ANALISI MATEMATICA B
LEZIONE 37 - 17.12.2021

Teo ① Se $\sum |a_k| < +\infty$ allora per ogni $\sigma : \mathbb{N} \to \mathbb{N}$ bigettiva

$$\sum a_k = \sum a_{\sigma(k)}$$

(Esempio: $\sum \frac{(-1)^k}{k}$ convergente $\qquad \sum \left|\frac{(-1)^k}{k}\right| = \sum \frac{1}{k} = +\infty$

$a_k = \frac{(-1)^k}{k}$ $\qquad \forall x \in \mathbb{R} \ \exists \sigma : \mathbb{N} \to \mathbb{N}$ t.c. $\sum a_{\sigma(k)} = x$.)

Teo ② Se $\sum_{k=0}^{+\infty} \sum_{j=0}^{+\infty} |a_{kj}| < +\infty$

$$\sum_{k=0}^{+\infty} \sum_{j=0}^{+\infty} a_{k,j} = \sum_{n=0}^{+\infty} \sum_{k=0}^{n} a_{k,n-k}$$

Teorema (della coda) Se $\sum a_k$ è convergente allora

$$\lim_{n \to +\infty} \sum_{k=n}^{+\infty} a_k = 0$$

dim

$$S_n = \sum_{k=0}^{n-1} a_k \qquad S_n \to S \in \mathbb{R}.$$

$$S - S_n \to 0$$

$$\overset{\shortparallel}{\sum_{k \geq n}^{+\infty} a_k}$$

$\square$

dim (1)

Hyp: $\sum |a_k|$

$$\sum a_k \overset{?}{=} \sum a_{\sigma(k)}$$

$n=8$

$$\begin{array}{cccccccc} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \\ a_{\sigma(4)} & a_{\sigma(5)} & a_{\sigma(6)} & a_{\sigma(2)} & & a_{\sigma(3)} & & a_{\sigma(1)} & \quad a_{\sigma(7)} & \quad a_{\sigma(0)} \end{array}$$

$$(*) = \left| \sum_{k=0}^{n-1} a_k - \sum_{j=0}^{n-1} a_{\sigma(j)} \right| < \varepsilon$$

$k = \sigma(j) \qquad j = \sigma^{-1}(k)$

i termini che non si cancellano sono quelli che hanno indice nell'insieme:

$\rightarrow A_1 = \{k : k < n \text{ e } \sigma^{-1}(k) \geq n\}$ stanno nella prima somma ma non nella seconda

$\rightarrow A_2 \{k : k \geq n \text{ ma } \sigma^{-1}(k) < n\}$

$$(*) \leq \sum_{k \in A_1 \cup A_2} |a_k|$$

Siccome $\sum_{k=0}^{+\infty} |a_k|$ è convergente $\sum_{k=N}^{+\infty} |a_k| \to 0$ per $N \to +\infty$

$$\forall \varepsilon > 0 \quad \exists N : \sum_{k=N}^{+\infty} |a_k| < \varepsilon .$$

$$\forall N\ \exists n : \quad A_1 \cup A_2 \subseteq \{N, N+1, N+2, \dots\} = \{n : n \geq N\}$$

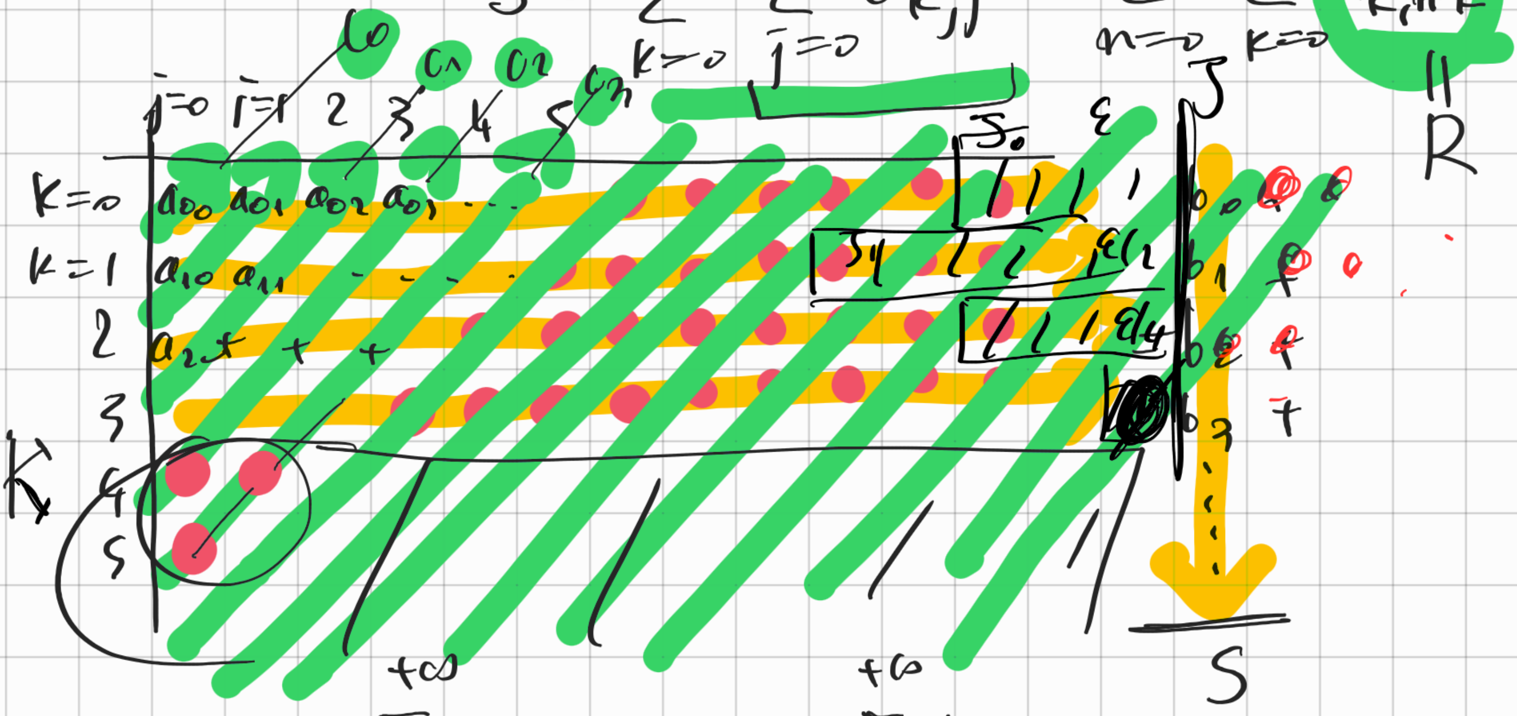
$$\{\sigma(0), \dots, \sigma(n)\} \supseteq \{0, 1, \dots, N\}$$

$$n = \max \sigma^{-1}(\{0,1,\dots N\})$$

$\square$

<u>dém</u> ②

$$S = \sum_{k=0}^{+\infty} \sum_{j=0}^{+\infty} a_{k,j} \overset{?}{=} \sum_{n=0}^{+\infty} \sum_{k=0}^{n} a_{k,n-k} = R$$



$$b_k = \sum_{j=0}^{+\infty} a_{k,j} \qquad S = \sum_{k=0}^{+\infty} b_k$$

$$\text{Hyp}: \sum_{k=0}^{+\infty} \sum_{j=0}^{+\infty} |a_{k,j}| < +\infty \qquad \forall k : \sum_{j=0}^{+\infty} |a_{k,j}| < +\infty$$

$$\geq \sum |b_k| < +\infty \qquad \sum_{k=0}^{+\infty} |b_k| < +\infty$$

$$(\#\mathbb{N} \times \mathbb{N} = \#\mathbb{N})$$

$$c_n = \sum_{k=0}^{n} a_{k,n-k}$$

$j = n-k$
$k+j = n$

$$R = \sum_{n=0}^{+\infty} c_n \qquad S \overset{?}{=} R$$

$$\left| \sum_{k=0}^{K} b_k - \sum_{n=0}^{N} c_n \right| < 3\varepsilon$$

$\forall \varepsilon > 0$ siccome $\sum_{k=0}^{+\infty} \sum_{j=0}^{+\infty} |a_{kj}| < +\infty$ $\exists K : \sum_{k=K}^{+\infty} \sum_{j=0}^{+\infty} |a_{kj}| < \varepsilon$

$\Rightarrow$ I termini di $\sum_{n=0}^{N} c_n$ che non stanno in $\sum_{k=0}^{K} b_k$ hanno somma piccola.

Ora fissato k $\sum_{j=0}^{+\infty} |a_{kj}| < +\infty$

$\Downarrow$ coda

$$\sum_{j=J_k}^{+\infty} |a_{kj}| < \frac{\varepsilon}{2^k}$$

Scelgo $N = K + J$ $\max\{J_0, J_1, \dots, J_K\}$

così se $k \leq K$ ma $(k+j) \geq N$ allora $j \geq J \Rightarrow j \geq J_k$

$$\sum |a_{kj}| \leq \sum_{k=0}^{K} \frac{\varepsilon}{2^k} = 2\varepsilon \qquad \square$$

Esempio rilevante

$$\exp(z) = \sum_{k=0}^{+\infty} \frac{z^k}{k!} \quad \text{è convergente}$$

$$\boxed{\exp(z)\exp(w) \stackrel{?}{=} \exp(z+w) = R}$$

$$\Vert$$

$$\rightarrow \left(\sum_{k=0}^{+\infty} \frac{z^k}{k!} \right) \cdot \left(\sum_{j=0}^{+\infty} \frac{w^j}{j!} \right)$$

$$\Vert$$

$$\sum_{k=0}^{+\infty} \left(\sum_{j=0}^{+\infty} \frac{w^j}{j!} \right) \cdot \frac{z^k}{k!}$$

$$\Vert$$

$$S = \underbrace{\sum_{k=0}^{+\infty} \sum_{j=0}^{+\infty} \underbrace{\left(\frac{z^k}{k!} \frac{w^j}{j!} \right)}}_{a_{k,j}}$$

$$R = \sum_{n=0}^{+\infty} \frac{(z+w)^n}{n!} = \sum_{n=0}^{+\infty} \sum_{k=0}^{n} \binom{n}{k} \frac{z^k w^{n-k}}{\cancel{n!}}$$

$$\binom{n}{k} = \frac{\cancel{n!}}{k!(n-k)!}$$

$$= \sum_{n=0}^{+\infty} \sum_{k=0}^{n} \underbrace{\frac{z^k}{k!} \frac{w^{n-k}}{(n-k)!}}_{a_{n,n-k}} \qquad a_{k,j} = \frac{z^k}{k!} \frac{w^j}{j!}$$

$$\sum_{k=0}^{+\infty} \sum_{j=0}^{+\infty} |a_{k,j}| = \sum_{k=0}^{+\infty} \sum_{j=0}^{+\infty} \left| \frac{z^k}{k!} \frac{w^j}{j!} \right| = \sum_{k=0}^{+\infty} \sum_{j=0}^{+\infty} \frac{|z|^k}{k!} \frac{|w|^j}{j!}$$

$$\ldots\ldots = \left(\sum_{k=0}^{+\infty} \frac{|z|^k}{k!} \right) \cdot \left(\sum_{j=0}^{+\infty} \frac{|w|^j}{j!} \right) = \exp|z| \cdot \exp|w| < +\infty$$

sono convergenti!

Cosa sappiamo di $\exp(z)$?

1. $\exp(0) = 1 \quad \Leftarrow \quad \exp(0) = \sum_{k=0}^{+\infty} \frac{0^k}{k!} = 1.$

2. $\exp(\bar{z}) = \overline{\exp(z)}$

$$\sum_{k=0}^{N} \frac{\bar{z}^k}{k!} = \sum_{k=0}^{N} \overline{\left(\frac{z^k}{k!} \right)} = \overline{\sum_{k=0}^{N} \frac{z^k}{k!}}$$

$$\downarrow \qquad\qquad \downarrow N \to +\infty$$

$$\sum_{k=0}^{+\infty} \frac{\bar{z}^k}{k!} \qquad\qquad \overline{\sum_{k=0}^{+\infty} \frac{z^k}{k!}}$$

$$\shortparallel \qquad\qquad \shortparallel$$

$$\exp(\bar{z}). \qquad\qquad \overline{\exp(z)}$$

(3) $\exp(z+w) = \exp(z)\cdot\exp(w)$

$\Rightarrow$ (4) $\exp(-z) = \dfrac{1}{\exp(z)}$

$$1 = \exp(z-z) = \exp(z)\cdot\underbrace{\exp(-z)}$$

in particolare $\exp(z) \neq 0 \quad \forall z \in \mathbb{C}$.

(5) $\exp(x) = a^x$

$$\underbrace{\exp(x+y) = \exp(x)\exp(y)}$$

$x \geq y \overset{?}{\Rightarrow} \exp(x) \geq \exp(y)$

crescente

$$\exp(x) - \exp(y) \geq 0$$

$$\exp(y)\left(\frac{\exp(x)}{\exp(y)} - 1\right) \geq 0$$

$$\exp(y)\underbrace{(\exp(x-y) - 1)} \geq 0$$

$t = x-y \geq 0 \Rightarrow \exp(x-y) \geq 1$

$$\exp t = \sum_{k=0}^{+\infty} \frac{t^k}{k!} = 1 + \sum_{k=1}^{+\infty} \frac{t^k}{k!} \geq 1.$$

$\exp(y) > 0$

Se $y > 0$ $\quad \exp(y) \geq 1.$

$$\exp(-y) = \frac{1}{\exp(y)} \geq 0$$

$$\exp(x) = a^x \qquad a = \exp(1).$$

(6) $\exp(z)$ è continua.

$$\lim_{z\to 0} \frac{e^z - 1}{z} = 1$$

$$\| \quad \sum_{k=1}^{+\infty} \frac{z^{k-1}}{k!}$$

Più in generale: $\boxed{\sum a_k z^k}$ serie di potenze.