ANALISI MATEMATICA B
LEZIONE 66 — 14.3.2022

Integrali che si riconducono a funzioni razionali

$\int \frac{P(x)}{Q(x)}\,dx$ lo sappiamo calcolare.

<u>Caso 1</u>. $\int f(e^{\lambda x})\,dx$ $\quad \lambda \in \mathbb{R}$ $\quad f$ razionale $= \frac{P}{Q}$.

P, Q polinomi

$$\begin{cases} y = e^{\lambda x} \\ \lambda x = \ln y \\ \underline{dx} = \underline{\frac{1}{\lambda y}\,dy} \end{cases} \qquad \int f(e^{\lambda x})\,dx = \int \frac{f(y)}{\lambda y}\,dy = \dots$$

<u>ES</u>

$$\int \frac{2\sqrt{e^x} + e^{2x}}{e^x - 4}\,dx = \int \frac{2e^{\frac{x}{2}} + e^{2x}}{e^x - 4}\,dx =$$

$$\begin{cases} y = e^{\frac{x}{2}} \\ x = 2\ln y \\ dx = \frac{2}{y}\,dy \end{cases}$$

$y = \boxed{e^{\frac{x}{2}}} \qquad e^{2x} = \left(e^{\frac{x}{2}}\right)^4 = y^4$

$e^x = \left(e^{\frac{x}{2}}\right)^2 = y^2$

$$= \int \frac{2\cancel{y} + y^{\cancel{4}\,3}}{y^2 - 4}\,\frac{2}{\cancel{y}}\,dy = 2\int \frac{2 + y^3}{y^2 - 4}\,dy$$

$$\begin{array}{r|l} y^3 + 2 & y^2 - 4 \\ \hline y^3 - 4y & y \\ \hline 4y + 2 & \end{array}$$

$$= 2\int \left[y + \frac{4y + 2}{y^2 - 4}\right] dy$$

$$= y^2 + 2\int \left[\frac{A}{y + 2} + \frac{B}{y - 2}\right] dy = (*)$$

$$\begin{cases} \dfrac{A}{y+2} + \dfrac{B}{y-2} = \dfrac{4y+2}{y^2-4} \\ A(y-2) + B(y+2) = 4y+2 \\ \qquad\qquad 4B = 8+2 \\ -4A = -8+2 \qquad 4A = 6 \end{cases} \qquad \begin{cases} B = \dfrac{5}{2} \\ A = \dfrac{3}{2} \end{cases}$$

$$(4) = y^2 + 3\int \frac{1}{y+2}\,dy + 5\int \frac{1}{y-2}\,dy$$

$$= y^2 + 3\ln|y+2| + 5\ln|y-2|$$

$$= y^2 + \ln\left|(y+2)^3\cdot(y-2)^5\right| = e^x + \ln\left|\left(e^{\frac{x}{2}}+2\right)^3\left(e^{\frac{x}{2}}-2\right)^5\right|.$$

<u>Caso 2</u> $\quad \int f(\sin^2 x, \cos^2 x, \sin x\cos x)\,dx$

f razionale.

$$1+\operatorname{tg}^2 x = \frac{1}{\cos^2 x}$$

$$\begin{cases} t = \operatorname{tg} x \\ \sin x\cdot\cos x = \operatorname{tg} x\cdot\cos^2 x \\ x = \operatorname{arctg} t \end{cases}$$

$$\begin{cases} \cos^2 x = \dfrac{1}{1+t^2} \\ \sin x\cdot\cos x = \dfrac{t}{1+t^2} \\ \sin^2 x = \dfrac{t^2}{1+t^2} \end{cases}$$

$$dx = \frac{1}{1+t^2}\,dt$$

Esempio

$$\int \frac{1}{\cos x \cdot (\sin x + \cos x)} \, dx$$

$$= \int \frac{1}{\sin x \cos x + \cos^2 x} \, dx \quad = \quad t = \operatorname{tg} x$$

$$= \int \frac{1}{\frac{t}{1+t^2} + \frac{1}{1+t^2}} \cdot \frac{1}{1+t^2} \, dt$$

$$= \int \frac{1}{t+1} \, dt = \ln|t+1| = \ln|\operatorname{tg} x + 1|$$

Caso 3

$$\int f(\sin x, \cos x) \, dx$$

$$t = \operatorname{tg} \frac{x}{2} \qquad \cos x = \cos\left(\frac{x}{2} + \frac{x}{2}\right) = \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}$$

$$= 2\cos^2 \frac{x}{2} - 1$$

$$= \frac{2}{1+t^2} - 1 = \frac{1-t^2}{1+t^2}$$

$$\sin x = \sin\left(\frac{x}{2} + \frac{x}{2}\right) = 2 \sin \frac{x}{2} \cos \frac{x}{2} = \frac{2 \cdot t}{1+t^2}$$

$$\begin{cases} t = \operatorname{tg} \frac{x}{2} \\ \sin x = \frac{2t}{1+t^2} \\ \cos x = \frac{1-t^2}{1+t^2} \end{cases} \qquad \begin{cases} \frac{x}{2} = \operatorname{arctg} t \\ dx = \frac{2}{1+t^2} \leftarrow \end{cases}$$

Esempio $$\int \frac{1}{\sin x}\,dx = \int \frac{\cancel{1+t^2}}{2t}\cdot\frac{2}{\cancel{1+t^2}}\,dt$$

$$\begin{cases} t = \operatorname{tg}\frac{x}{2} \\ \sin x = \frac{2t}{1+t^2} \\ dx = \frac{2}{1+t^2} \end{cases}$$

$$= \ln|t| = \ln\left|\operatorname{tg}\frac{x}{2}\right|$$

Caso 4 $$\int f(\sqrt[n]{x})\,dx$$ f razionale.

$y = \sqrt[n]{x}$

$x = y^n$

$dx = n y^{n-1}\,dy$

Esempio $$\int \frac{\sqrt[4]{x}}{\sqrt[2]{x} + \sqrt[3]{x}}\,dx = \int \frac{y^3}{y^6 + y^4}\cdot 12 y^{11}\,dy$$

$n = 12$ $$x^{1/4}, \quad x^{1/2}, \quad x^{1/3}$$
$$x^{\frac{3}{12}} \qquad x^{6/12} \qquad x^{\frac{4}{12}}$$

$y = \sqrt[12]{x} = x^{\frac{1}{12}}$

$x = y^{12}$

$dx = 12\, y^{11}\,dy$

$$= 12\int \frac{y^{14}}{y^6+y^4}\,dy = 12\int \frac{y^{10}}{y^2+1}\,dy = (*)$$

$$
\begin{array}{l|l}
y^{10} & y^2+1 \\
\hline
y^{10}+y^8 & y^8-y^6+y^4-y^2+1 \\
\hline
-y^8 & \\
-y^8-y^6 & \\
\hline
y^6 & \\
y^6+y^4 & \\
\hline
-y^4 & \\
-y^4-y^2 & \\
\hline
y^2 & \\
y^2+1 & \\
\hline
-1 &
\end{array}
$$

$$\frac{y^{10}}{y^2+1} = y^8-y^6+y^4-y^2+1-\frac{1}{y^2+1}$$

$$(*) = 12\left[\frac{y^9}{9}-\frac{y^7}{7}+\frac{y^5}{5}-\frac{y^3}{3}+y\right]$$

$$-12\operatorname{arctg} y$$

$$= \frac{4}{3}\sqrt[4]{x^3}-\frac{12}{7}\sqrt[12]{x^7}+\frac{12}{5}\sqrt[12]{x^5}-4\cdot\sqrt[4]{x}+$$

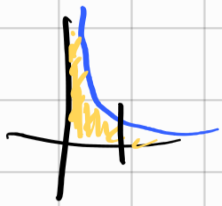
$$+ 12\sqrt[12]{x} - 12 \operatorname{arctg}\sqrt[12]{x}$$

INTEGRALI IMPROPRI o GENERALIZZATI

$\int_a^b f(x)\,dx$ $\quad$ $f: \underbrace{[a,b]} \to \mathbb{R}$, f limitata.

$\int_{-\infty}^{+\infty} e^{-x^2} dx = ?$

$\int_0^{+\infty} \frac{1}{x}\, dx = ?$



$\int_0^1 \frac{1}{x}\, dx = ?$

Def $f: A \subseteq \mathbb{R} \to \mathbb{R}$ diremo che f è localmente Riemann integrabile se per ogni $[a,b] \subseteq A$ f risulta limitata e essere Riemann integrabile su $[a,b]$.

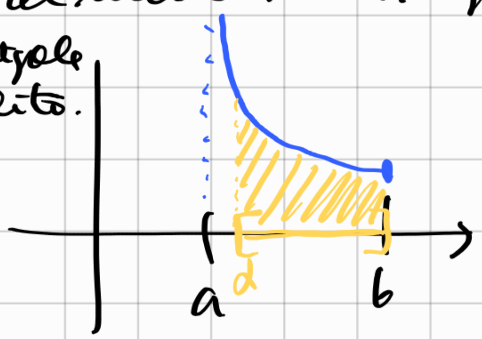
Oss Se $f: A \to \mathbb{R}$ è continua allora è localmente R-integrabile.

Def (integrale improprio) $-\infty \leq a < b < +\infty$

① Se $f: (a,b] \to \mathbb{R}$, localmente R-integrabile

definiamo: $\int_a^b f \overset{\cdot}{=} \lim_{d \to a^+} \int_d^b f$ $\leftarrow$ integrale finito.

$\uparrow$ è un integrale improprio



Es 1

$$\int_0^1 \frac{1}{x}\,dx = \lim_{\alpha\to 0^+}\int_\alpha^1 \frac{1}{x}\,dx = \lim_{\alpha\to 0^+}\left[\ln x\right]_\alpha^1$$

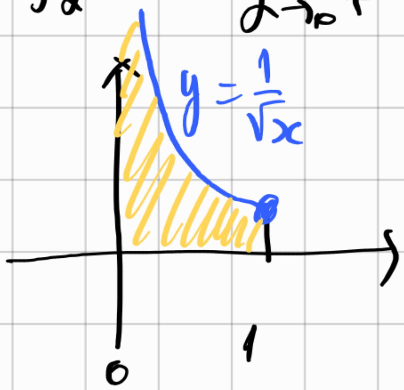
$$= \lim_{\alpha\to 0^+}\left[\ln 1 - \ln\alpha\right] = -\lim_{\alpha\to 0^+}\ln\alpha = +\infty$$

Es 2

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{\alpha\to 0^+}\int_\alpha^1 x^{-\frac{1}{2}}\,dx$$

$$= \lim_{\alpha\to 0^+}\left[\frac{x^{1/2}}{1/2}\right]_\alpha^1 = 2\lim_{\alpha\to 0^+}\left[\sqrt{1}-\sqrt{\alpha}\right]$$

$$= 2.$$



f è integrabile in senso improprio su $(a,b]$

Terminologia:

l'integrale esiste

(a) Se il limite esiste finito diremo che l'integrale (improprio) converge

(b) Se il limite esiste ed è infinito diremo che l'integrale (improprio) diverge

(c) Se il limite è indeterminato diremo che l'integrale (improprio) è indeterminato

carattere

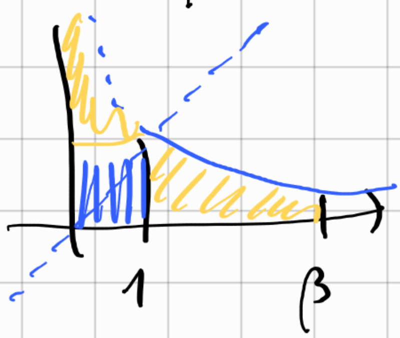
$$-\infty < a < b \le +\infty$$

(2) $f : [a,b) \to \mathbb{R}$ è analogo

$$\int_a^b f = \lim_{\beta \to b^-} \int_a^\beta f$$

ES
$$\int_1^{+\infty} \frac{1}{x}\,dx = \lim_{\beta\to+\infty} \int_1^\beta \frac{1}{x}\,dx$$

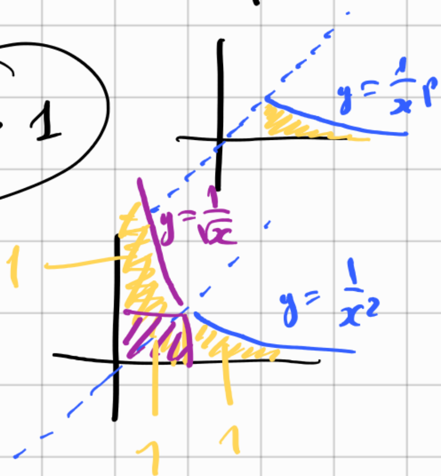
$$= \lim_{\beta\to+\infty} \Big[\ln x\Big]_1^\beta = \lim_{\beta\to+\infty} \ln\beta = +\infty$$



$$\int_1^{+\infty} \frac{1}{x^p}\,dx = \lim_{\beta\to+\infty} \int_1^\beta \frac{1}{x^p}\,dx$$

$$= \lim_{\beta\to+\infty} \left[\frac{x^{1-p}}{1-p}\right]_1^\beta = \lim_{\beta\to+\infty} \left[\frac{\beta^{1-p}}{1-p} - \frac{1}{1-p}\right] \qquad p \neq 1.$$

$$= \begin{cases} \dfrac{1}{p-1} & \text{se } p > 1 \\ +\infty & \text{se } p < 1 \end{cases}$$



$$\int_0^1 \frac{1}{x^p}\,dx = \begin{cases} +\infty & \text{se } p \geq 1 \\ \text{converge} & \text{se } p < 1 \end{cases}$$

<u>Caso 3</u>: $f:(a,b)\to\mathbb{R}$ $\quad -\infty \le a < b \le +\infty$

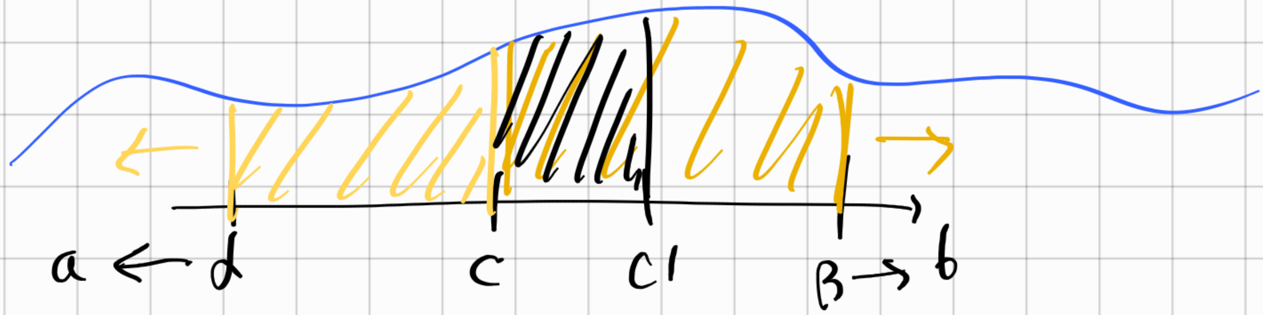
Se f è localmente R-integrabile posso spezzare prendendo $c\in(a,b)$

$$(a,b) = (a,c] \cup [c,b)$$

$$\int_a^b f = \int_a^c f + \int_c^b f$$ ← se gli integrali esistono e anche la somma

sono integrali impropri

$$= \lim_{\alpha\to a^+}\int_\alpha^c f + \lim_{\beta\to b^-}\int_c^\beta f.$$



$$\int_\alpha^c + \int_c^\beta + \int_\alpha^{c'} + \int_{c'}^\beta .$$

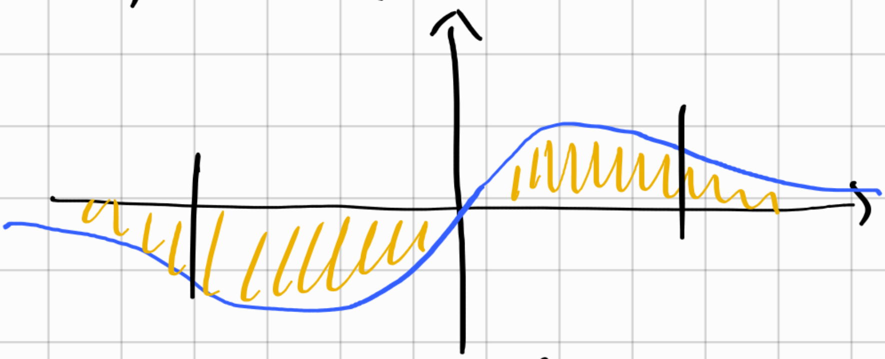
$$\left(\quad \text{"}(+\infty) + (-\infty) = \nexists\text{"} \right)$$

<u>Es</u> $$\int_{-\infty}^{+\infty} \frac{2x}{1+x^2}\,dx = \lim_{\alpha\to-\infty}\int_\alpha^0 \frac{1x}{1+x^2}\,dx + \lim_{\beta\to+\infty}\int_0^\beta \frac{2x}{1+x^2}\,dx$$

$$= \lim_{\alpha\to-\infty}\left[\ln(1+x^2)\right]_\alpha^0 + \lim_{\beta\to+\infty}\left[\ln(1+x^2)\right]_0^\beta$$

$$= \lim_{\alpha \to -\infty} \left[-\ln(1+\alpha^2) \right] + \lim_{\beta \to +\infty} \ln(1+\beta^2)$$

$$= (-\infty) + (+\infty) \quad \text{non esiste}$$



Oss

$$\boxed{\lim_{\alpha \to +\infty} \int_{-\alpha}^{\alpha} \frac{2x}{1+x^2}\, dx = 0}$$

NON è la nostra definizione