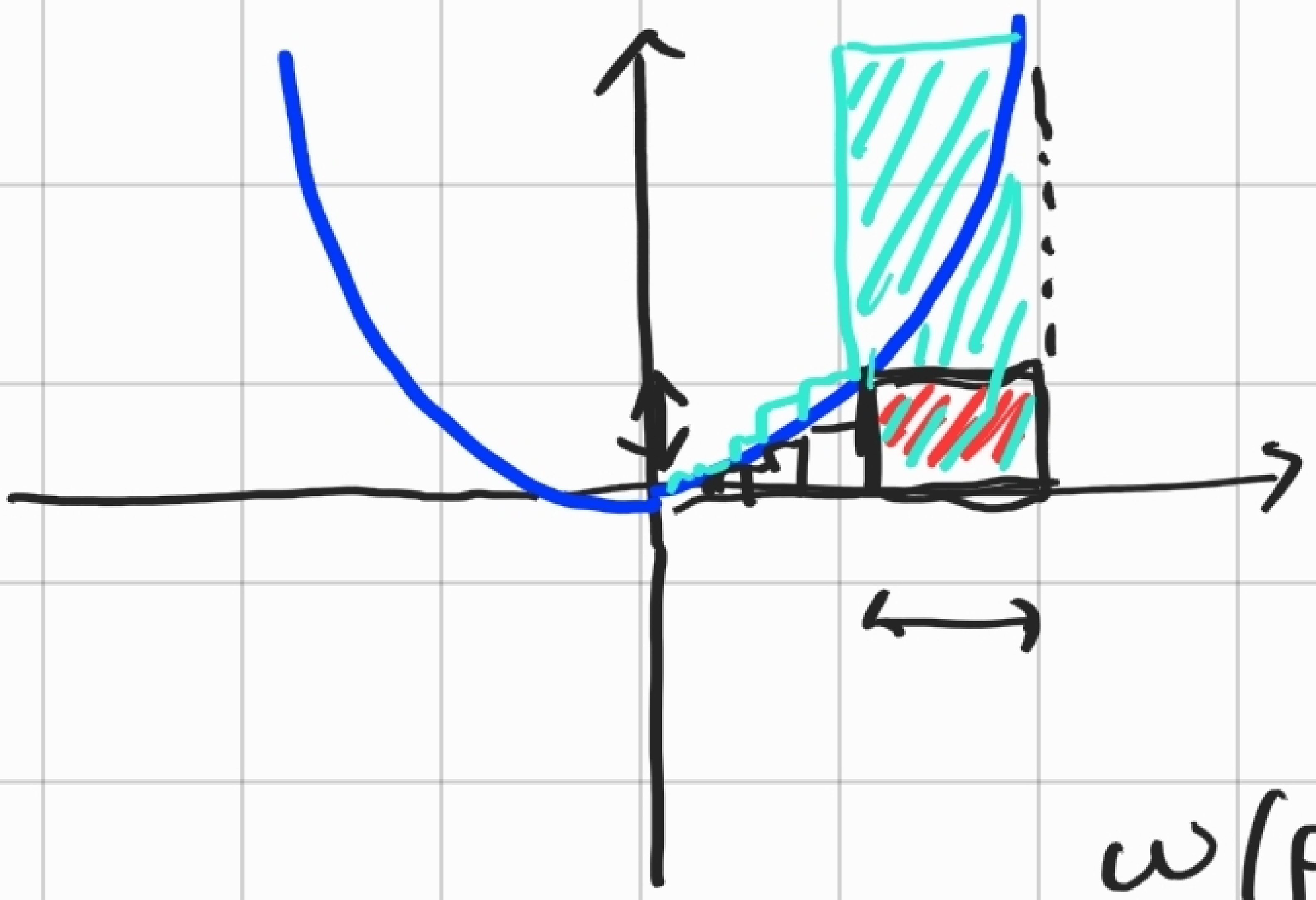
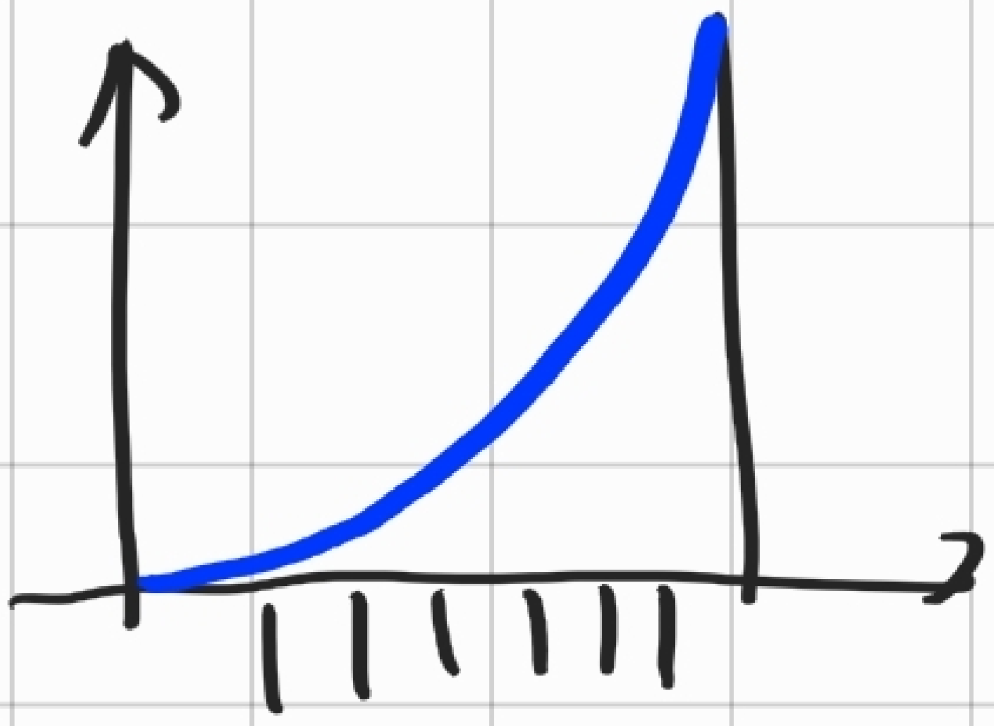
ANALISI MATEMATICA B
LEZIONE 64 - 12.3.2021





$$\omega(\rho) = \sup_{|x-y|\le \rho} |f(x)-f(y)|$$

f è uniformemente continua $\Leftrightarrow$ $\omega(\rho) \to 0$

Integrazione di funzioni razionali

P, Q polinomi

$$\int \frac{P(x)}{Q(x)}\,dx$$

① se $\deg P \geq \deg Q$ $\to$ posso fare la divisione:

$[\,P = N \cdot Q + R\,]$
$\deg R < \deg P$.

$$\frac{P(x)}{Q(x)} = N(x) + \frac{R(x)}{Q(x)}$$

$\deg R(x) < \deg Q(x)$

Es $\int \frac{x^3+2}{x+1}\,dx = \int \left(x^2-x+1+\frac{1}{x+1}\right) dx =$

$$\begin{array}{r|l} x^3+2 & x+1 \\ x^3+x^2 & \overline{x^2-x+1} \\ \hline -x^2+2 & \\ -x^2-x & \\ \hline x+2 & \\ x+1 & \\ \hline 1 & \end{array}$$

x^2-x+1 ← quoziente

1 ← resto

VERIFICA

$$\frac{x^3 - x^2 + x + x^2 - x + 1 + 1}{x+1} = \frac{x^3+2}{x+1}$$

$$= \frac{x^3}{3} - \frac{x^2}{2} + x + \ln(x+1)$$

Se $\deg Q = 1$ $\qquad \int \frac{a}{x+c} = a\ln(x+c)$

Se $\deg Q = 2$

$$\int \frac{dx+e}{x^2+bx+c}\,dx$$

$$\Delta = b^2 - 4c$$

Se $\Delta > 0$ $\qquad x^2+bx+c = (x-\lambda_1)(x-\lambda_2)$

$$\lambda_{12} = \frac{-b \pm \sqrt{\Delta}}{2}$$

Se $\Delta < 0$

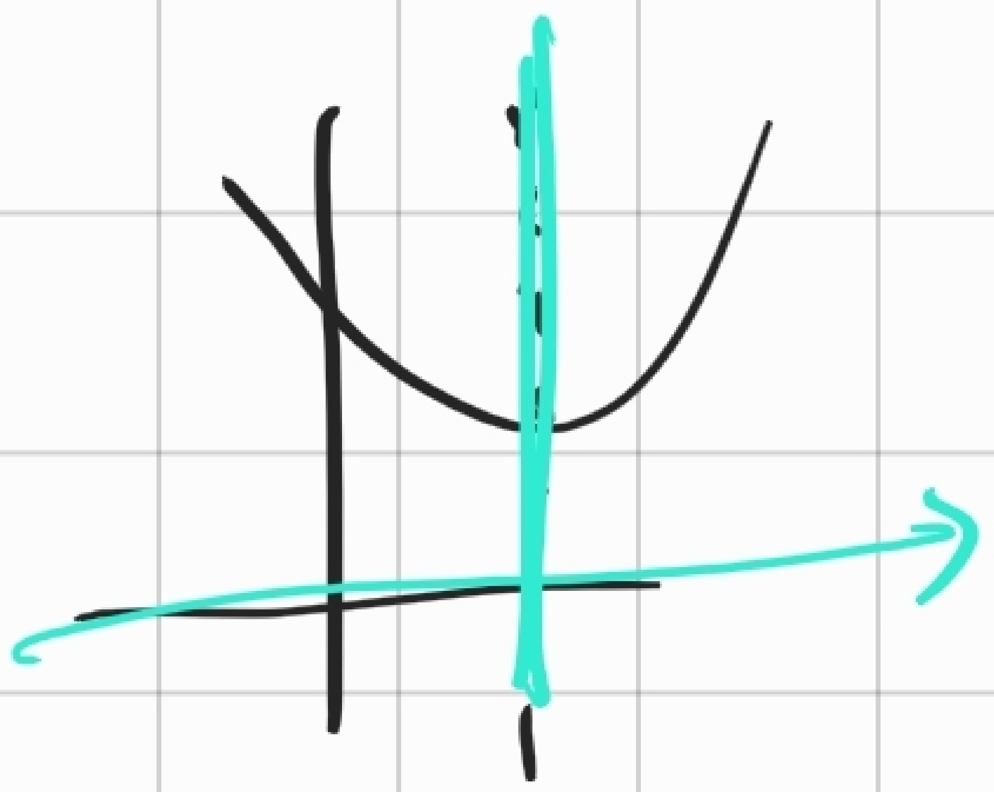
<u>Es</u> $\int \frac{1-x}{x^2+1}\, dx$

$$= \int \frac{1}{x^2+1}\, dx - \frac{1}{2}\int \frac{2x}{x^2+1}\, dx$$

$$= \arctan x - \frac{1}{2} \ln (x^2+1)$$

In generale $\frac{dx+e}{x^2+bx+c}$ $\quad \Delta < 0$

$$x^2+bx+c = (x-\lambda)^2 + k$$



$$= k\left[\left(\frac{x-\lambda}{\sqrt{k}}\right)^2 + 1\right]$$

$$y = \frac{x-\lambda}{\sqrt{x}} \qquad y^2+1$$

ES $\displaystyle\int \frac{x}{x^2+x+1}\,dx$

$\Delta = 1-4<0$

$$x^2+x+1 = \left(x+\frac{1}{2}\right)^2 + 1 - \frac{1}{4}$$

↑ ↑ doppio prodotto

$$= \left(x+\frac{1}{2}\right)^2 + \frac{3}{4}$$

$$= \frac{3}{4}\left[\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)^2 + 1\right]$$

$$= \frac{3}{4}\left[\left(\frac{2x+1}{\sqrt{3}}\right)^2+1\right] = \frac{3}{4}\left[y^2+1\right]$$

$$y = \frac{2x+1}{\sqrt{3}}, \quad x = \frac{\sqrt{3}y-1}{2}$$

$$dx = \frac{\sqrt{3}}{2}dy$$

$$\int \frac{x}{x^2+x+1}\,dx = \frac{4}{6}\int \frac{\sqrt{3}y-1}{y^2+1}\,\frac{\sqrt{3}}{2}\,dy$$

$= (*)$

$$\int \frac{1}{x^2+\frac{3}{4}}\,dx = ? = \frac{4}{3}\int \frac{1}{\frac{4}{3}x^2+1}\,dx$$

$$= \frac{4}{3} \operatorname{arctg}\left(\frac{2}{\sqrt{3}} x\right) \cdot \frac{\sqrt{3}}{2}$$

$$= \frac{2}{\sqrt{3}} \operatorname{arctg}\left(\frac{2}{\sqrt{3}} x\right)$$

<u>oppure</u>

$$\int \frac{x}{x^2+x+1} dx = \frac{1}{2} \int \frac{\overbrace{2x+1}-1}{x^2+x+1} dx$$

$$= \frac{1}{2} \ln\left(x^2+x+1\right) - \frac{1}{2} \int \frac{1}{x^2+x+1} dx$$

$= \ldots\ldots$

$$\frac{4}{6} \int \frac{\sqrt{3}y-1}{y^2+1} \frac{\sqrt{3}}{2} dy = \frac{1}{\sqrt{3}} \int \frac{\sqrt{3}y-1}{y^2+1} dy$$

$$= \frac{\sqrt{3}}{\sqrt{3}} \cdot \frac{1}{2} \int \frac{2y - \frac{2}{\sqrt{3}}}{y^2+1} dy$$

$$= \frac{1}{2} \ln\left(y^2+1\right) - \frac{1}{2} \frac{2}{\sqrt{3}} \operatorname{arctg} y$$

$$= \frac{1}{2} \ln\left(\left(\frac{2x+1}{\sqrt{3}}\right)^2+1\right) - \frac{1}{\sqrt{3}} \arctg \frac{2x+1}{\sqrt{3}}$$

$$= \frac{1}{2} \ln\left[\frac{4}{3}\left(x^2+ x+1\right)\right] - \frac{1}{\sqrt{3}} \arctg \frac{2x+1}{\sqrt{3}}$$

Q generico

Più in generale se $\deg Q > 2$

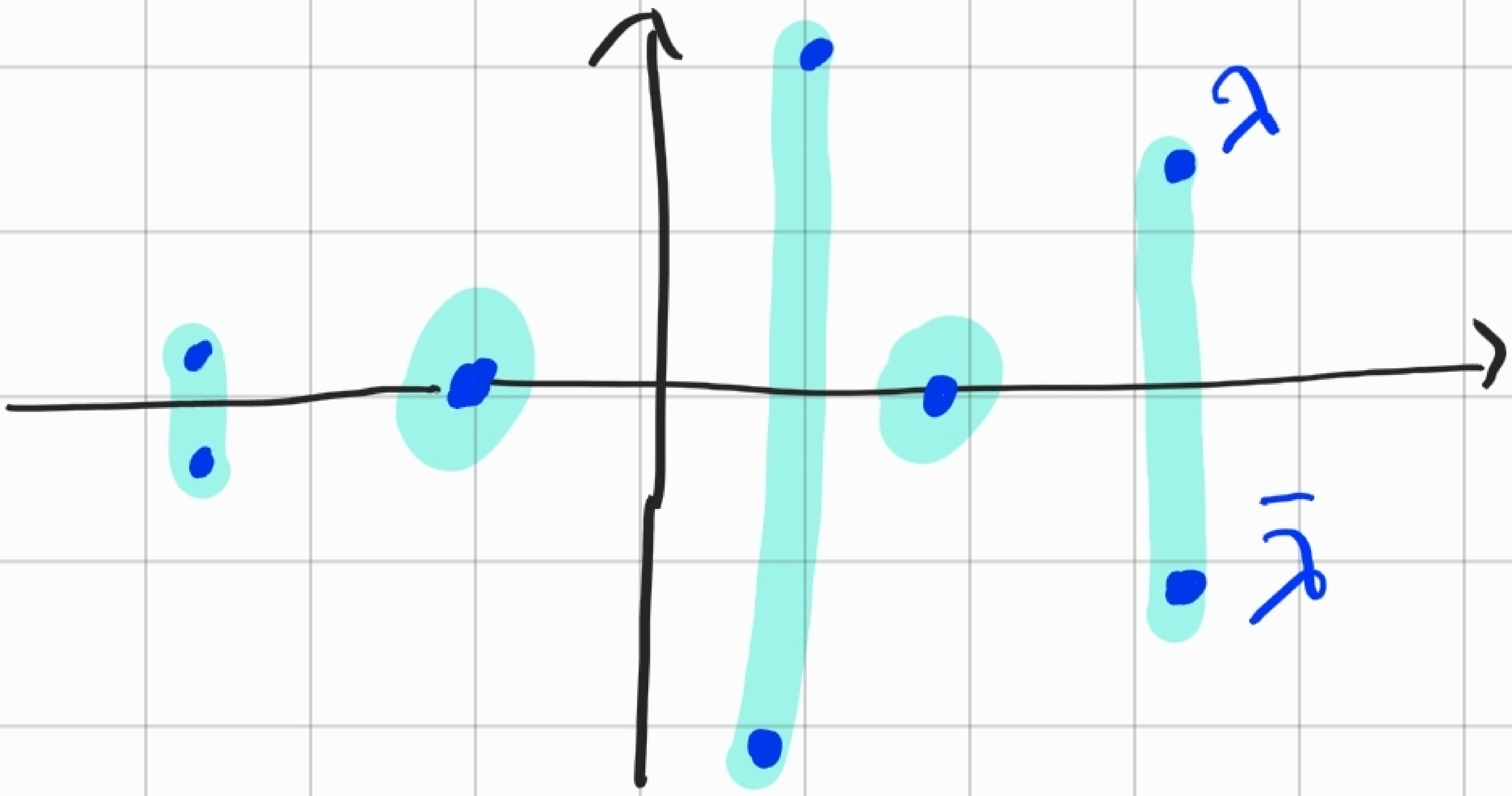
$$Q(x) = (x-\lambda_1)(x-\lambda_2)\cdots(x-\lambda_n)$$

$$\lambda_1, \dots, \lambda_n \in \mathbb{C}.$$

Se Q ha coefficienti reali

Se $Q(\lambda) = 0$
allora $Q(\bar{\lambda}) = 0$

$\overline{Q(\lambda)} = \bar{0} = 0$
$\quad \shortparallel$
$Q(\bar{\lambda})$



Es:

$ax^2+bx+c=0$

$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

se $\Delta = b^2-4ac < 0$

$$\frac{-b \pm i\sqrt{-\Delta}}{2a}$$

$$(x-\lambda)(x-\bar{\lambda}) = x^2 - (\lambda+\bar{\lambda})x + \lambda\bar{\lambda}$$

$$= x^2 - 2\,\mathrm{Re}\lambda \cdot x + |\lambda|^2$$

polinomio a coefficienti reali

$$Q(x) = (x-\lambda_1)\cdots(x-\lambda_n)\cdot(x^2+b_1x+c_1)\cdots(x^2+b_1+c_1)$$

radici reali — coppie di radici complesse coniugate.

$$= (x-\lambda_1)^{p_1}\cdots(x-\lambda_n)^{p_n}\,(x^2+b_1x+c_1)^{q_1}\cdots(\quad)^{q_m}$$

Scomposizione di Hermite

$$\deg P < \deg Q = N = p_1+\dots+p_n+2q_1+\dots+2q_m$$

$$Q(x) = (x-\lambda_1)^{p_1}\cdots(x-\lambda_n)^{p_n}\cdot(x^2+\alpha_1x+\beta_1)^{q_1}\cdots(x^2+\alpha_mx+\beta_m)^{q_m}$$

$$\frac{P(x)}{Q(x)} = \sum_{k=1}^{n}\frac{A_k}{x-\lambda_k} + \sum_{k=1}^{m}\frac{B_kx+C_k}{x^2+\alpha_kx+\beta_k} + \left(\frac{R(x)}{\tilde{Q}(x)}\right)'$$

$$n+2m+(N-n-2m) = N$$

dove

$$\tilde{Q}(x) = (x-\lambda_1)^{p_1-1} \cdots (x-\lambda_n)^{p_n-1} \cdot (x^2+\alpha_1 x+\beta_1)^{q_1-1} \cdots (x^2+\alpha_m x+\beta_m)^{q_m-1}$$

$R(x)$ è un polinomio da determinare

$$\deg R < \deg \tilde{Q}$$

idea di un

$$\frac{1}{x-\lambda_k}, \frac{1}{(x-\lambda_k)^2}, \frac{1}{(x-\lambda_k)^3} \ldots$$

sono vettori indipendenti

$$\frac{1}{x^2} = \left(-\frac{1}{x}\right)'$$

Esempio (esageratamente complicato)

$$\int \frac{x^{10}-3x^5+1}{x^8-x^7+2x^6-2x^5+x^4-x^3}\,dx \quad \leftarrow Q(x)$$

$$\begin{array}{l|l} x^{10}-3x^5+1 & Q(x) \\ \hline \vdots & x^2+x \\ \hline x^4+1 & \end{array}$$

$\deg Q = 8$

$$\int \frac{P(x)}{Q(x)} = \int (x^2 + x)\,dx + \int \frac{x^4+1}{Q(x)}\,dx$$

$$\| \quad \frac{x^3}{3} + \frac{x^2}{2}$$

$$Q(x) = x^3\left(x^5 - x^4 + 2x^3 - 2x^2 + x - 1\right)$$

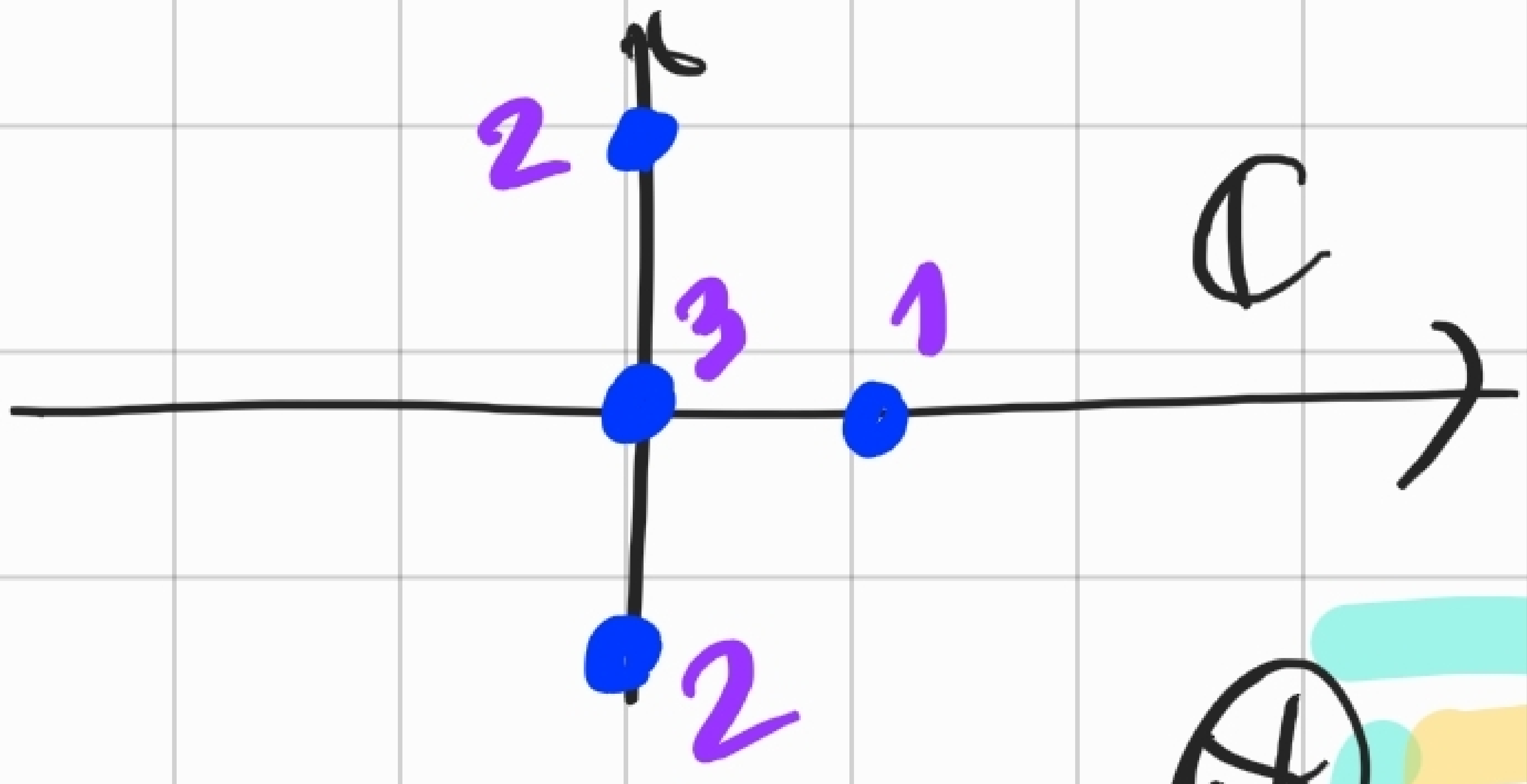
$$1 - 1 + 2 - 2 + 1 - 1 = 0$$

$$= x^3 (x-1)(x^4 + 2x^2 + 1)$$

$$= x^3 (x-1)(x^2+1)^2$$

$\lambda_1 = 0 \quad \lambda_2 = 1 \quad \lambda_{3,4} = \pm i$

$p_1 = 3 \quad p_2 = 1 \quad q_1 = 2$



$\deg R < 4$

$R(x)$

Th

$$\frac{P(x)}{Q(x)} = \frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{x^2+1} + \left(\frac{Ex^3+Fx^2+Gx+H}{x^2(x^2+1)}\right)'$$

(*)

$$\longrightarrow \frac{\tilde{E}}{x^2} + \frac{\tilde{F}}{x^3} + \frac{\tilde{G} + \tilde{H}x}{(x^2+1)^2}$$

$$\int \frac{1}{(x^2+1)^2}\,dx = \left(\frac{ax+b}{x^2+1}\right)'$$

$$\left(\frac{Ex^3+Fx^2+Gx+H}{x^2(x^2+1)}\right)' = \frac{\%_0}{x^2}\cdot\frac{1}{(x^2+1)}$$

$$= \left(\frac{\%_0}{x^2}\right)'\cdot\frac{1}{(x^2+1)} + \left(\frac{\%_0}{x^2}\right)\cdot\left(\frac{1}{x^2+1}\right)'$$

$$= \frac{\cdots}{x^3(x^2+1)} + \frac{\cdots}{x^2(x^2+1)^2}$$

$$= \frac{\cdots}{x^3(x^2+1)^2}$$

$$\frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{x^2+1} + \frac{\cdots}{x^3(x^2+1)^2}$$

$$= \frac{Z(x)}{x^3(x-1)(x^2+1)^2}$$

per curiosità:

$$Z(x) = (A+B+C)x^7 + (-A-C+D-E)x^6 + (2A+2B+C-D+E-2F)x^5 +$$

$$+ (-2A-C+D+E+2F-3G)x^4 + (A+B-D-E+3G-4H)x^3$$

$$+ (-A-G+4H)x^2 + (G-2H)x + 2H$$