INTEGRALI DI FUNZIONI RAZIONALI

$$\int \frac{P(x)}{Q(x)}\,dx \qquad P, Q \text{ polinomi}$$

$$Q(x) = \sum_{k=0}^{n} a_k x^k, \quad a_n = 1. \quad a_k \in \mathbb{R}$$

Teorema fondamentale dell'algebra:

$$Q(z) = (z-\lambda_1)(z-\lambda_2)\cdots(z-\lambda_n)$$

$$\lambda_1, \dots, \lambda_n \in \mathbb{C}.$$

$$Q(\bar{z}) = \sum_{k=0}^{n} a_k \bar{z}^k \qquad \bar{a}_k = a_k$$

$$= \sum_{k=0}^{n} \bar{a}_k \bar{z}^k$$

$$= \overline{\sum_{k=0}^{n} a_k z^k} = \overline{Q(z)}$$

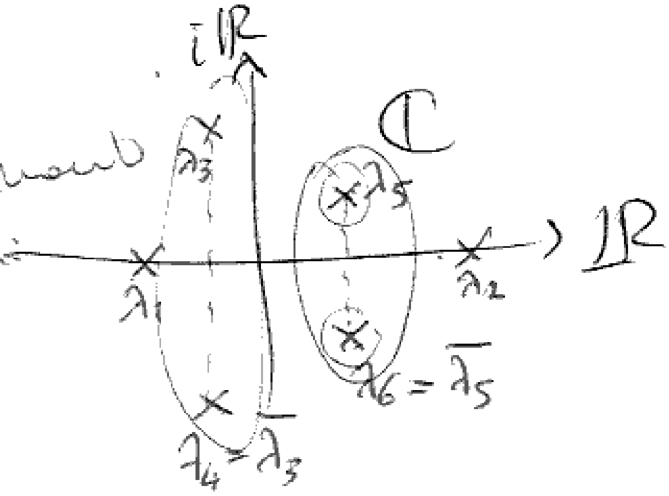
Se $\lambda \in \mathbb{C}$ è una radice di Q

$$Q(\lambda) = 0 = \bar{0} = \overline{Q(\lambda)} = Q(\bar{\lambda})$$

anche $\bar{\lambda} \in \mathbb{C}$ è una radice di Q.

ES $\quad Q(x) = x^2 + 1$

Radici di un polinomio a coefficienti reali



$$Q(x) = (x-\lambda_1)(x-\lambda_2)\cdots(x-\lambda_m)\underbrace{(x-\lambda_{m+1})}_{\mu}\underbrace{(x-\lambda_{m+2})}_{\bar{\mu}}\cdots(x-\lambda_n)$$

Se $\mu \in \mathbb{C} \setminus \mathbb{R}$

$$(x-\mu)(x-\bar{\mu}) = x^2 - (\mu+\bar{\mu})x + \mu\bar{\mu}$$

$$\mu+\bar{\mu} = 2\,\mathrm{Re}\,\mu \in \mathbb{R}$$
$$\mu\bar{\mu} = |\mu|^2 \in \mathbb{R}$$

$$Q(x) = (x-\lambda_1)\cdots(x-\lambda_m)(x^2+\beta_1 x+\alpha_1)\cdots(x^2+\beta_\ell x+\alpha_\ell)$$

$\Delta < 0$

$$m + 2\ell = n$$

$$P(x) = S(x)\,Q(x) + R(x)$$

quoziente, resto

(1) $$\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

$\deg R < \deg Q$

(2) $\deg P < \deg Q$. $Q(x) = (x-\lambda_1)\cdots(x-\lambda_m)$
$(x^2+\beta_1 x+\alpha_1)\cdots(x^2+\beta_\ell x+\alpha_\ell)$

$$\frac{P(x)}{Q(x)} = \frac{A_1}{x-\lambda_1} + \frac{A_2}{(x-\lambda_1)^2} + \dots + \frac{B_1}{(x^2+\beta_1 x+\alpha_1)} + \dots$$

se $\lambda_2 = \lambda_1$

ES
$$\frac{P(x)}{(x-1)^2(x-2)(x^2+x+1)^3} \overset{?}{=} \boxed{\frac{A_1}{x-1} + \frac{A_2}{(x-1)^2}} + \frac{A_3}{x-2} +$$
$$+ \frac{B_1+C_1x}{(x^2+x+1)} + \frac{B_2+C_2x}{(x^2+x+1)^2} + \frac{B_3+C_3x}{(x^2+x+1)^3}$$

$$\frac{\text{Polinomio di grado 5}}{(x^2+x+1)^3}$$

$$\frac{A_1(x-1)+A_2}{(x-1)^2}$$

$$\frac{1}{x-1} \Big/ \frac{1}{(x-1)^2} \Big/ \frac{1}{(x-2)} \Big| \frac{1}{x^2+x+1} \quad \frac{x}{x^2+x+1} \quad \dots$$

Devo mostrare che $Q(x) = (x-1)^2(x-2)(x^2+x+1)^3$

$$\frac{1}{x-1},\ \frac{1}{(x-1)^2},\ \frac{1}{(x-2)},\ \boxed{\frac{1}{x^2+x+1},\ \frac{x}{(x^2+x+1)}},\ (*)$$

$$\frac{1}{(x^2+x+1)^2},\ \frac{x}{(x^2+x+1)^2},\ \frac{1}{(x^2+x+1)^3},\ \frac{x}{(x^2+x+1)^3}$$

sono una base dello spazio vettoriale

$$V_Q = \left\{ \frac{P(x)}{Q(x)} : P \text{ polinomio},\ \deg P < \deg Q \right\}$$

$$= \frac{\mathbb{R}[x]_{<n}}{Q(x)}$$

$$\dim V_Q = n = 9$$

Basta dimostrare che le 9 funzioni $(*)$ sono linearmente indipendenti.

$$\mathrm{span}_{\mathbb{C}}\left\{\frac{1}{x^2+x+1}, \frac{x}{x^2+x+1}\right\} = \mathrm{span}_{\mathbb{C}}\left\{\frac{1}{x-\mu}, \frac{1}{x-\bar{\mu}}\right\}$$

$$x^2+x+1 = (x-\mu)(x-\bar{\mu})$$

$$\left(\begin{array}{l} \mu = \frac{-1+\sqrt{1-3}}{2} = \frac{-1+i\sqrt{3}}{2} \\ \bar{\mu} = \frac{-1-i\sqrt{3}}{2} \end{array}\right)$$

Esercizio di algebra lineare

Mostriamo che sono linearmente indipendenti

$$\frac{1}{x-1}, \quad \frac{1}{(x-1)^2}, \quad \frac{1}{(x-2)}$$

$$f(x) = \frac{c_1}{x-1} + \frac{c_2}{(x-1)^2} + \frac{c_3}{(x-2)} = \underline{0}$$

$$0 \equiv f(x)\cdot(x-2) = \frac{c_1(x-2)}{x-1} + \frac{c_2(x-2)}{(x-1)^2} + c_3$$

f non è definita per $x=2$

per $x=2$ $= 0 + 0 + c_3$

$c_3 = 0$

$$0 \equiv f(x)\cdot(x-1)^2 = \cancel{c_1 + c_2}$$

$$c_1\cdot(x-1) + c_2 + \frac{c_3(x-1)^2}{x-2}$$

per $x=1$ $= 0 + c_2 + 0 \Rightarrow c_2 = 0$

Problema

$$\boxed{\int \frac{1}{(1+x+x^2)^2}\,dx = ?}$$

Decomposizione di Hermite: $Q(x) = (x-1)^2(x-2)(x^2+x+1)^3$

$\deg Q = 9$

Residui

$$\boxed{\frac{P(x)}{(x-1)^2(x-2)(x^2+x+1)^3}} = \boxed{\frac{A_1}{x-1} + \frac{A_2}{x-2} + \frac{A_3 + A_4 x}{x^2+x+1}} + \left(\frac{R(x)}{(x-1)(x^2+x+1)^2}\right)'$$

$$\tilde{Q}(x) = (x-1)(x^2+x+1)^2$$

$$\deg \tilde{Q} = 5$$

allora $\deg R < 5$.

Idea: $\quad \frac{d}{dx}\left(\frac{1}{(x-\lambda)^p}\right)' = -p\,\frac{1}{(x-\lambda)^{p+1}}$

$$\left(\frac{R(x)}{(x-1)(x^2+x+1)^2}\right)' = \left(\frac{R(x)}{x-1}\right)' \frac{1}{(x^2+x+1)^2} + \frac{R(x)}{x-1}\left(\frac{1}{(x^2+x+1)^2}\right)'$$

<u>Idea</u> $\quad \dfrac{1}{(x-\lambda)^p} \xrightarrow{D} \dfrac{-p}{(x-\lambda)^{p+1}}$

$$\ln|x-\lambda| \xrightarrow{D} \boxed{\frac{1}{x-\lambda}}$$

$$\int \frac{1}{(1+x+x^2)^2}\,dx =$$

$$\frac{1}{(1+x+x^2)^2} = \frac{A+Bx}{1+x+x^2} + \left(\frac{Cx+D}{1+x+x^2}\right)'$$

$$= \frac{(A+Bx)(1+x+x^2) + C(1+x+x^2) - (Cx+D)(2x+1)}{(1+x+x^2)^2}$$

$$= \frac{Bx^3 + (A+B+C-2C)x^2 + (A+B+C-C-2D)x + A+C-D}{(1+x+x^2)^2}$$

$$= \frac{Bx^3 + (A+B-C)x^2 + (A+B-2D) + A+C-D}{(1+x+x^2)^2}$$

$$\begin{cases} B=0 \\ A+B-C=0 \\ A+B-2D=0 \\ A+C-D=1 \end{cases} \qquad \begin{cases} B=0 \\ C=A \\ D=A/2 \\ A+A-\frac{A}{2}=1 \end{cases} \qquad \begin{cases} B=0 \\ C=2/3 \\ D=1/3 \\ A=\frac{2}{3} \end{cases}$$

$$\frac{1}{(1+x+x^2)^2} = \frac{\frac{2}{3}\cancel{a}}{1+x+x^2} + \left(\frac{\boxed{2/3}\,x + 1/3}{1+x+x^2}\right)'$$

$$\int \frac{1}{(1+x+x^2)^2}\,dx = \frac{2}{3}\int \frac{1}{1+x+x^2}\,dx \boxed{+\frac{1}{3}\,\frac{2x+1}{1+x+x^2}}$$

si riconduce a $\int \frac{1}{1+y^2}\,dy$

INTEGRALI che si riconducono a integrali di funzioni razionali

<u>Caso 1.</u>

$$f(x) = R\left(e^{\lambda x}\right) \quad , (\lambda \neq 0)$$

$$t = e^{\lambda x}$$

$$\lambda x = \ln t \qquad x = \frac{\ln t}{\lambda} \qquad dx = \frac{1}{\lambda t}\, dt$$

<u>Esempio</u>

$$\int \frac{2\sqrt{e^x} + e^{2x}}{e^x - 4}\, dx \qquad \lambda = \frac{1}{2}$$

$$= \int \frac{2 e^{\frac{x}{2}} + e^{2x}}{e^x - 4}\, dx = \int \frac{2 e^{\frac{x}{2}} + \left(e^{\frac{x}{2}}\right)^4}{\left(e^{\frac{x}{2}}\right)^2 - 4}\, dx$$

$$R(t) = \frac{2t + t^4}{t^2 - 4} \qquad = \int \frac{2t + t^4}{t^2 - 4}\, \frac{2}{t}\, dt$$

$$= 2\int \frac{2 + t^3}{t^2 - 4}\, dt \ \ldots\ldots$$

$$\int R\left(e^{\lambda x}\right) dx = \int \frac{R(t)}{\lambda \cdot t}\, dt = \ldots\ldots$$

<u>Caso 2</u> Funzioni razionali in
$\sin^2 x$, $\sin x \cos x$, $\cos^2 x$

$R(u, v, z)$ razionale

$f(x) = R(\sin^2 x, \sin x \cos x, \cos^2 x)$

<u>Esempio</u> $\displaystyle\int \frac{1}{\cos x \cdot (\sin x + \cos x)} \, dx$

$\displaystyle = \int \frac{1}{\sin x \cos x + \cos^2 x} \, dx = \boxed{\int \frac{1}{\frac{t}{1+t^2} + \frac{1}{1+t^2}} \cdot \frac{1}{1+t^2} \, dt}$

$\displaystyle \ln|\operatorname{tg} x + 1| = \ln|t+1| = \int \frac{1}{t+1} \, dt =$

razionali

Si risolve $\boxed{t = \operatorname{tg} x}$

$\displaystyle \cos^2 x = \frac{1}{1+\operatorname{tg}^2 x} = \frac{1}{1+t^2}$

$\displaystyle \sin x \cos x = \operatorname{tg} x \cdot \cos^2 x = \frac{\operatorname{tg} x}{1+\operatorname{tg}^2 x} = \frac{t}{1+t^2}$

$\displaystyle \sin^2 x = \operatorname{tg}^2 x \cdot \cos^2 x = \frac{t^2}{1+t^2}$

razionale

$x = \operatorname{arctg} t$

$\displaystyle dx = \frac{1}{1+t^2} \, dt$

Caso 3 Esp.

$$R(\cos x, \sin x)$$

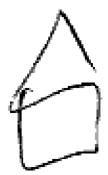
$$t = \mathrm{tg}\,\frac{x}{2} \qquad \frac{x}{2} = \mathrm{arctg}\,t$$

$$\begin{cases} \cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = \dfrac{1-t^2}{1+t^2} \\ \sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = \dfrac{2t}{1+t^2} \end{cases}$$

$$\cancel{dx = d\,2\,\mathrm{arctg}\,t = \dots} \quad \frac{}{1+t^2}$$

$$\int R(\cos x, \sin x)\,dx$$

$$= \int R\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)\frac{2}{1+t^2}\,dt$$



$$\int \frac{1}{\sin x}\,dx$$

$$\left(\text{Es} \qquad \int \frac{\sin x}{\cos^2 x}\,dx \right)$$