Non-Standard Methods and Reverse Mathematics

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From Hilbert's Program to Reverse Math.

Hilbert's Reductionism Program:

Reduce the whole math to finite math.

Reverse Math Program (Friedman-Simpson):

Reduce a stronger system (WKL_0)
to a weaker system (*PRA*).

How much math can be developed
within the stronger system?

Practice of Reverse Mathematics

0. *Fix a base theory* T $(:= RCA_0)$.

1. *Pick a theorem* τ.

2. *Find the weakest axiom* α *s.t.*

$$T + \alpha \vdash \tau.$$

3. *Very often, we can show*

$$T \vdash \alpha \leftrightarrow \tau.$$

Reverse Mathematics

- *Reverse math classifies mathematical theorems, according to which set existence axioms are need to prove them.*

Framework: $\textit{Second order arithmetic} \ \ Z_2$

$$= \ \textit{Basic axioms for } (+, \cdot, 0, 1, <)$$

$$+ \ \textit{Comprehension } (CA): \ \exists X \forall x (x \in X \leftrightarrow \varphi(x))$$

$$+ \ \textit{Induction}: \ \varphi(0) \wedge \forall x (\varphi(x) \rightarrow \varphi(x+1)) \rightarrow \forall x \varphi(x)$$

Classes of Formulas

- Bounded formulas (Σ^0_0), only with $\forall x < t, \exists x < t$

- Arithmetical formulas (Σ^1_0) , with no set quantifiers

$$\Sigma^0_n : \exists\overrightarrow{x_1}\forall\overrightarrow{x_2}\cdots Q\overrightarrow{x_n}\ \varphi\ \mathit{with}\ \varphi\ \mathit{bounded}.$$

$$\Pi^0_n : \forall\overrightarrow{x_1}\exists\overrightarrow{x_2}\cdots Q\overrightarrow{x_n}\ \varphi\ \mathit{with}\ \varphi\ \mathit{bounded}.$$

- Analytical formulas:

$$\Sigma^1_n : \exists\overrightarrow{X_1}\forall\overrightarrow{X_2}\cdots Q\overrightarrow{X_n}\ \varphi\ \mathit{with}\ \varphi\ \mathit{arithmetic}.$$

$$\Pi^1_n : \forall\overrightarrow{X_1}\exists\overrightarrow{X_2}\cdots Q\overrightarrow{X_n}\ \varphi\ \mathit{with}\ \varphi\ \mathit{arithmetic}.$$

Big five subsystems

$$RCA_0 = \Delta^0_1\text{-}CA + \Sigma^0_1\text{-}ind$$

$$WKL_0 = RCA_0 + \ weak\ K\ddot{o}nig's\ lemma$$
$$for\ infinite\ binary\ trees$$

$$ACA_0 = RCA_0 + \Sigma^0_1\text{-}CA$$

$$ATR_0 = RCA_0 + \ iteration\ of\ \Sigma^0_1\text{-}CA$$
$$along\ any\ well\ ordering$$

$$\Pi^1_1\text{-}CA_0 = RCA_0 + \Pi^1_1\text{-}CA$$

Some results of R. M.

$Over\ RCA_0$

$$
\begin{aligned}
WKL_0 &\leftrightarrow the\ maximum\ principle \\
&\leftrightarrow the\ Cauchy\text{-}Peano\ theorem \\
&\leftrightarrow Brouwer's\ fixed\ point\ theorem \\[1em]
ACA_0 &\leftrightarrow the\ Bolzano\text{-}Weierstrass\ theorem \\
&\leftrightarrow the\ Ascoli\ lemma \\[1em]
ATR_0 &\leftrightarrow the\ Luzin\ separation\ theorem \\
&\leftrightarrow \Sigma^0_1\text{-}determinacy \\[1em]
\Pi^1_1\text{-}CA_0 &\leftrightarrow the\ Cantor\text{-}Bendixson\ theorem \\
&\leftrightarrow \Sigma^0_1 \wedge \Pi^0_1\text{-}determinacy
\end{aligned}
$$

Remarks.

- While doing reverse mathematics, one often needs to invent a new proof or modify an old proof for the popular theorem, so that it suits for a weaker subsystem.
- For instance, to prove the Peano existence theorem wihin WKL_0 , Simpson (1984) invented a new proof which does not depend on the Ascoli lemma, which is known to be stronger than WKL_0 .
- Non-standard methods are useful in many cases.

A remark after a nonstandard proof of the Peano existence theorem in

Albeverio, et al., “Nonstandard methods in stochastic analysis and mathematical physics (1986)”, p. 31.

- Nonstandard *praxis* is remarkably constructive.
- In the standard approach one uses in the final step the Ascoli lemma. This part of the argument is lacking in the nonstandard proof.
- It is possible to recast the nonstandard proof to give a proof of the Peano existence theorem where the only nonrecursive element is the weak Konig’s Lemma.

Non-standard methods

- Conservation:

$$T + \alpha \vdash \varphi \Rightarrow T \vdash \varphi$$

- Inner models:

$$T \vdash (\varphi \leftrightarrow M \models \varphi)$$

- Outer models:

$$M \models \varphi \Leftrightarrow {}^{*}M \models {}^{*}\varphi$$

Conservation results

- Shoenfield:

  $$ZF + V = L \vdash \sigma \Rightarrow ZF \vdash \sigma \ \ for \ \ \sigma \in \Sigma^1_2 \cup \Pi^1_2$$

- Barwise-Schlipf:

  $$\Sigma^1_1\text{-}AC_0 \vdash \sigma \Rightarrow ACA_0 \vdash \sigma \ \ for \ \ \sigma \in \Pi^1_2$$

- Harrington:

  $$WKL_0 \vdash \sigma \Rightarrow RCA_0 \vdash \sigma \ \ for \ \ \sigma \in \Pi^1_1$$

- Simpson-T.-Yamazaki (2002):

  $$WKL_0 \vdash \sigma \Rightarrow RCA_0 \vdash \sigma$$

  $$for \ \ \sigma \equiv \forall X \exists ! Y \varphi(X, Y) \ \ with \ \ \varphi \ \ arith.$$

Application of Simpson-T.-Yamazaki's result

The fundamental theorem of algebra *(FTA):*

Any complex polynomial of a positive degree

has a unique factorization into linear terms.

$$WKL_0 \models \ FTA \ with \ standard \ polynomials$$

$$\therefore \ RCA_0 \models \ FTA \ with \ standard \ polynomials$$

$$or \ RCA \models \ FTA$$

Inner model methods

- Count. corded β_n-models and reflection.
- Resplendency and recursive saturation.

- Defining the satisfaction relation on $\mathbb{R}$.

Defining the real number system $\mathbb{R}$

The following definitions are made in RCA_0.

- ✓ Using the pairing function, we define $\mathbb{N}$ and $\mathbb{Q}$.
- ✓ The basic operations on $\mathbb{N}$ and $\mathbb{Q}$ are also naturally defined.
- ✓ A real number is an infinite sequence $\{q_n\}$ of rationals such that $|q_n - q_m| \leq 2^{-n}$ for all $m > n$.
- ✓ The operations on $\mathbb{R}$ are also defined so that the resulting structure is a real closed order field.

Satisfaction on $\mathbb{R}$

- ✓ Simpson-T.-Yamazaki

  $Sat_{\mathbb{R}}(\ulcorner\varphi(\overrightarrow{x})\urcorner, \overrightarrow{\xi})$ *can be defined as a* Δ^0_2 *formula.*

  In $RCA_0, Sat_{\mathbb{R}}$ *satisfies the Tarski clauses for the standard formulas.*

- ✓ Sakamoto-T. (2004)

  In $RCA_0, Sat_{\mathbb{R}}$ *satisfies the Tarski clauses for all the formulas. In particular,*

  $$Sat_{\mathbb{R}}(\ulcorner\exists\overrightarrow{x}\varphi(\overrightarrow{x},\overrightarrow{y})\urcorner, \overrightarrow{\beta}) \leftrightarrow \exists\overrightarrow{\alpha}\, Sat_{\mathbb{R}}(\ulcorner\varphi(\overrightarrow{x},\overrightarrow{y})\urcorner, \overrightarrow{\alpha}, \overrightarrow{\beta})$$

- ❖ The following fact (called *strong FTA*) is essential:

  $$RCA_0 \vdash \forall p(x) \in \mathbb{Q}[x]\exists\overrightarrow{\alpha} \in \mathbb{C}^{<\mathbb{N}} p(x) = \Pi_i(x-\alpha_i)$$

Applications of Sakamoto-T's result

$$RCA_0 \vdash \ Hilbert's\ Nullstellensatz:$$

$$p_1, \cdots, p_m \in \mathbb{C}[\overrightarrow{x}]\ have\ no\ common\ zeros$$

$$\Rightarrow \exists q_1 \cdots \exists q_m \in \mathbb{C}[\overrightarrow{x}]\ p_1 q_1 + \cdots + p_m q_m = 0$$

$$RCA_0 \vdash \ strong\ FTA$$

Outer Model Method

Theorem (H. Friedman, Kirby-Paris)

$$Suppose\ M \models\ PRA,\ countable.$$

$$Suppose\ b \ll_M c\ (i.e.,\ f(b) <_M c\ for\ all\ prim.\ rec.\ f).$$

$$Then\ \exists I \subseteq_e M\ \ s.t.\ b \in I, c \notin I\ \ and\ \ I \models \mathrm{I}\Sigma_1$$

$$Moreover,\ if\ C(M) = \{X \subseteq M : \exists a \in M\ codes\ X\},$$

$$(I, C(M) \restriction I) \models WKL_0.$$

Theorem (T.) A converse to the above holds.

$$Suppose\ (M, S) \models WKL_0,\ countable,\ M \neq \omega.$$

$$Then\ \exists^* M \supseteq_e M\ \ s.t.\ ^*M \models \mathrm{I}\Sigma_1\ and\ S = C(M^*) \restriction M.$$

Self-Embedding Theorems

Thm. (self-embedding for WKL_0 , T. 1997)

$$Suppose\ (M,S) \models WKL_0,\ countable,\ M \neq \omega.$$

$$Then\ \exists I \subsetneq_e M\ s.t.\ (M,S) \simeq (I, S\Gamma I).$$

- History of self embedding results.

  $H.Friedman\ (1970's)\ for\ PA.$

  $Ressayre,\ Dimitracopoulous\ and\ Paris$

  $(1980's)\ for\ \mathrm{I}\Sigma_1.$

(Proof) By a back-and-forth argument.

Cor. $Suppose\ (M,S)\ \models WKL_0,\ countable,\ M \neq \omega.$

$$Then\ \exists^* M \supsetneq_e M, \exists^* S\ s.t.\ (^*M, ^*S) \models WKL_0$$

$$and\ S =\ ^*S \restriction M.$$

Application (the maximum principle)

$$WKL_0 \vdash \ Any\ cont.\ function\ f : [0,1] \to [0,1]\ has\ a\ max.$$

(Proof)

$$\begin{array}{ccc|c}
 & V = (M, S) & & {}^*V = ({}^*M, {}^*S) \\
\hline
 & f : \ [0,1] \cap \mathbb{Q} \to [0,1] & \Longrightarrow & {}^*f : \{q_i\}_{i<a} \to 2^b \\
 & \quad \| \qquad\qquad\quad \| & & (a, b \in {}^*M - M, f = {}^*f \cap M) \\
 & \{q_i\}_{i \in M} \qquad 2^M & & \Downarrow \\
 & {}^*m \cap M\ \ is\ \ \sup f & \Longleftarrow & {}^*m = \max\{{}^*f(q_i)\}_{i<a}
\end{array}$$

Application (New)

$$WKL_0 \vdash \ Strong\ FTA.$$

(Proof) $V = (M, S)$	${}^*V = ({}^*M, {}^*S)$
$f : \mathbb{Q}[x] \to (\mathbb{C} \cap \mathbb{Q}^2)^{<\mathbb{N}}$ $\Longrightarrow$	${}^*f : \{p_i\}_{i<a} \to ({}^*\mathbb{C} \cap {}^*\mathbb{Q}^2)^{<b}$
$\|\|$	
$\{p_i\}_{i \in M}$ *with infinite repetition*	$(a, b \in {}^*M - M, f = {}^*f \cap M)$
s.t. $f(p_i)$ *is a list of rational approximations of the roots of* p_i *with error* $< 2^{-i}$.	$\Downarrow$
${}^*f(p_{j_i}) \Gamma M$ *is the list of roots of* p_i. $\Longleftarrow$	$\{p_i\}_{i \in M} = \{p_{j_i}\}_{j_i \notin M, i \in M}$

Other applications

$$WKL_0 \vdash \mathit{The\,Cauchy{-}Peano\,theorem}\ (\mathit{Tanaka},\ 1997)$$

$$WKL_0 \vdash \mathit{The\ existence\ of\ Haar\ measure}$$

$$\mathit{for\ a\ compact\ group}\ (\mathit{Tanaka\text{-}Yamazaki},\ 2000)$$

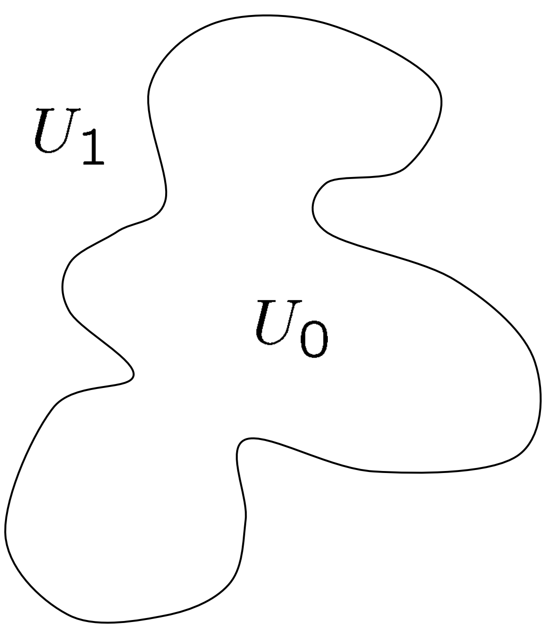
$$WKL_0 \vdash \mathit{The\ Jordan\ curve\ theorem}$$

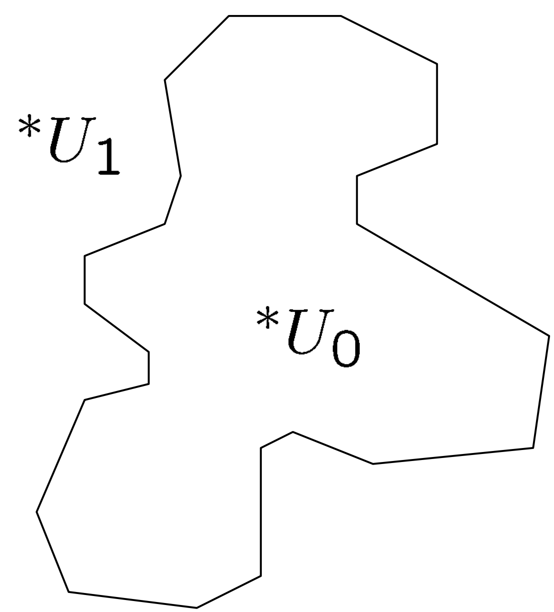
$$(\mathit{Sakamoto\text{-}Yokoyama},\ \mathit{to\ appear})$$

Application (Sakamoto, Yokoyama)

$$WKL_0 \vdash \ The\ Jordan\ Curve\ Theorem$$

(Proof) $V = (M, S)$ | ${}^*V = ({}^*M, {}^*S)$





Outer model method for ACA_0

$$Suppose\ (M,S) \models ACA_0,\ countable,\ M \neq \omega.$$

$$Then\ \exists^* M \supsetneq_e M \exists^* S$$

$$s.t.\ ({}^*M, {}^*S) \models ACA_0, S = {}^*S \restriction M$$

$$and\ \exists * : S \to {}^*S\ \forall \varphi(x, X) \in \Sigma^1_1 \cup \Pi^1_1$$

$$(M,S) \models \varphi(m, A) \leftrightarrow ({}^*M, {}^*S) \models \varphi(m, {}^*A)$$

This easily follows from

Theorem (Gaifman): $Every\ model\ M\ of\ PA\ has\ a\ conservative\ extension\ K,\ i.e.,\ (the\ sets\ definable\ in\ K) \restriction M\ =\ the\ sets\ definable\ in\ M.$

Applications

$$ACA_0 \vdash \ Any\ Cauchy\ sequence\ converges.$$

(Proof) $\quad V = (M, S)$	${}^*V = ({}^*M, {}^*S)$
$\{a_i\}_{i \in M}$ *a Cauchy seq.* $\Longrightarrow$	${}^*(\{a_i\}_{i\in M}) = \{({}^*a)_i\}_{i \in {}^*M}.$
	Pick $j \in {}^*M - M.$
$\forall n \exists m \forall k > m \lvert a_k - b \rvert < 2^{-n}$ $\Longleftarrow$	$\forall n \in M \exists m \in M \forall k > m$ $\lvert({}^*a)_k - ({}^*a)_j\rvert < 2^{-n}.$
	${}^*b \approx ({}^*a)_j.$

- $ACA_0 \vdash \ The\ Riemann\ mapping\ theorem.$

  $(Yokoyama,\ to\ appear)$

THANK YOU