{F}$ which is Scott complete and $F$ is dense in $\hat{F}$.

$X \in \mathsf{Class}(M) \mapsto \Sigma_{i \in X} 2^{-(i+1)}$

For each $a \in M$, $s_a = \Sigma_{i \in a \cap X} 2^{-(i+1)}$.

$I_X = \{x \in \mathbb{R}^M : \exists a \in M(x < s_a)\}$ is Dedekindean.

$\sup(I_X) = r(X)$.

**Proposition.** *For any model $M$ of* PA, *$\mathbb{R}^M$ is real closed and $|\widehat{\mathbb{R}^M}| = |\mathsf{Class}(M)|$.*

**Proposition.** *$\mathbb{R}^M$ is Scott complete iff $\mathsf{Class}(M) = \mathsf{Def}(M)$.*

**Definition.** $M$ *is rather classless if* $\mathrm{Def}(M) = \mathrm{Class}(M)$

**Theorem. (Schmerl 81)** *Let* $T$ *be a completion of* $\mathrm{PA}^*$ *in a countable language* $\mathcal{L}$. *Then, for every cardinal* $\kappa$ *with* $\mathrm{cf}(\kappa) > \aleph_0$, $T$ *has a* $\kappa$*-like rather classless model.*

**Theorem. (Kaufmann 77** $(\Diamond)$**, Shelah 78)** *There is a recursively saturated rather classless* $\omega_1$*-like model of* PA.

**Theorem. (Schmerl 02)** *For all regular* $\lambda < \mu$, *there is rather classless* $M \models \mathrm{PA}$ *such that* $|M| = \mu$ *and* $|M|$ *is* $\lambda$*-saturated.*

- Conservative extensions

**Definition.** *The extension* $M \prec N$ *is conservative if for every* $X \in \mathrm{Def}(N)$, $X \cap M \in \mathrm{Def}(M)$.

**Theorem. (MacDowel-Specker 61)** *Every model of* PA* *for countable language has a conservative elementary (end) extension.*

**Theorem. (Mills 78)** *Every countable nonstandard model* $M \models$ PA *has an expansion to a model of* PA* *with no conservative extension.*

**Theorem. (Enayat 06)** *There is* $\mathfrak{X} \subseteq \mathcal{P}(\mathbb{N})$ *such that* $(\mathbb{N}, \mathfrak{X})$ *has no conservative extension.*

Let $T$ be a completion of PA.

$p(v)$ is *unbounded* if $(v > t) \in p(v)$ for each closed Skolem term $t$.

**Theorem. (Gaifman, 65-76)** *For* $p(v) \in S_1(T)$ *t.f.a.e.*

- $p(v)$ *is minimal*
- $p(v)$ *is indiscernible and unbounded*
- $p(v)$ *is rare and end-extensional*
- $p(v)$ *is selective and definable*
- $p(v)$ *is 2-indiscernible and unbounded [Schmerl]*
- $p(v)$ *is strongly indiscernible and unbounded*

- If $p(v)$ is a minimal type of Th($M$), then for every linearly ordered set $(I, <)$ $M$ has a canonical $I$-extension generated over $M$ by a set of (indiscernible) elements realizing $p(v)$.

- A problem: If $M \prec_{\mathsf{end}} N$ and $N$ is recursively saturated, then the extension is not conservative.

- A way out: Minimal types of Th$(M, S)$, where $S$ is a truth extension of $M$.