[*International Congress of Nonstandard Methods in Mathematics*, Pisa, 25-31 May 2006]

Independent Random Matching

Darrell Duffie, Stanford University

and

Yeneng Sun, National University of Singapore

Dynamic Random Matching

Let $S = \{1, 2, \ldots, K\}$ be a finite set of types.

A discrete-time dynamical system $\mathcal{D}$ with random mutation, partial matching and type changing

The initial distribution of types is p^0.

In each time period $n \geq 1$,

• first, each type-k agent randomly mutates to an agent of type l with probability b_{kl}.

• Then, each agent of type k is either not matched, with probability q_k, or is matched to a type-l agent with a probability proportional to the fraction of type-l agents in the population immediately after the random mutation step.

- When an agent is not matched, she keeps her type.

- When a type-k agent is matched with a type-l agent, the type-k agent becomes type r with probability $\nu_{kl}(r)$, where ν_{kl} is a probability distribution on S, and similarly for the type-l agent.

- When I is finite, the independence condition cannot be imposed even for static full matchings.
- Correlation reduces to zero when the population is large.
- Independent random matching in a continuum population (i.e., a non-atomic measure space of agents) is widely used (explicitly and implicitly) in economic literature and also in evolutionary biology.
- However, a mathematical foundation has been lacking.

Formal Inductive Definition of Dynamic Random Matching

Let $\alpha^0 : I \to S = \{1, \ldots, K\}$ be an initial type function with distribution p^0 on S.

For time period $n \geq 1$, a random mutation is modeled by a process h^n from $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ to S. Given a $K \times K$ probability transition matrix b, we require that, for each agent $i \in I$,

$$P\left(h_i^n = l \,|\, \alpha_i^{n-1} = k\right) = b_{kl},$$

the specified probability with which an agent i of type k at the end of time period $n-1$ mutates to type l.

Let $\overline{p}^{\,n-1/2}$ be the expected cross-sectional type distribution immediately after the random mutation. The random partial matching function π^n at time n is defined by:

1. For any $\omega \in \Omega$, $\pi^n_\omega(\cdot)$ is a full matching on $I - (\pi^n_\omega)^{-1}(\{J\})$.
2. Extending h^n so that $h^n(J, \omega) = J$ for any $\omega \in \Omega$, let $g^n(i, \omega) = h^n(\pi^n(i, \omega), \omega)$.
3. Let $q \in [0, 1]^S$. For each agent $i \in I$,

$$P\left(g^n_i = J \,|\, h^n_i = k\right) = q_k,$$

$$P\left(g^n_i = l \,|\, h^n_i = k\right) = \frac{(1 - q_k)(1 - q_l)\overline{p}^{\,n-1/2}_l}{\sum_{r=1}^{K}(1 - q_r)\overline{p}^{\,n-1/2}_r}.$$

Let $\nu : S \times S \to \Delta$ specify the probability distribution $\nu_{kl} = \nu(k,l)$ of the new type of a type-k agent after she is matched with a type-l agent.

We require that the type function α^n after the partial matching satisfies, for each agent $i \in I$,

$$P\left(\alpha_i^n = r \,|\, h_i^n = k, g_i^n = J\right) = \delta_k^r,$$
$$P\left(\alpha_i^n = r \,|\, h_i^n = k, g_i^n = l\right) = \nu_{kl}(r),$$

where δ_k^r is one if $r = k$, and zero otherwise.

Markov Conditional Independence

• an independent random mutation follows from the previous period,

• followed by an independent random partial matching,

• for matched agents, there is independent random type changing.

• Formally, the random mutation is Markov conditionally independent if, for λ-almost all $i, j \in I$, for all types $k, l \in S$

$$P(h_i^n = k, h_j^n = l \,|\, \alpha_i^0, \ldots, \alpha_i^{n-1}; \alpha_j^0, \ldots, \alpha_j^{n-1})$$
$$= P(h_i^n = k \,|\, \alpha_i^{n-1}) P(h_j^n = l \,|\, \alpha_j^{n-1}).$$

Define a mapping Γ from Δ to Δ such that, for each $p = (p_1, \ldots, p_K) \in \Delta$, the r-th component of Γ is

$$\Gamma_r(p_1, \ldots, p_K) = q_r \sum_{m=1}^{K} p_m b_{mr} + \sum_{k,l=1}^{K} \frac{\nu_{kl}(r)(1-q_k)(1-q_l) \sum_{m=1}^{K} p_m b_{mk} \sum_{j=1}^{K} p_j b_{jl}}{\sum_{t=1}^{K} (1-q_t) \sum_{j=1}^{K} p_j b_{jt}}.$$

Theorem 1. Let $\mathbb{D}$ be any dynamical system with random mutation, partial matching and type changing whose parameters are (p^0, b, q, ν) that is Markov conditionally independent. Then:

(1) For time $n \geq 1$, the expected cross-sectional type distribution is given by $\overline{p}^{\,n} = \Gamma(\overline{p}^{\,n-1}) = \Gamma^n(p^0)$, and $\overline{p}^{\,n-1/2}_{\,k} = \sum_{l=1}^{K} b_{lk}\overline{p}^{\,n-1}_{\,l}$, where Γ^n is the composition of Γ with itself n times, and where $\overline{p}^{\,n-1/2}$ is the expected cross-sectional type distribution after the random mutation.

(2) For λ-almost all $i \in I$, $\{\alpha_i^n\}_{n=0}^{\infty}$ is a Markov chain with transition matrix z^n at time $n-1$ defined by

$$z_{kl}^n = q_l b_{kl} + \sum_{r,j=1}^{K} \nu_{rj}(l) b_{kr} \frac{(1-q_r)(1-q_j)\overline{p}_j^{\,n-1/2}}{\sum_{r'=1}^{K}(1-q_{r'})\overline{p}_{r'}^{\,n-1/2}}.$$

(3) For λ-almost all $i, j \in I$, the Markov chains $\{\alpha_i^n\}_{n=0}^{\infty}$ and $\{\alpha_j^n\}_{n=0}^{\infty}$ are independent.

(4) For P-almost all $\omega \in \Omega$, the cross-sectional type process $\{\alpha_\omega^n\}_{n=0}^{\infty}$ is a Markov chain with transition matrix z^n at time $n-1$.

(5) For P-almost all $\omega \in \Omega$, at each time period $n \geq 1$, the realized cross-sectional type distribution after the random mutation $\lambda(h_\omega^n)^{-1}$ is its expectation $\bar{p}^{\,n-1/2}$, and the realized cross-sectional type distribution at the end of period n, $p^n(\omega) = \lambda(\alpha_\omega^n)^{-1}$, is equal to its expectation $\bar{p}^{\,n}$, and thus, P-almost surely, $p^n(\omega) = \Gamma^n(p^0)$.

(6) There is a stationary distribution p^*. That is, with initial cross-sectional type distribution $p^0 = p^*$, for every $n \geq 1$, the realized cross-sectional type distribution p^n at time n is p^*, P-almost surely, and $z^n = z^1$. In particular, all of the relevant Markov chains are time-homogeneous with a constant transition matrix having p^* as a fixed point.

Theorem 2. Fixing any parameters
p^0 for initial cross-sectional type distribution,
b for mutation probabilities,
$q \in [0,1]^S$ for no-match probabilities,
ν for type-changing probabilities,

there exists a dynamical system $\mathbb{D}$ with random mutation, partial matching and type changing that is Markov conditionally independent with these parameters.

The six properties in Theorem 1 hold for any Markov conditionally independent dynamical matching (not just for the particular examples shown in Theorem 2).

That is analogous to the fact that the classical law of large numbers hold for any sequence of random variables satisfying independence (or uncorrelatedness) with some moment conditions (not just for a particular example showing the existence of a sequence of independent random variables).

The Proof of Theorem 1

is based on the exact law of large numbers.

Let f be any real-valued process on $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$. If f is square integrable and essentially uncorrelated, then

$$P\left(\omega \in \Omega : \mathbb{E}(f_\omega) = \mathbb{E}f\right) = 1.$$

Based on that, it is easy to show that if f is essentially pairwise independent, then

$$P\left(\omega \in \Omega : \lambda(f_\omega)^{-1} = (\lambda \boxtimes P)f^{-1}\right) = 1.$$

Converse law of large numbers: the necessity of uncorrelatedness or independence (both are the standard conditions).

Law of large numbers for *-independent random variables

Let $\{X_i\}_{i=1}^n$ be a hyperfinite sequence of $*$-independent random variables on an internal probability space $(\Omega, \mathcal{F}_0, P_0)$ with internal mean zero and variances bounded by a common standard positive number C,

The elementary Chebyshev's inequality says that for any positive hyperreal number ϵ,

$$P_0\left(|X_1 + \ldots + X_n|/n \geq \epsilon\right) \leq C/n\epsilon^2,$$

which implies $P_0\left(|X_1 + \ldots + X_n|/n \simeq 0\right) \simeq 1$.

Loeb Transition Probability

- $(I, \mathcal{I}_0, \lambda_0)$ a hyperfinite internal probability space

- $\{(\Omega, \mathcal{F}_0, P_{0i}) : i \in I\}$ an internal collection of hyperfinite internal probability measures

- **Define** τ_0 **on** $(I \times \Omega, \mathcal{I}_0 \otimes \mathcal{F}_0)$ **by letting** $\tau_0(\{(i, \omega)\}) = \lambda(\{i\})P_{0i}(\{\omega\})$ **for** $(i, \omega) \in I \times \Omega$.

- Let $(I, \mathcal{I}, \lambda)$, $(\Omega, \mathcal{F}_i, P_i)$, and $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \tau)$ be the Loeb spaces corresponding respectively to $(I, \mathcal{I}_0, \lambda_0)$, $(\Omega, \mathcal{F}_0, P_{0i})$, and $(I \times \Omega, \mathcal{I}_0 \otimes \mathcal{F}_0, \tau_0)$.

The following result presents a Fubini type theorem for the Loeb transition probability $P_i, i \in I$, which generalizes Keisler's Fubini Theorem for the case that P_i equals a Loeb probability measure P for all $i \in I$.

Proposition 1. Let f be a real-valued integrable function on $(I \times \Omega, \sigma(\mathcal{I}_0 \otimes \mathcal{F}_0), \tau)$. Then,

1. f_i is $\sigma(\mathcal{F}_0)$-measurable for each $i \in I$ and integrable on $(\Omega, \sigma(\mathcal{F}_0), P_i)$ for λ-almost all $i \in I$;
2. $\int_\Omega f_i(\omega) dP_i(\omega)$ is integrable on $(I, \sigma(\mathcal{I}_0), \lambda)$;
3. $\int_I \int_\Omega f_i(\omega) dP_i(\omega) d\lambda(i) = \int_{I \times \Omega} f(i, \omega) d\tau(i, \omega)$.

The Proof of Theorem 2

is based on an infinite product of Loeb transition probabilities.

- For each $m \geq 1$, let Ω_m be a hyperfinite set with its internal power set $\mathcal{F}_m$.

- Ω^n, Ω^∞, and Ω_n^∞ denote $\prod_{m=1}^{n} \Omega_m$, $\prod_{m=1}^{\infty} \Omega_m$, and $\prod_{m=n}^{\infty} \Omega_m$ respectively.

- $\{\omega_m\}_{m=1}^{n}$, $\{\omega_m\}_{m=1}^{\infty}$, and $\{\omega_m\}_{m=n}^{\infty}$ denoted by ω^n, ω^∞, and ω_n^∞ respectively.

• For each $n \geq 1$, let Q_n be an internal transition probability from Ω^{n-1} to $(\Omega_n, \mathcal{F}_n)$, that is, for each $\omega^{n-1} \in \Omega^{n-1}$, $Q_n(\omega^{n-1})$ is a hyperfinite internal probability measure on $(\Omega_n, \mathcal{F}_n)$.

• $Q_1 \otimes Q_2 \otimes \cdots \otimes Q_n$ defines an internal probability measure on $(\Omega^n, \otimes_{m=1}^n \mathcal{F}_m)$.

• Denote $Q_1 \otimes Q_2 \otimes \cdots \otimes Q_n$ by Q^n, and $\otimes_{m=1}^n \mathcal{F}_m$ by $\mathcal{F}^n$. Then Q^n is the internal product of the internal transition probability Q_n with the internal probability measure Q^{n-1}.

• Let P^n and $P_n(\omega^{n-1})$ be the corresponding Loeb measures, which are defined respectively on $\sigma(\mathcal{F}^n)$ and $\sigma(\mathcal{F}_n)$ (the proceeding one is much richer). P^n is the Loeb product $P_1 \boxtimes P_2 \boxtimes \cdots \boxtimes P_n$ of the Loeb transition probabilities $P_1, P_2, \ldots, P_n$.

• Let $\mathcal{F}^\infty = \cup_{n=1}^\infty \left[\mathcal{F}^n \times \Omega_{n+1}^\infty\right]$, which is an algebra of sets in Ω^∞. One can define a measure P^∞ on this algebra by letting $P^\infty(E_n \times \Omega_{n+1}^\infty) = P^n(E_n)$ for each $E_n \in \mathcal{F}^n$.

Proposition 2. There is a unique countably additive probability measure on $\sigma(\mathcal{F}^\infty)$ that extends the set function P^∞ on $\mathcal{E}$.

Proposition 3. The usual product of the probability spaces $(I, \mathcal{I}, \lambda)$ and $(\Omega^\infty, \sigma(\mathcal{F}^\infty), P^\infty)$ has a Fubini extension $\left(I \times \Omega^\infty, \sigma\left(\cup_{n=1}^{\infty}(\mathcal{I}_0 \otimes \mathcal{F}^n) \times \Omega_{n+1}^{\infty}\right), \boxtimes_{m=0}^{\infty} P_m\right)$.

Note that Keisler's Fubini Theorem is not applicable to $(I, \mathcal{I}, \lambda)$ and $(\Omega^\infty, \sigma(\mathcal{F}^\infty), P^\infty)$ since $\mathcal{F}^\infty$ is not an internal algebra.

Some details on the Static Case

Let $\alpha : I \to S$ be an $\mathcal{I}$-measurable type function with type distribution $p = (p_1, \ldots, p_K)$ on S. **An independent random partial matching** π with no-match probabilities $q_1, \ldots, q_K$ in $[0,1]$ is a mapping from $I \times \Omega$ to $I \cup \{J\}$ (J denotes "no match") such that

1. $\forall \omega \in \Omega$, $\pi_\omega|_{I-\pi_\omega^{-1}(\{J\})}$ is a full matching on $I - \pi_\omega^{-1}(\{J\})$.
2. Let $g(i,\omega) = \alpha(\pi(i,\omega))$ with $\alpha(J) = J$. g is essentially pairwise independent and measurable from $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ to $S \cup \{J\}$.
3. for λ-almost all $i \in \{i \in I : \alpha(i) = k\}$,

$$P(g_i = J) = q_k$$

$$P(g_i = l) = \frac{(1-q_k)p_l(1-q_l)}{\sum_{r=1}^{K} p_r(1-q_r)}.$$

Proposition 4. There is an atomless probability space $(I, \mathcal{I}, \lambda)$ of agents such that for any given $\mathcal{I}$-measurable type function β from I to S, and for any $q \in [0,1]^S$, there exists an independent-in-types random partial matching π from $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ to I with $q = (q_1, \ldots, q_K)$ as the no-match probabilities.

Ideas of the Proof

• Pick $M \in {}^*\mathbb{N}_\infty$. Let $I = \{1, 2, ..., M\}$, and $(I, \mathcal{I}_0, \lambda_0)$ the internal counting probability space.

• For any $\mathcal{I}$-measurable type function β from I to $S = \{1, \ldots, K\}$, find an internal lifting α from I to S. $\forall k \in S$, $A_k = \alpha^{-1}(k)$ has M_k elements with $\lambda(A_k) = p_k \simeq M_k/M$.

• Pick $m_k \in {}^*\mathbb{N}_\infty$ such that $q_k \simeq m_k/M_k$, and $N = \sum_{l=1}^{K}(M_l - m_l) \in {}^*\mathbb{N}_\infty$ is even.

$$\frac{N}{M} = \sum_{l=1}^{K} \frac{M_l}{M}\left(1 - \frac{m_l}{M_l}\right) \simeq \sum_{l=1}^{K} p_l(1 - q_l).$$

• Let $\mathcal{P}_{m_k}(A_k)$ be the collection of all such internal subsets of A_k with m_k elements.

• For given $B_k \in \mathcal{P}_{m_k}(A_k)$ for $k = 1, 2, \ldots, K$, let $\pi^{B_1, B_2, \ldots, B_K}$ be a (full) matching on $I - \cup_{k=1}^{K} B_k$ produced by pairwise draws; there are $(N-1)!!$ such matchings.

• Let Ω be the set of all ordered tuples $(B_1, B_2, \ldots, B_K, \pi^{B_1,B_2,\ldots,B_K})$ such that $B_k \in \mathcal{P}_{m_k}(A_k)$ for each $k \in S$, and $\pi^{B_1,B_2,\ldots,B_K}$ is a matching on $I - \cup_{k=1}^{K} B_k$.

• Ω has $((N-1)!!)\prod_{k=1}^{K}\binom{M_k}{m_k}$ many elements in total. Let $(\Omega, \mathcal{F}_0, P_0)$ be the internal counting probability space.

• Let J represent non-matching.
For $\omega = (B_1, B_2, \ldots, B_K, \pi^{B_1,B_2,\ldots,B_K})$,
let $\pi(i,\omega)$ be

$$\begin{cases} J, & \exists k \in S, i \in B_k, \\ \pi^{B_1,B_2,\ldots,B_K}(i), & i \in I - \cup_{r=1}^{K} B_r. \end{cases}$$

Thanks!