Esercizio 1

$$\lim_{n \to +\infty} \frac{\log\left(\sqrt{n^3} - \sqrt{n}\right)}{\log\left(\sqrt{n} + \log n\right)}$$

$$\frac{\log\left(\sqrt{n^3} - \sqrt{n}\right)}{\log\left(\sqrt{n} + \log n\right)} = \frac{\log\left(n^{3/2}\left(1 - \frac{1}{n}\right)\right)}{\log\left(n^{1/2}\left(1 + \frac{\log n}{\sqrt{n}}\right)\right)}$$

$$= \frac{\frac{3}{2}\log n + \log\left(1 - \frac{1}{n}\right)}{\frac{1}{2}\log n + \log\left(1 + \frac{\log n}{\sqrt{n}}\right)} = 3 \cdot \frac{1 + \overbrace{\frac{\log\left(1 - \frac{1}{n}\right)}{\log n}}^{\to 0}}{1 + \underbrace{\frac{\log\left(1 + \frac{\log n}{\sqrt{n}}\right)}{\log n}}_{\to 0}} \to 3$$

Esercizio 2

$$f(x) = \begin{cases} (1-\cos x)\,\mathrm{sen}\frac{1}{x} & \text{se } x \neq 0 \\ 0 & \text{se } x = 0 \end{cases} \qquad f: \mathbb{R} \to \mathbb{R}$$

In $\mathbb{R}\setminus\{0\}$ f è composizione di funzioni continue e derivabili infinite volte. Quindi su $\mathbb{R}\setminus\{0\}$ f è continua, derivabile e la derivata è continua.

In $x=0$? Mostriamo che f è derivabile in $x=0$.

$$\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{(1-\cos h)\,\mathrm{sen}\frac{1}{h} - 0}{h}$$

$$= \lim_{h\to 0} \frac{(1-\cos h)}{h}\,\mathrm{sen}\frac{1}{h} = 0$$

$\cos x = 1 - \frac{x^2}{2} + o(x^2)$

in quanto $\frac{1-\cos h}{h} = \frac{o(h)}{h} \to 0$

e $\left|\mathrm{sen}\frac{1}{h}\right| \leq 1$

dunque $\left|\frac{1-\cos h}{h}\,\mathrm{sen}\frac{1}{h}\right| \leq \left|\frac{1-\cos h}{h}\right| \to 0.$

La derivata è continua in $x=0$?
Calcolo la derivata $f'(x)$ per $x \neq 0$:

$$f'(x) = \sin x\,\mathrm{sen}\frac{1}{x} + (1-\cos x)\left(\cos\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)$$

$$= \mathrm{sen}\,x\,\mathrm{sen}\frac{1}{x} - \frac{1-\cos x}{x^2}\cos\frac{1}{x}$$

osserviamo ora che per $x \to 0$:

$$\underbrace{\operatorname{sen} x}_{\to 0} \, \underbrace{\operatorname{sen} \frac{1}{x}}_{\text{limitata}} \longrightarrow 0 \qquad \frac{1-\cos x}{x^2} = \frac{\frac{x^2}{2}+o(x^2)}{x^2} \to \frac{1}{2}$$

Dunque $\lim_{x\to 0} f'(x)$ non esiste

perché se (per assurdo) fosse $\lim_{x\to 0} f'(x) = \ell$

si avrebbe per $x \to 0$:

$$\cos\frac{1}{x} = \frac{\operatorname{sen} x \operatorname{sen}\frac{1}{x} - f'(x)}{\frac{1-\cos x}{x^2}} \longrightarrow \frac{0-\ell}{\frac{1}{2}} = -2\ell$$

mentre sappiamo che $\lim_{x\to 0} \cos\frac{1}{x}$ non esiste.

Esercizio 3

$$f(x) = \frac{xe^{-x}}{x-1}$$

$D(f) = \mathbb{R} \setminus \{1\}$

$\lim_{x \to +\infty} f(x) = 0 \qquad y=0$ AS. ORIZZONTALE

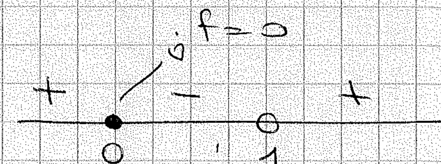
$\lim_{x \to -\infty} f(x) = +\infty$

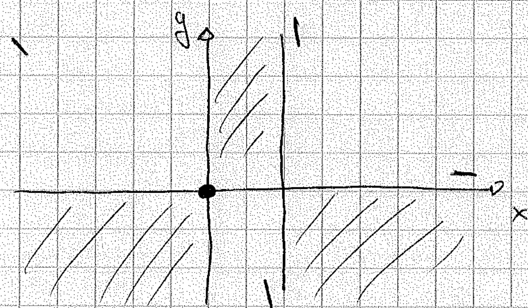
$\lim_{x \to -\infty} \frac{f(x)}{x} = \lim_{x \to -\infty} \frac{e^{-x}}{x-1} = -\infty \qquad \nexists$ ASINTOTO

$\lim_{x \to 1^+} f(x) = +\infty \qquad x = 1$ AS. VERTICALE

$\lim_{x \to 1^-} f(x) = -\infty$

$f(x) \gtrless 0 \iff \frac{x}{x-1} \gtrless 0$

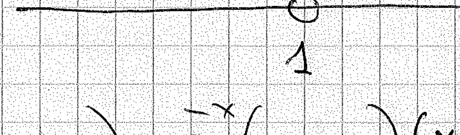




$$f'(x) = \frac{(x-1)(e^{-x} - xe^{-x}) - xe^{-x} \cdot 1}{(x-1)^2} =$$

$$= \frac{e^{-x}}{(x-1)^2}\left(-\cancel{x}^2 + 2x - 1 - x\right) = \frac{-e^{-x}(x^2 - x + 1)}{(x-1)^2}$$

$\Rightarrow f'(x) < 0 \quad \forall x \in D(f)$

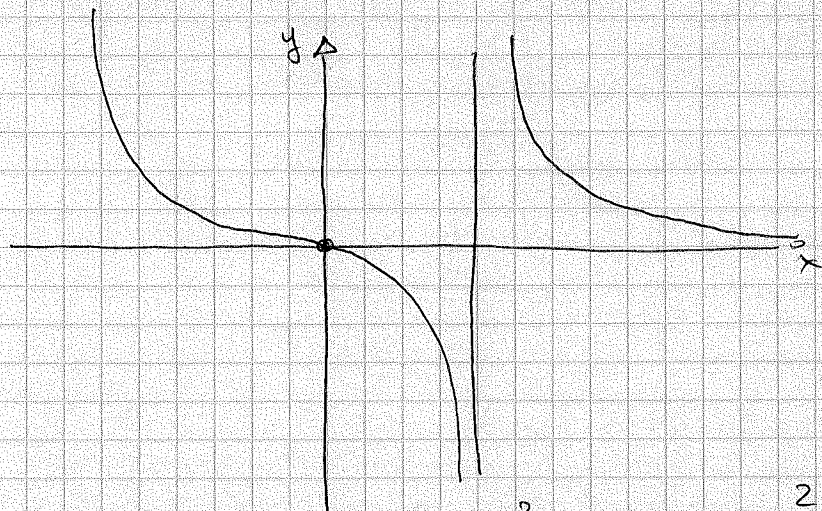


$$f''(x) = \frac{1}{(x-1)^{\cancel{4}3}}\left((x-1)^{\cancel{2}} e^{-x}(x^2 - x + 1) - e^{-x}(2x-1)(\cancel{x}-1)^{\cancel{2}} + e^{-x}(x^2 - x + 1)\, 2\cancel{(x-1)}\right)$$

$$= \frac{e^{-x}}{(x-1)^3}\left(x^3 - x^2 + x - x^2 + x - \cancel{1} - \cancel{2x^2} + \cancel{2x} + x - \cancel{1} + \cancel{2x^2} - \cancel{2x} + \cancel{2}\right)$$

$$= \frac{e^{-x}\,x}{(x-1)^3}\left(x^2-2x+3\right) = \frac{x e^{-x}}{(x-1)^3}\left((x-1)^2+2\right)$$

$$\begin{array}{ccccc} \cup & & \cap & & \cup \\ \hline & 0 & & 1 & \end{array}$$



Sia $c \in (1,2) \Rightarrow \displaystyle\int_c^2 f(x)\,dx = \int_c^2 \frac{x e^{-x}}{x-1}\,dx >$

$$> \int_c^2 \frac{1 \cdot e^{-2}}{x-1}\,dx = e^{-2}\ln(x-1)\Big|_{x=c}^{x=2} = e^{-2}\left(0-\ln(c-1)\right)$$

$\to +\infty$ quando $c \to 1^+ \Rightarrow \displaystyle\int_1^2 f(x)\,dx = +\infty$

Sia $M>2 \Rightarrow \displaystyle\int_2^M f(x)\,dx$ è una funzione monotona crescente di M.

$\Rightarrow \exists \displaystyle\lim_{M\to+\infty}\int_2^M f(x)\,dx = \sup_{M>2}\int_2^M f(x)\,dx$

Inoltre $\displaystyle\int_2^M f(x)\,dx = \int_2^M \frac{x e^{-x}}{x-1}\,dx =$

$$= \int_2^M e^{-x}\,\frac{x-1+1}{x-1}\,dx = \int_2^M e^{-x}\left(1+\frac{1}{x-1}\right)dx <$$

$$< \int_2^M e^{-x}\,2\,dx = -2e^{-x}\Big|_{x=2}^{x=M} = 2\left(e^{-2}-e^{-M}\right) \to 2e^{-2}$$

quando $M\to+\infty$

$\Rightarrow \displaystyle\lim_{M\to+\infty}\int_2^M f(x)\,dx$ esiste finito cioè

$\displaystyle\int^{+\infty} f(x)\,dx$ converge.

$$\sum_{n=1}^{\infty} \frac{\log(1+\alpha^n)}{n^\alpha}$$

<u>Esercizio 4</u>

È une serie a termini positivi

$\lim_{n\to\infty} n^\alpha = +\infty \quad \forall \alpha > 0$

$$\lim_{n\to\infty} \alpha^n = \begin{cases} 0 & \alpha \in (0,1) \\ 1 & \alpha = 1 \\ +\infty & \alpha > 1 \end{cases}$$

<u>Per $\alpha \in (0,1)$</u> abbiamo $\lim_{n\to\infty} \frac{\log(1+\alpha^n)}{\alpha^n} = 1$

$\Rightarrow$ la serie ha lo stesso carattere di $\sum \frac{\alpha^n}{n^\alpha}$

Poiché $\frac{\alpha^n}{n^\alpha} < \frac{\alpha^n}{1} = \alpha^n$ e $\sum \alpha^n$ converge abbiamo che $\sum \frac{\alpha^n}{n^\alpha}$ converge e dunque anche la serie in esame converge.

<u>Per $\alpha = 1$</u> $\quad \frac{\log(1+\alpha^n)}{n^\alpha} = \frac{\log(2)}{n}$. Poiché la serie armonica $\sum \frac{1}{n}$ diverge, anche la serie in esame diverge.

<u>Per $\alpha > 1$</u>

$$n\log(\alpha) = \log(\alpha^n) < \log(1+\alpha^n) \overset{*}{<} \log(2\alpha^n) = \log(2) + n\log(\alpha) \overset{*}{<} 2n\log(\alpha)$$

(*) queste disuguaglianze valgono definitivamente.

$$\Rightarrow \frac{\log(\alpha)}{n^{\alpha-1}} = \frac{n\log(\alpha)}{n^\alpha} < \frac{\log(1+\alpha^n)}{\alpha^n} < \frac{2n\log(\alpha)}{n^\alpha} = \frac{2\log(\alpha)}{n^{\alpha-1}}$$

$\Rightarrow$ la serie ha lo stesso carattere di $\sum \frac{1}{n^{\alpha-1}}$ $\Rightarrow$

diverge per $\alpha - 1 \le 1$ cioè $1 < \alpha \le 2$

converge per $\alpha - 1 > 1$ cioè $\alpha > 2$

<u>RIASSUMENDO</u>

La serie converge per $\alpha \in (0,1) \cup (2,+\infty)$

diverge per $\alpha \in [1,2]$.