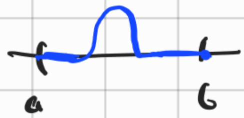
ELEMENTI di CALCOLO delle VARIAZIONI
LEZIONE 8 - 23.3.2023

Volta scorsa: $u \in L^1$, $\int_a^b u \cdot \varphi' = 0$, $\forall \varphi \in C_c^\infty(a,b) \Rightarrow u = c$ q.o.

__Corollario__ (lemma fondamentale I)

Se $u \in L^1$, $\quad \int_a^b u \cdot \varphi = 0 \quad \forall \varphi \in C_c^\infty \Rightarrow u = 0$ q.o.

__dim__ $\quad \int_a^b u \cdot \varphi' = 0 \overset{\text{II}}{\Longrightarrow} u = c$ q.o.



Ora basta prendere $\quad \varphi \in C_c^\infty(a,b) \quad \int_a^b \varphi \neq 0$.

$$0 = \int_a^b u \cdot \varphi = \int_a^b c \cdot \varphi = c \cdot \int_a^b \varphi \qquad \text{se } \int_a^b \varphi \neq 0$$

allora $c = 0$. $\square$

__Teorema__ (Du Bois-Reymond) $\qquad \mathcal{F}(u) = \int_a^b F(x, u, u')\,dx$

$F \in C^1$, $u \in C^1$, $\quad \delta\mathcal{F}(u, \varphi) = 0 \quad \forall \varphi \in C_c^\infty(a,b)$.

Allora $\quad x \longmapsto F_p(x, u(x), u'(x))$ è C^1 e

E.L. $\qquad \dfrac{d}{dx} F_p(x, u, u') = F_z(x, u, u')$

__Esempio__

$$\begin{cases} \mathcal{F}(u) = \int_{-1}^{1} u^2(x) \cdot (2x - u'(x))^2\,dx \\ u(-1) = 0 \\ u(1) = 1 \end{cases}$$

$$u_0(x) = \begin{cases} 0 & \text{se } x \in [-1, 0] \\ x^2 & \text{se } x \in [0, 1] \end{cases} \qquad u_0 \in C^1 \setminus C^2$$

$\mathcal{F} \geq 0 \quad$ e $\quad \mathcal{F}(u_0) = 0$.

$F(x, z, p) = z^2 \cdot (2x - p)^2 \qquad \Bigg| \begin{array}{l} F_p = -2z^2(2x - p) \\ F_z = 2z\,(2x - p)^2 \end{array}$

E.L.: $\frac{d}{dx}\left(-2u^2\cdot(2x-u')\right) = 2u(2x-u')^2$

$$\frac{d}{dx}\, 0 = 0$$

<u>dim</u> $\forall \varphi \in C_c^\infty$

$$0 = \delta \mathcal{F}(u,\varphi) = \frac{d}{d\varepsilon}\left[\int_a^b F(x, u+\varepsilon\varphi, (u+\varepsilon\varphi)')\, dx\right]_{\varepsilon=0}$$

$$= \int_a^b \left[F_z(x,u,u')\cdot\varphi + F_p(x,u,u')\cdot\varphi'\right] dx = (*)$$

$$\left[\begin{array}{ll} H(x) = \int_a^x F_z(t,u(t),u'(t))\,dt & F_z \in C^0,\ u,u' \in C^0 \\ H'(x) = F_z(x,u(x),u'(x)) & \end{array}\right]$$

$$(*) \underset{\text{per parti}}{=} \left[\cancel{H(x)\cdot\varphi}\right]_a^b - \int_a^b H(x)\cdot\varphi' + \int_a^b F_p\cdot\varphi'$$

$$= \int_a^b \left[F_p(x,u,u') - H(x)\right]\varphi'(x)\,dx = 0$$

per il lemma fondamentale II $\quad F_p(x,u,u') - H(x) = c.$

$$F_p(x,u(x),u'(x)) = c + H(x) \in C^1.$$

$$\frac{d}{dx} F_p(x,u,u') = H'(x) = F_z(x,u,u') \qquad \square$$

[idea: dovrebbe valere lo stesso se $u \in L^1$ e $u' \in L^1$]

<u>Stesso teorema versione in più variabili</u> $\quad u : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$

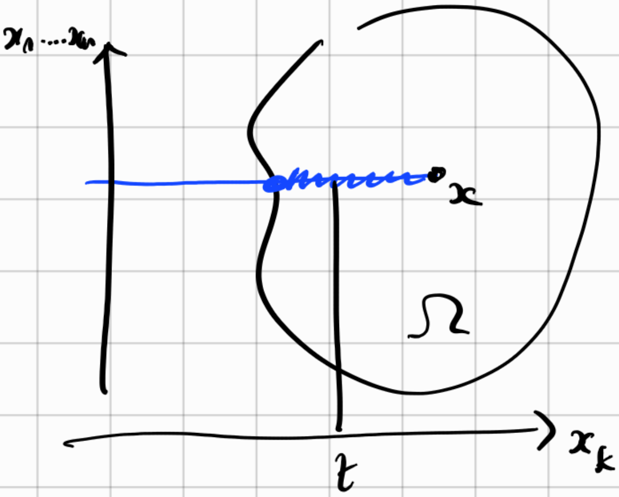
$$\mathcal{F}(u) = \int_\Omega F(\underline{x}, u, \nabla u)\, d\underline{x} \qquad u \in C^1,\ F \in C^1$$

$\cdot\ \delta\mathcal{F}(u,\varphi) = 0 \quad \forall \varphi \in C_c^\infty(\Omega)$

allora $\quad \operatorname{div} F_p(x,u,\nabla u) = F_z(x,u,\nabla u).$

dim

$$0 = \delta \mathcal{F}(u,\varphi) = \int_\Omega \left[F_z(\underline{x},u,\nabla u)\cdot\varphi + F_p(\underline{x},u,\nabla u)\cdot\nabla\varphi \right] d\underline{x} = (*)$$



$$H_k(x) = \frac{1}{n}\int_{-\infty}^{x_k} F_z(x_1,\dots,t,\dots x_n, u(\dots),\nabla u(\dots))\,dt$$

$$\frac{\partial H}{\partial x_k}(x) = \frac{1}{n} F_z(x,u(x),\nabla u(x)).$$

$$\mathrm{div}\,\underline{H} = \sum \frac{\partial}{\partial x_k} H_k = F_z(x,u,\nabla u)$$

$$(*) = -\int_\Omega \underline{H}(x)\cdot\nabla\varphi + \int_\Omega F_p\cdot\nabla\varphi$$

$$= \int_\Omega \left[F_p(x,u,\nabla u) - H \right]\cdot\nabla\varphi$$

Lemma fondamentale in + variabili

$$F_p - H = \underline{c} \quad \text{su ogni cc.}$$

$$\mathrm{div}\,F_p = 0 + \mathrm{div}\,\underline{H} = F_z .$$

Condizioni al secondo ordine. (condizione di Legendre)

Teo Se u_0 è minimo di $\mathcal{F}(u)=\int_a^b F(x,u(x),u'(x))\,dx$ $\quad F\in C^2$

con $u(a)$ e $u(b)$ fissati $\quad u\in C^1$.

allora $\quad F_{pp}(x,u_0(x),u_0'(x))\geq 0$

Nel caso $\underline{u}\in\mathbb{R}^n$ $\quad F_{pp}$ è una matrice e si intende

$$\boxed{F_{p_ip_j}(x,u_0(x),u_0'(x))\,\xi_i\xi_j\geq 0\quad \forall x\ \ \forall\,\underline{\xi}\in\mathbb{R}^n.}$$

condizione di Legendre

dim

$$\mathcal{F}(u_0+\varepsilon\varphi)=\int_a^b F(x,u_0+\varepsilon\varphi,u_0'+\varepsilon\varphi')\,dx$$

per E.L.

$$=\int_a^b F(x,u_0,u_0')+\varepsilon\cancel{\int_a^b\left(F_z\cdot\varphi+F_p\cdot\varphi'\right)}+$$

$$+\frac{1}{2}\varepsilon^2\int_a^b\left[(F_{zz}\cdot\varphi,\varphi)+2\left(F_{zp}\cdot\varphi,\varphi'\right)+\left(F_{pp}\cdot\varphi',\varphi'\right)\right]$$

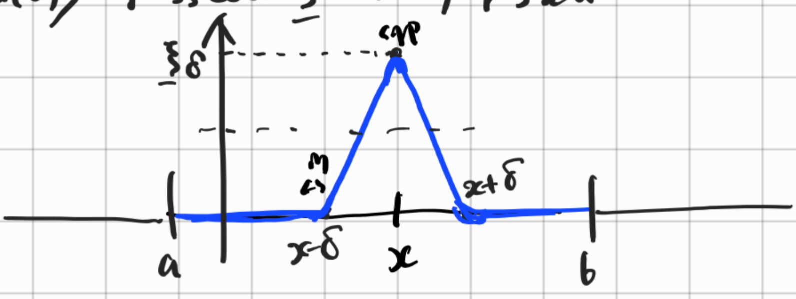
$$+o(\varepsilon^2)$$

$$\mathcal{F}(u_0+\varepsilon\varphi)=\mathcal{F}(u_0)+\frac{1}{2}\varepsilon^2\int_a^b[\,\ldots\ldots\,]+o(\varepsilon^2)$$

Se u_0 è minimo $\longrightarrow\ \geq 0\quad\forall\varphi.$

Fissato $x\in(a,b)$, fissato $\underline{\xi}\in\mathbb{R}^n$, fissato $\delta>0$

Scegliamo φ:



$(*)$

$$0 \le \int_a^b [\dots] = \int_{x-\delta}^{x+\delta} \left[(F_{zz}\varphi,\varphi) + 2(F_{zp}\varphi,\varphi') + (F_{pp}\varphi',\varphi') \right]$$

$$\le \int_{x-\delta}^{x+\delta} \left[|F_{zz}|\delta^2 |\xi|^2 + 2|F_{zp}|\delta|\xi|^2 + \boxed{(F_{pp}\xi,\xi)} \right]$$

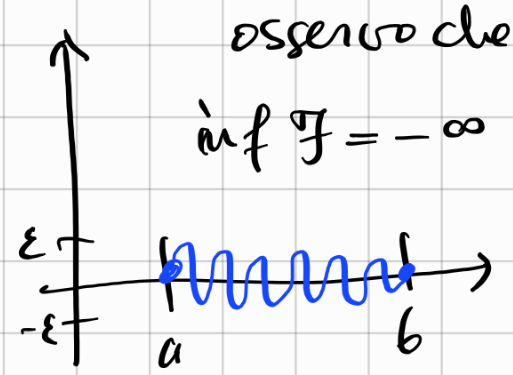
$$\varphi'(x) = \begin{cases} \xi & \text{se } x \in (x-\delta, x) \\ -\xi & \text{se } x \in (x, x+\delta) \end{cases}$$

ma $(F_{pp}(-|\xi|, -|\xi|)) = (F_{pp}\xi,\xi)$

Se $(F_{pp}\xi,\xi)$ fosse negativo per qualche x e qualche ξ troverei δ molto piccolo per cui si viola $(*)$ $\square$

<u>Esempio</u>

$$\begin{cases} \mathcal{F}(u) = \frac{1}{2}\int_a^b \left[u^2(x) - (u')^2(x)\right] dx \\ u(a) = 0 \\ u(b) = 0 \end{cases}$$



osservo che $\inf \mathcal{F} = -\infty$

$$F(x,z,p) = \frac{1}{2}z^2 - \frac{1}{2}p^2$$

$$F_z = z \qquad F_p = -p$$

E.L.: $\frac{d}{dx}(-u') = u \qquad u'' + u = 0$

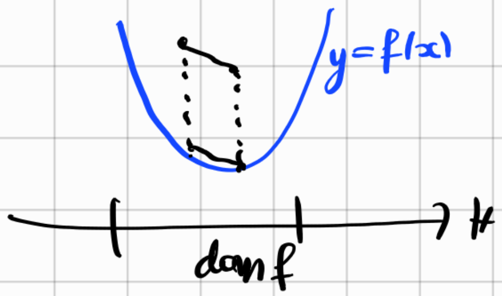
se $u(x) = A\cos x + B\sin x$ e $u(a) = 0$, $u(b) = 0$.

E.L. ha soluzione. Ma $F_{pp} = -1$.

dunque le soluzioni non sono minimi.

DUALITÀ CONVESSA

$+\infty$ $\qquad$ $+\infty$



H spazio di Hilbert separabile

$f : H \to (-\infty, +\infty] = \overline{\mathbb{R}}$

$\operatorname{dom} f = \{ x \in H : f(x) < +\infty \}$

$\operatorname{epi} f = \{ (x, y) \in H \times \mathbb{R} : y \geq f(x) \}$

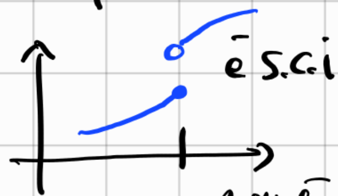
def f è convessa se $\forall \lambda \in [0,1]$ $f((1-\lambda)x + \lambda y) \leq (1-\lambda) f(x) + \lambda f(y)$

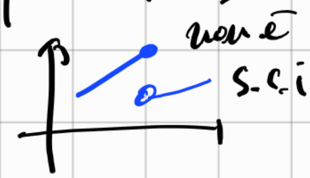
$\forall x, y \in H$ o $\forall x,y \in \operatorname{dom} f$.

oss f è convessa $\iff$ $\operatorname{epi} f$ è convesso.

def f è semicontinua inferiormente (s.c.i) se quando

$x_k \to x \implies f(x) \leq \liminf_k f(x_k)$





oss f è s.c.i $\iff$ $\operatorname{epi} f$ è chiuso

dim

"$\implies$" Siano $(x_k, y_k) \xrightarrow{H \times \mathbb{R}} (x, y)$ $\qquad x_k \to x \qquad y_k \to y$

$y_k \geq f(x_k)$ $\qquad f(x) \underset{\text{s.c.i}}{\leq} \liminf f(x_k) \leq y$ ok

"$\impliedby$" $x_k \to x$

$y = \liminf f(x_k)$ $\qquad y_j = f(x_{k_j}) \to y$

$(x_{k_j}, y_j) \in \operatorname{epi} f$ è chiuso $\implies y \geq f(x)$ $\implies$ s.c.i. $\square$

$\downarrow$

(x, y) $\qquad\qquad y = \liminf f(x_k)$