ANALISI MATEMATICA B
LEZIONE 33 - 12.12.2022

Convergenza incondizionata

Teorema (convergenza condizionata) Sia $a_k \in \mathbb{R}$ t.c. $\sum a_k$ è convergente ma non assolutamente convergente.
(ES $\sum \frac{(-1)^k}{k}$)

Dato qualunque $x \in \overline{\mathbb{R}}$ esiste un riordinamento dei termini $b_k = a_{\sigma(k)}$ $\sigma : \mathbb{N} \to \mathbb{N}$ bigettiva

$$\sum b_k = x.$$

dim (ES: $1 \quad -\frac{1}{2} \quad +\frac{1}{3} \quad -\frac{1}{4} \quad +\frac{1}{5} \quad -\frac{1}{6}$)

$$P = \{k : a_k \geq 0\}, \quad N = \{k : a_k < 0\}$$

$$= \{p_1, p_2, \dots, p_k, \dots\} \qquad \{n_1, n_2, \dots, n_k, \dots\}$$

$$\sum a_{p_k} = +\infty \qquad \sum a_{n_k} = -\infty$$

$$\underbrace{\sum a_k}_{\text{convergente}} = \sum a_{p_k} + \sum a_{n_k}$$

se fosse convergente → $\sum a_{p_k}$; sarebbe convergente → $\sum a_{n_k}$

$$\sum |a_k| = \sum a_{p_k} - \sum a_{n_k}$$

sarebbe convergente! ← $\sum |a_k|$

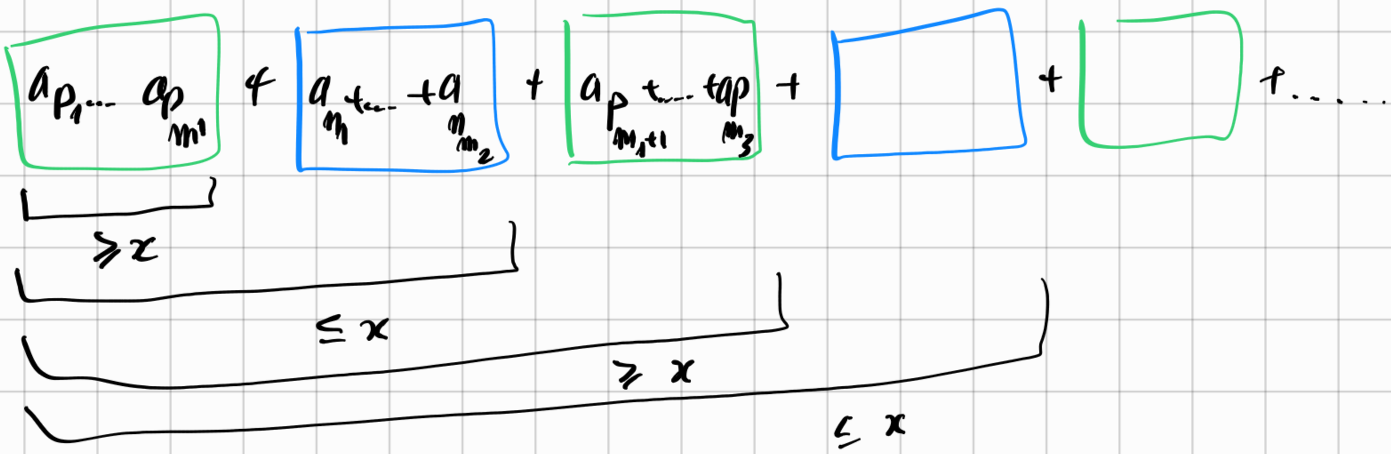
se fossero entrambe convergenti ← $\sum a_{p_k}$, $\sum a_{n_k}$

Se voglio avere somma $x \in \mathbb{R}$.

Comincio a sommare i termini positivi $\overbrace{a_{p_1} + a_{p_2} + \dots + a_{p_{m_1}}} \geq x$

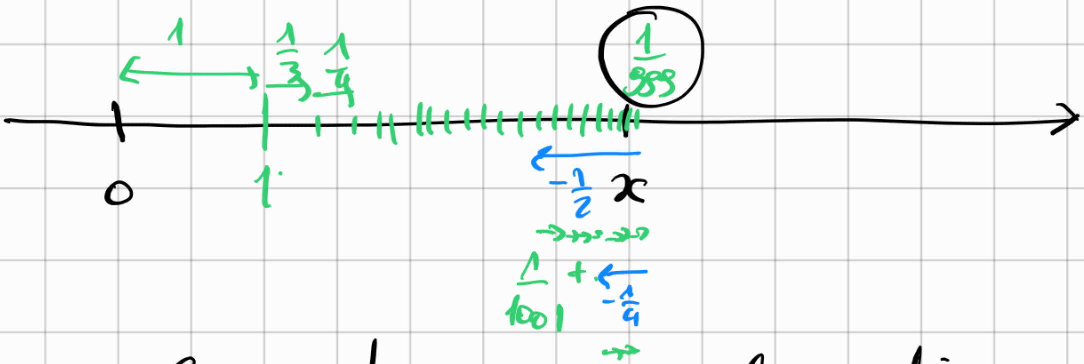
poi sommo i termini negativi $a_{p_1} + \dots + a_{p_{m_1}} + a_{n_1} + a_{n_2} + \dots + a_{n_{m_2}} \leq x$

... continuo in questo modo:



$\sum a_k$ convergente $a_k \to 0 \Rightarrow a_{p_k} \to 0 \quad a_{n_k} \to 0$

$$\sum \frac{(-1)^k}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \dots$$



□

In generale non ha senso parlare di somme di un insieme di numeri

$\sum \left\{ \frac{(-1)^n}{n} : n \in \mathbb{N} \right\}$ non ha senso

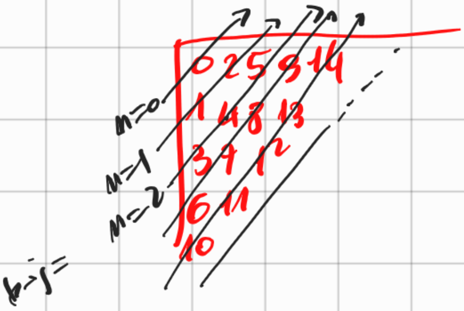
Teorema (convergenza incondizionata) Se $\sum |a_k|$ è convergente $(a_k \in \mathbb{C})$.
e $\sigma : \mathbb{N} \to \mathbb{N}$
allora $\sum a_{\sigma(k)} = \sum a_k$.

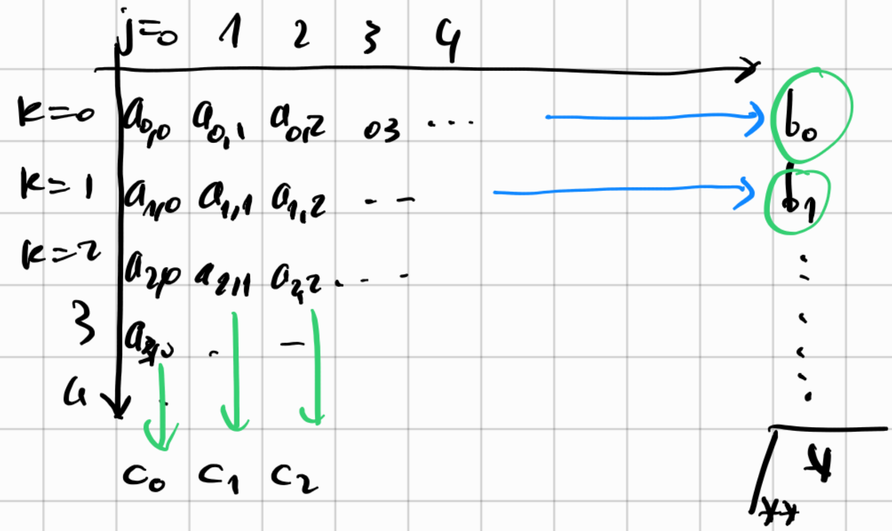
Teorema (scambio delle somme) $a : (\mathbb{N} \times \mathbb{N}) \to \mathbb{C}$
$a_{k,j} = a(k,j)$

Tesi $(*) = \sum_{k=0}^{+\infty} \underbrace{\sum_{j=0}^{+\infty} a_{k,j}}_{b_k} = \sum_{j=0}^{+\infty} \sum_{k=0}^{+\infty} a_{k,j} \quad (**)$

Ipotesi $\sum_{k=0}^{+\infty} \sum_{j=0}^{+\infty} |a_{k,j}| < +\infty$

Oss: $\mathbb{N} \times \mathbb{N} \longleftrightarrow \mathbb{N}$
↑
Cantor
(primo metodo diagonale)





Teorema (Somme alla Cauchy)

Tesi $\sum_{k=0}^{+\infty} \sum_{j=0}^{+\infty} a_{k,j} = \sum_{n=0}^{+\infty} \sum_{k=0}^{n} a_{k,n-k}$

↑ righe ↑ colonne ↑ diagonali

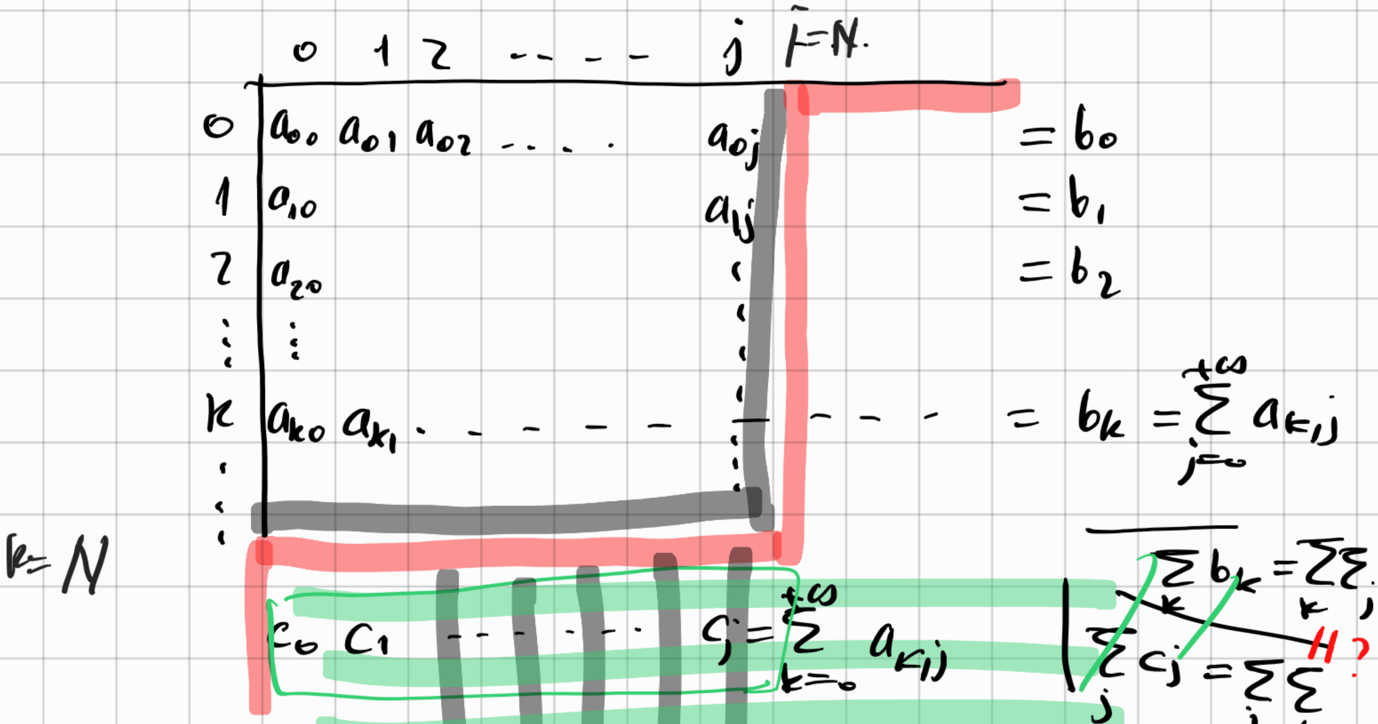
Ipotesi $\sum_{k=0}^{+\infty} \sum_{j=0}^{+\infty} |a_{k,j}| < +\infty$.

Esempio negativo

$$\sum_k \sum_j \neq \sum_j \sum_k$$
$$\| \qquad \|$$
$$0 \qquad +\infty$$

$$
\begin{array}{c|cccccccccc}
k=0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 = \sum_{j=0}^{+\infty} a_{0,j} \\
1 & 0 & 2 & -2 & 0 & 0 & 0 & 0 & 0 & & 0 \\
\vdots & 0 & 0 & 3 & -3 & 0 & 0 & 0 & 0 & 0 & 0 \\
k & 0 & 0 & 0 & 4 & -4 & 0 & 0 & 0 & 0 & 0 = \sum a_{kj} \\
 & 0 & 0 & 0 & 0 & 5 & -5 & 0 & 0 & 0 & 0 \\
 & 0 & 0 & 0 & 0 & 0 & 6 & -6 & 0 & 0 & \\
\hline
 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & & S = \sum_k \sum_j
\end{array}
$$

Dimostriamo (solo l'idea) lo scambio $\sum_k \sum_j = \sum_j \sum_k$.



Ipotesi $\sum_k \sum_j |a_{kj}| < +\infty$

(1) Verso $\exists$ una matrice $N \times N$ tale che

$$\sum_{k=0}^{N-1} \sum_{j=N}^{+\infty} |a_{kj}| + \sum_{k=N}^{+\infty} \sum_{j=0}^{+\infty} |a_{kj}| < \varepsilon.$$

[**Teorema della coda**: Se $\sum a_k$ è convergente allora $\forall \varepsilon > 0 \ \exists N$ t.c. $\sum_{k=N}^{+\infty} a_k < \varepsilon$.]

L'ipotesi che abbiamo è $\sum_k \sum_j |a_{kj}|$ è convergente

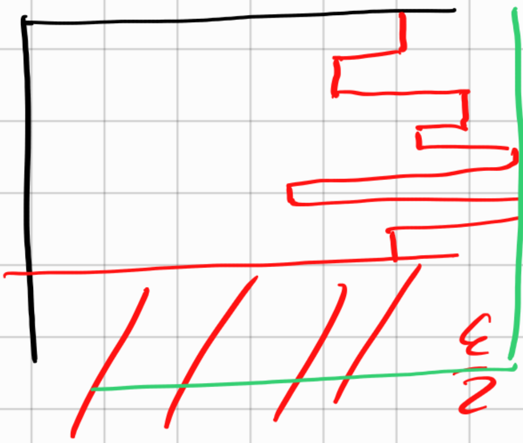
$$= \sum_k B_k$$

$\forall k$: $B_k = \sum_j |a_{kj}|$ è convergente $\Rightarrow$ $b_k = \sum a_{kj}$ è convergente

$$\forall \varepsilon : \forall k : \exists N_k : \sum_{j=N_k}^{+\infty} |a_{kj}| < \frac{\varepsilon}{2^{k+1}}$$

• $\forall \varepsilon : \exists M : \sum_{k=M}^{+\infty} B_k < \frac{\varepsilon}{2}$

$$= \sum_{k=M}^{+\infty} \sum_{j=0}^{+\infty} |a_{kj}|$$



$$N = \max\{M, N_0, N_1, \dots, N_M\} < +\infty$$

Conclusione: $S \doteq \underbrace{\sum_{k=0}^{+\infty} \sum_{j=0}^{+\infty} a_{kj}}$ tesi: $\sum_{j=0}^{+\infty} \sum_{k=0}^{+\infty} a_{kj} \overset{?}{=} S$

[$S \in \mathbb{R}$ perché $b_k = \sum_{j=0}^{+\infty} a_{kj}$ $\quad |b_k| \le \sum_{j=0}^{+\infty} |a_{kj}| = B_k$

$S = \sum_{k=0}^{+\infty} b_k$ $\quad \sum B_k < +\infty$

$|S| \le \sum |b_k| = \sum B_k < +\infty$.]

Basta dimostrare che $\lim_{N \to +\infty} \sum_{j=0}^{N} \underbrace{\sum_{k=0}^{+\infty} a_{kj}}_{c_j} = S$

Basta mostrare che $\forall \varepsilon>0 \ \exists N$ t.c.

$$\left|\sum_{j=0}^{N-1} c_j - S\right| < \varepsilon.$$

Un po' di fiducia → $\frac{\varepsilon}{2}$ ✓ $\sum_{k=N}^{+\infty} |a_{kj}|$ ✓

$$\sum_{j=0}^{N-1} c_j = \sum_{j=0}^{N-1}\sum_{k=0}^{+\infty} a_{kj} = \sum_{j=0}^{N-1}\left[\sum_{k=0}^{N-1} a_{kj} + \sum_{k=N}^{+\infty} a_{kj}\right]$$

$$= \sum_{j=0}^{N-1}\sum_{k=0}^{N-1} a_{kj} \pm \frac{\varepsilon}{2}$$

$$= \sum_{k=0}^{N-1}\sum_{j=0}^{N-1} a_{kj} \pm \frac{\varepsilon}{2}$$

$$= \sum_{k=0}^{N-1}\left[\sum_{j=0}^{+\infty} a_{kj} \pm \frac{\varepsilon}{2^k}\right] \pm \frac{\varepsilon}{2} = \sum_{k=0}^{N-1} b_k \pm \underbrace{\sum_{k=0}^{N-1}\frac{\varepsilon}{2^k}} \pm \frac{\varepsilon}{2}$$

$$= \sum_{k=0}^{N-1} b_k \pm \underbrace{2\varepsilon \pm \frac{\varepsilon}{2}}$$

$\downarrow$

S

ok □

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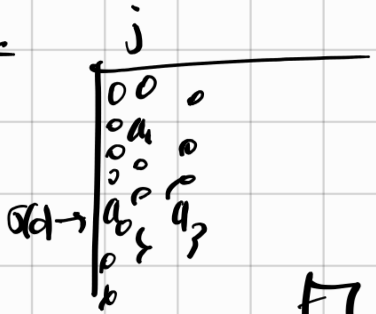
dim (convergenza incondizionata)

$\sum a_{\sigma(k)} = \sum a_k$ se $\sum |a_k| < +\infty$

$$\sum_j a_{\sigma(j)} = \sum_j \sum_k b_{k,j} = \sum_k \sum_j b_{k,j} = \sum_k a_k$$

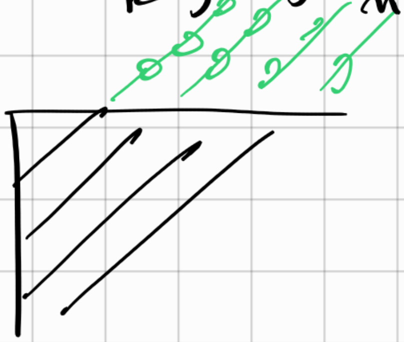
$$b_{kj} = \begin{cases} a_k & \text{se } k=\sigma(j) \\ 0 & \text{altrimenti} \end{cases}$$



□

dim (Somme alla Cauchy)

$$\sum_k \sum_j a_{kj} = \sum_n \sum_k a_{k,n-k}$$



$$b_{k,n} = \begin{cases} a_{k,n-k} & \text{se } k \le n \\ 0 & \text{se } k > n \end{cases}$$

Teo ?

$$\sum_k \sum_n b_{k,n} = \sum_k \sum_{n=k}^{+\infty} a_{k,n-k} =$$

$$= \sum_n \sum_k a_{k,n-k} \qquad \square$$