ANALISI MATEMATICA (B)

LEZIONE 71

25.3.2020

INIZIO ORE 11:10

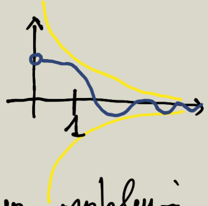
CONVERGENZA ASSOLUTA

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx < +\infty$$

Esempio (analogo per le serie: $\sum_{k=1}^{+\infty} \frac{(-1)^k}{k}$)

$$\int_0^{+\infty} \frac{\sin x}{x}\,dx$$

è convergente ma non assolutamente



Per $x \to 0^+$ non ci sono problemi (la funzione è limitata).

Consideriamo $x \to +\infty$. $\left|\frac{\sin x}{x}\right| \le \frac{1}{x}$ ma $\int_1^{+\infty} \frac{1}{x}\,dx = +\infty$.

$$\int_1^{+\infty} \frac{\sin x}{x}\,dx$$

IDEA INTEGRO PER PARTI

$$= \left[-\frac{\cos x}{x}\right]_1^{+\infty} - \int_1^{+\infty} \frac{\cos x}{x^2}\,dx$$

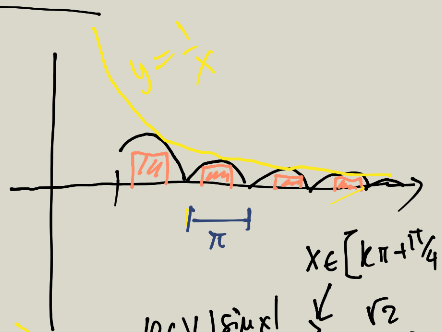
$\left[-\frac{\cos x}{x}\right]_1^{+\infty} \to \frac{\cos 1}{1}$

$\int_1^{+\infty} \frac{\cos x}{x^2}\,dx$ è assolutamente convergente!

$\left|\frac{\cos x}{x^2}\right| \le \frac{1}{x^2}$ e $\int_1^{+\infty} \frac{1}{x^2}\,dx < +\infty$

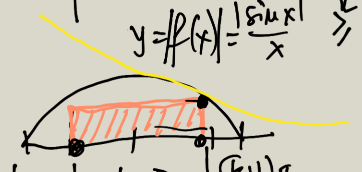
Ma $\int_1^{+\infty} \frac{|\sin x|}{x} = +\infty$

$f(x) = \frac{\sin x}{x}$



$x \in \left[k\pi + \frac{\pi}{4}, k\pi + \frac{3}{4}\pi\right]$

$y = |f(x)| = \frac{|\sin x|}{x} \ge \frac{\sqrt{2}}{2\left(k\pi + \frac{3}{4}\pi\right)}$



$$\int_{k\pi}^{k\pi+\pi} |f(x)| \ge \int_{k\pi+\frac{\pi}{4}}^{k\pi+\frac{3}{4}\pi} \frac{\sqrt{2}}{2}\,\frac{1}{k\pi+\frac{3}{4}\pi}\,dx$$

$$= \frac{\sqrt{2}}{2}\,\frac{\pi/2}{k\pi+\frac{3}{4}\pi} = \frac{\sqrt{2}}{4\left(k+\frac{3}{4}\right)}$$

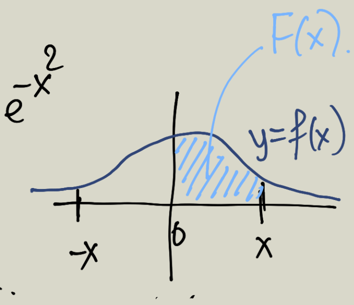
Studio di funzioni integrali

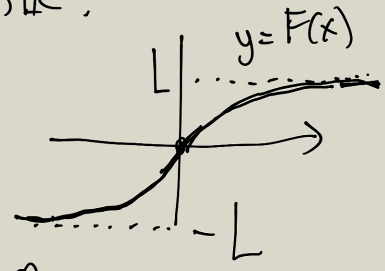
$$F(x) = \int_0^x e^{-t^2}\,dt \qquad f(x) = e^{-x^2}$$

nuova funzione elementare (erf)

è definita $\forall x \in \mathbb{R}$.

$F : \mathbb{R} \to \mathbb{R}$.





$F(0) = 0$

Se $x>0$ $F(x)>0$

Se $x<0$ $F(x)<0$

$$\lim_{\substack{x\to+\infty \\ (-)}} F(x) = \int_0^{+\infty} e^{-t^2}\,dt < +\infty$$

$$e^{-t^2} << \frac{1}{t^2} \qquad \int_1^{+\infty} \frac{1}{t^2}\,dt < +\infty$$

F è dispari $F(-x) = -F(x)$.

chiaro geometricamente ma si può dimostrare analiticamente

$$F(-x) = \int_0^{-x} e^{-t^2}\,dt \underset{\substack{s=-t \\ ds=-dt}}{=} -\int_0^{x} e^{-(-s)^2}\,ds = -\int_0^x e^{-s^2}\,ds = -F(x).$$

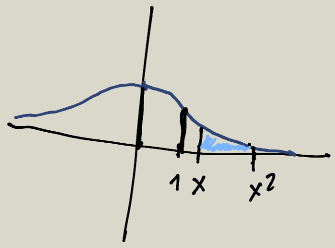
F è crescente (per motivi geometrici)

analiticamente: $F'(x) = e^{-x^2} > 0$ ... $F''(x) = -2x\,e^{-x^2}$ ..

F è strettamente crescente

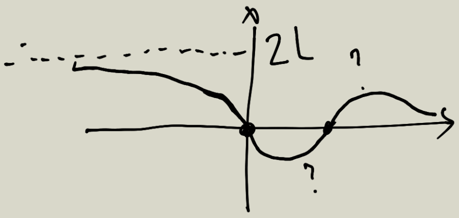
Esercizio

$$F(x) = \int_x^{x^2} e^{-t^2}\,dt$$



per quali x

$F:\mathbb{R}\to\mathbb{R}$ $\quad F(x) \overset{?}{=} 0 \iff x = x^2 \iff x=0,\ x=1$



$$\lim_{x\to+\infty} F(x) \overset{?}{=} 0 \qquad F(x) = \int_x^{x^2} e^{-t^2}dt \le \int_x^{+\infty} e^{-t^2}dt$$

Teorema della coda

Se $\int_a^{+\infty} f(x)\,dx$ converge allora $\lim_{x\to+\infty} \int_x^{+\infty} f(x)\,dx = 0$

$$\int_1^{+\infty} e^{-t^2}dt < +\infty$$

$$\| \quad \lim_{x\to+\infty} \int_1^x e^{-t^2}dt = L < +\infty$$

$$\lim_{x\to-\infty} F(x) = \int_{-\infty}^{+\infty} e^{-t^2}dt = 2L$$

con L visto prima

coda $\Rightarrow$ $\displaystyle\int_x^{+\infty} e^{-t^2}dt = \underbrace{\int_1^{+\infty} e^{-t^2}dt}_{= L} - \underbrace{\int_1^{x} e^{-t^2}dt}_{\to L} \to 0$

$$F'(x) = ?$$

$$G(x) = \int_0^x e^{-t^2}dt, \quad G'(x) = e^{-x^2}$$

$$F(x) = \big[G(t)\big]_x^{x^2} = G(x^2) - G(x)$$

$$F'(x) = G'(x^2)\cdot 2x - G'(x) = e^{-(x^2)^2}\cdot 2x - e^{-x^2}$$

$$= e^{-x^4}\cdot 2x - e^{-x^2}.$$

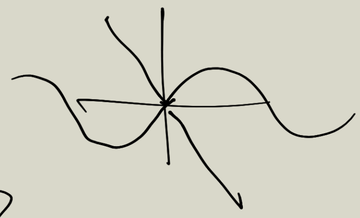
$$\left[\ln|t|\right]_x^{2x} = \ln|2x| - \ln|x| = \ln\frac{|2x|}{|x|} = \ln 2$$

...ucato?

$$F(x) = \int_x^{2x} \frac{1}{t+\sin t}\,dt$$

$$f(x) = \frac{1}{x+\sin x}$$

$-x \neq \sin x$

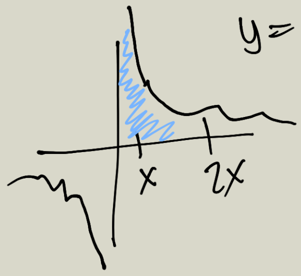


$x \neq 0.$

$f: \mathbb{R}\setminus\{0\} \to \mathbb{R}$

$\frac{1}{t+\sin t} \sim \frac{1}{2t}$ per $t \to 0$

$\int_0^{\varepsilon} \frac{1}{2t} = +\infty$



è pari?

$F: \mathbb{R}\setminus\{0\} \to \mathbb{R}$

$$\lim_{x\to 0^+} F(x) = ? = \frac{\ln 2}{2}$$

$\lim_{x\to 0^+} F(x)$

$\lim_{x\to +\infty} F(x)$

$$\int_x^{2x} \frac{1}{t+\sin t}\,dt = \underbrace{\int_x^{2x} \frac{1}{2t}\,dt}_{=\frac{\ln 2}{2}} + \underbrace{\int_x^{2x}\left[\frac{1}{t+\sin t} - \frac{1}{2t}\right]dt}$$

$$\int_x^{2x} \frac{2t - t - \sin t}{2t(t+\sin t)}\,dt = \int_x^{2x} \frac{\frac{t^3}{6}(1+o(1))}{4t^2(1+o(1))}\,dt$$

$\sin t = t - \frac{t^3}{6} + \dots$

$$1-\varepsilon \le \frac{1+o(1)}{1+o(1)} \le 1+\varepsilon$$

$\Uparrow$ se x abbastanza piccolo.

$$\frac{t}{24}\,\frac{1+o(1)}{1+o(1)}\,dt \longrightarrow \qquad \frac{1+o(1)}{1+o(1)} \to 1$$

$$\Downarrow$$

$$(1-\varepsilon)\cdot\underbrace{\int_x^{2x}\frac{t}{24}\,dt}_{\downarrow\, 0} \le (*) \le \underbrace{\int_x^{2x}\frac{t}{24}}_{\downarrow\, 0}$$

$$\downarrow 0 \qquad \downarrow 0$$

$$\int_x^{2x}\frac{t}{24}\,dt = \left[\frac{t^2}{2\cdot 24}\right]_x^{2x} = \frac{4x^2}{48} - \frac{x^2}{48} = \frac{3x^2}{48} \to 0$$

per $x \to 0$

1 per $t \to 0$

$\forall \varepsilon > 0 \ \exists \delta > 0$

$$\left| \frac{1+o(1)}{1+o(1)} - 1 \right| < \varepsilon$$

$$1-\varepsilon < \frac{1+o(1)}{1+o(1)} < 1+\varepsilon$$

$$\uparrow$$

$$\frac{1+o(1)}{1+o(1)} = 1 + o(1)$$

$$\int_x^{2x} (1+o(1))\, dt$$

$$\parallel$$

$$x + \int_x^{2x} o(1)\, dt$$

$$\left| \int_x^{2x} o(1) \right| < \varepsilon \cdot |x|$$

se $|x| < \delta$

$$\frac{1+o(1)}{1+o(1)} = 1+o(1)$$

$$(*) = \int_x^{2x} \frac{t}{24}\left(1+o(1)\right) dt$$

$$= \int_x^{2x} \left[\frac{t}{24} + o(t)\right] dt$$

$$= \int_x^{2x} \frac{t}{24}\, dt + \int_x^{2x} o(t)\, dt \longrightarrow 0$$

$$0 \xleftarrow{\text{pr } x \to 0} \cdots = \left[\frac{t^2}{48}\right]_x^{2x}$$

$o(t) \to 0$

$\forall \varepsilon > 0\ \exists \delta > 0 : |t| < \delta \Rightarrow |o(t)| < \varepsilon$

pr $x < \delta/2$

$\Rightarrow t \leq 2x < \delta$

$$\int_x^{2x} |o(t)|\, dt \leq \int_x^{2x} \varepsilon\, dt = \varepsilon \cdot x \to 0$$

pr $x \to 0$.