ANALISI MATEMATICA B
LEZIONE 21 - 9.11.2022

Successioni $\quad \underline{a} : \mathbb{N} \to \mathbb{R} \qquad \underline{a}(n) = a_n$

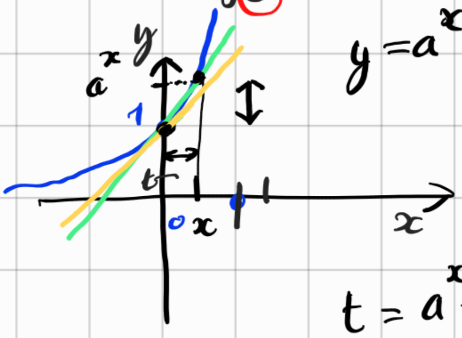
Carattere:

- regolare
  - convergenti $\qquad \lim a_n \in \mathbb{R}$
  - divergenti $\qquad \lim a_n \in \{+\infty, -\infty\}$
- indeterminata $\qquad \lim a_n$ non esiste

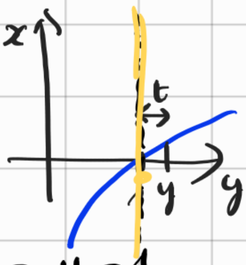
a_n monotona $\Rightarrow$ a_n regolare

$$\lim_{x \to 0} \boxed{\frac{a^x - 1}{x}} \qquad \left[\frac{0}{0}\right] \qquad [\log_e a]$$

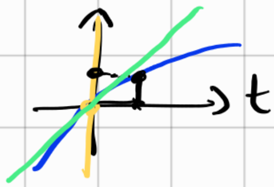
$$\begin{cases} f(x) \to \ell \\ g(x) \to m \\ \dfrac{f(x)}{g(x)} \to \dfrac{\ell}{m} \end{cases}$$



$y = a^x$



$$\frac{1}{g(x)} \to \frac{1}{m}$$



$$t = a^x - 1 = y - 1$$

$$\underbrace{1 + t = a^x} \qquad \underbrace{x = \log_a(1+t)} \qquad \begin{cases} t \to 0 \\ x \to 0 \end{cases}$$

$$\lim_{x \to 0} \frac{a^x - 1}{x} \underset{\uparrow}{=} \lim_{t \to 0} \frac{t}{\log_a(1+t)} = \frac{1}{\lim\limits_{t \to 0} \dfrac{\log_a(1+t)}{t}}$$

$y = f(x) \qquad x = g(t) = \log_a(1+t)$

$$\| \quad \lim_{x \to 0} \boxed{\frac{\log_a(1+x)}{x}} \qquad [\log_a e]$$

$x = \frac{1}{n}$

$$\lim_{\substack{n \to +\infty \\ n \in \mathbb{N}}} \frac{\log_a\left(1 + \frac{1}{n}\right)}{\frac{1}{n}}$$

$$n \cdot \log_a \left(1+\frac{1}{n}\right) = \log_a \left(1+\frac{1}{n}\right)^n$$

$$\boxed{\lim_{n\to+\infty} \left(1+\frac{1}{n}\right)^n} \quad [e]$$

$$\boxed{\text{La successione} \quad a_n = \left(1+\frac{1}{n}\right)^n \quad \text{converge?}}$$

1. a_n è monotona?

- puntuale $f(x_0) > b\infty$
- locale $f(x_0) \geq f(x)$ per $|x-x_0| < \delta$
- globale $f(x_0) \geq f(x)\ \forall x$

n	a_n	
1	2	2
2	9/4	2,25
3	64/27	2,30
4	⋮	
5	⋮	

81

$64 : 27 = 2,30\ldots$
54
100
81
19

2. a_n è crescente?

$$a_n = \left(1+\frac{1}{n}\right)^n = \left(\frac{n+1}{n}\right)^n$$

$\forall n \geq 2 \qquad a_{n-1} \overset{?}{\leq} a_n$

$$\left(\frac{n}{n-1}\right)^{n-1} \overset{?}{\leq} \left(\frac{n+1}{n}\right)^n$$

$$1 \leq \left(\frac{n+1}{n}\right)^n \cdot \underbrace{\left(\frac{n-1}{n}\right)^{n-1}}$$

$$\frac{n-1}{n} \overset{?}{\leq} \left(\frac{n+1}{n} \cdot \frac{n-1}{n}\right)^n = \left(\frac{n^2-1}{n^2}\right)^n = \left(1-\frac{1}{n^2}\right)^n$$

$$\frac{n-1}{n} = 1-\frac{1}{n}$$

$$\underbrace{\left(1-\frac{1}{n^2}\right)^n \overset{?}{\geq} 1-\frac{1}{n}}$$

Disuguaglianza di Bernoulli: $\boxed{(1+x)^n \geq 1+nx}$ se $x>-1$

$$\left[(1+x)^n = 1 + nx + \cancel{\frac{n(n-1)}{2}x^2} \cancel{+ \dots}\right]$$

Con $x = -\frac{1}{n^2}$

$$\left(1+\left(-\frac{1}{n^2}\right)\right)^n \geq 1 + n\left(-\frac{1}{n^2}\right) = 1-\frac{1}{n}$$

$$\left(1+\left(-\frac{1}{n^2}\right)\right)^n = \left(1-\frac{1}{n^2}\right)^n$$

a_n è crescente $\lim a_n$ esiste

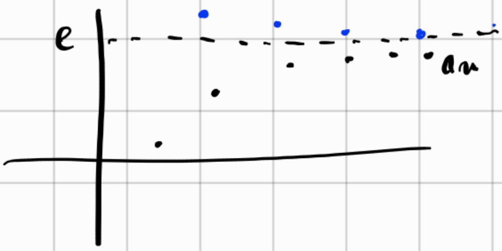
3. a_n è limitata? $\left(1+\frac{1}{n}\right)^n$

$$b_n = \left(1+\frac{1}{n}\right)^{n+1} \qquad b_n \geq a_n$$

$$= a_n \cdot \left(1+\frac{1}{n}\right) \geq a_n \qquad a_n \leq b_n \leq b_1$$

b_n è decrescente?

Sì stessa dimostrazione di prima.



$\lim a_n = \sup a_n$

$\lim b_n = \inf b_n$

$\left(1+\frac{1}{n}\right)^n$ è convergente

Def $$\boxed{e = \lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^n}$$

$e \geq \left(1+\frac{1}{2}\right)^2 > 2.$

$\left(1+\frac{1}{n}\right)^{n+1}$

n	b_n
1	4
2	$\frac{27}{8} = 3\dots$

$\boxed{2 < e < 4}$

4. $$\boxed{\lim_{\substack{x\to+\infty \\ x\in\mathbb{R}}} \left(1+\frac{1}{x}\right)^x \overset{?}{=} e}$$

$$\lfloor x\rfloor \le x < \lfloor x\rfloor + 1$$

with $\lfloor x\rfloor \in \mathbb{Z}$ and $\lfloor x\rfloor + 1 \in \mathbb{Z}$.

$$\frac{1}{\lfloor x\rfloor+1} \le \frac{1}{x} \le \frac{1}{\lfloor x\rfloor}$$

$$\left(1+\frac{1}{\lfloor x\rfloor+1}\right)^{\lfloor x\rfloor} \le \left(1+\frac{1}{x}\right)^x \le \left(1+\frac{1}{\lfloor x\rfloor}\right)^{\lfloor x\rfloor+1} = \underbrace{\left(1+\frac{1}{\lfloor x\rfloor}\right)^{\lfloor x\rfloor}}_{a_{\lfloor x\rfloor}\to e}\cdot \underbrace{\left(1+\frac{1}{\lfloor x\rfloor}\right)}_{\to 1} \to e$$

$$\left(1+\frac{1}{\lfloor x\rfloor+1}\right)^{\lfloor x\rfloor} = \frac{\overbrace{\left(1+\frac{1}{\lfloor x\rfloor+1}\right)^{\lfloor x\rfloor+1}}^{a_{\lfloor x\rfloor+1}\to e}}{\underbrace{\left(1+\frac{1}{\lfloor x\rfloor+1}\right)}_{\to 1}} \to e$$

$$\underset{\downarrow\, e}{h(x)} \le \underset{\downarrow\, e}{f(x)} \le \underset{\downarrow\, e}{g(x)}$$

squeeze

<u>Teoremi di confronto</u> lo facciamo venerdí

5. $$\lim_{x\to-\infty} \left(1+\frac{1}{x}\right)^x \overset{?}{=} e$$

$$\lim_{n\to+\infty} \left(1-\frac{1}{n}\right)^{-n} = e$$

$$\left(1-\frac{1}{n}\right)^{-n} = \left(\frac{n-1}{n}\right)^{-n} = \left(\frac{n}{n-1}\right)^{n} = \left(1+\frac{1}{n-1}\right)^{n}$$

$$\parallel$$

$$e = 1\cdot e \longleftarrow \left(1+\frac{1}{n-1}\right)\cdot\left(1+\frac{1}{n-1}\right)^{n-1}$$

Concludo come nel punto precedente.

$$\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^x = e$$

6. $$\boxed{\lim_{x\to0}(1+x)^{\frac{1}{x}} \overset{?}{=} e}$$

$$\begin{cases} \lim_{x\to0^+}(1+x)^{\frac{1}{x}} = \lim_{y\to+\infty}\left(1+\frac{1}{y}\right)^y = e \\ \lim_{x\to0^-}(1+x)^{\frac{1}{x}} = \lim_{y\to-\infty}\left(1+\frac{1}{y}\right)^y = e \end{cases}$$

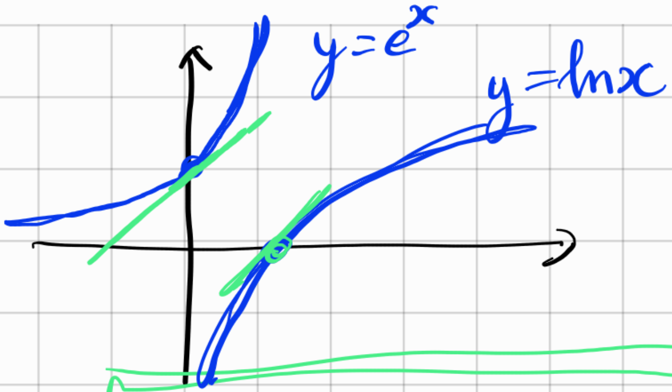
$y=\frac{1}{x}$

7. $$\boxed{\frac{\log_a(1+x)}{x}} = \log_a\left[(1+x)^{\frac{1}{x}}\right] \xrightarrow{x\to0} \boxed{\log_a e}$$

8. $$\lim_{x\to0}\frac{a^x-1}{x} = \frac{1}{\lim_{y\to0}\frac{\log_a(1+y)}{y}} = \frac{1}{\log_a e} = \log_e a$$

$$\boxed{\lim_{x\to0}\frac{e^x-1}{x} = 1}$$

$\log_e e$



$$\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1$$

e è la base naturale dell'esponenziale

$$\ln x = \log_e x$$

$$\begin{cases} \log x = \ln x \\ \log x = \log_{10} x \end{cases}$$