ANALISI MATEMATICA B
LEZIONE 60 - 6.2.2023

RICERCA delle PRIMITIVE

$$\int f(x)\,dx = \{\text{primitive di } f\} = \{F : F' = f\}$$

LINEARITA' $\quad \int(\lambda f + \mu g) = \lambda \int f + \mu \int g.$

dim Se $F \in \int f$, $G \in \int g$, devo verificare

che $\lambda F + \mu G \overset{?}{\in} \int(\lambda f + \mu g)$.

$$(\lambda F + \mu G)' = \lambda F' + \mu G' = \lambda f + \mu g \quad \square$$

ES
$$\int (x-1)^2\,dx = \int (x^2 - 2x + 1)$$
$$= \int x^2\,dx - 2\int x\,dx + \int 1\,dx$$
$$= \frac{x^3}{3} - 2\frac{x^2}{2} + x$$

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derivata funzione composta $\quad (F(g(x)))' = \underbrace{F'(g(x)) \cdot g'(x)}$

$$\int F'(g(x))\, g'(x)\,dx = F(g(x))$$

$f = F'$ $\qquad\qquad\qquad\qquad y = g(x)$

$\boxed{\text{sostituzione inversa:}}$
$$\int f(g(x))\, g'(x)\,dx = \left[\int f(y)\,dy\right]_{y=g(x)}$$

$$\begin{cases} y = g(x) \\ dy = g'(x)\,dx \end{cases}$$

Sostituzione diretta

$$\int f(y)\,dy = \left[\int f(g(x))\, g'(x)\, dx\right]_{x=g^{-1}(y)}$$

Integrali definiti

$$\int_{g(a)}^{g(b)} f(y)\,dy = \int_a^b f(g(x))\, g'(x)\, dx \quad \wedge \quad \begin{cases} y = g(x) \\ dy = g'(x)\, dx \end{cases}$$

Calcolare: $\int (e^x)^2\, dx$

$$\int (e^x)^2 dx = \int e^{2x} dx$$

$= \frac{1}{2} e^{2x}$ ← a occhio

sostituzione inversa

$= \frac{1}{2}\int e^{2x} \cdot 2dx = \frac{1}{2}\int e^y dy = \frac{1}{2} e^y = \frac{1}{2} e^{2x}$; $\begin{cases} y = 2x \\ dy = 2dx \end{cases}$

$= \int e^y \frac{1}{2} dy = \frac{1}{2}\int e^y$; $\begin{cases} x = \frac{1}{2} y \\ dx = \frac{1}{2} dy \end{cases}$

sostituzione diretta.

Calcolare: $\int \cos^2 x\, dx$

formula di addizione ; $\cos^2 + \sin^2 = 1$

$$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1$$

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\int \cos^2 x\, dx = \int \left[\frac{1}{2} + \frac{\cos 2x}{2}\right] dx = \frac{x}{2} + \frac{1}{2}\int \cos 2x\, dx$$

$$= \frac{x}{2} + \frac{1}{4} \sin(2x)$$

Es $\int \sqrt{1-x^2}\,dx = \left[\int \sqrt{1-\sin^2 t}\cdot\cos t \cdot dt\right]_{t=\boxed{\arcsin x}}$

$\left(\int \frac{1}{\sqrt{1-x^2}} = \arcsin x\right)$ $\boxed{\begin{cases} x = \sin t \\ dx = \cos t \cdot dt \end{cases}}$ ← sottointeso $t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ ↓ $\cos t \geq 0$ $|\cos t| = \cos t$

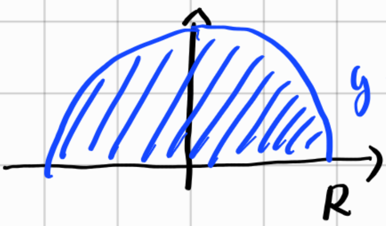
$$= \int \cos^2 t \, dt = \text{Es } 1$$

$$= \frac{t}{2} + \frac{1}{4}\sin 2t = \frac{t}{2} + \frac{1}{2}\sin t \cdot \cos t$$

$t = \arcsin x$

$$= \frac{1}{2}\arcsin x + \frac{1}{2}x \cdot \sqrt{1-x^2}$$

Oss



$x^2+y^2=R^2$

$y = \sqrt{R^2-x^2}$

Area del cerchio di raggio $R>0$

è: $2\int_{-R}^{R} \sqrt{R^2-x^2}\,dx = 2R\int_{-R}^{R} \sqrt{1-\left(\frac{x}{R}\right)^2}\,dx$

$\begin{cases} y = \frac{x}{R} \\ x = Ry \\ dx = R\,dy \end{cases}$

$$= 2R\int_{-1}^{1} \sqrt{1-y^2}\cdot R\cdot dy$$

$$= 2R^2 \int_{-1}^{1} \sqrt{1-y^2}\,dy = 2R^2\left[\frac{1}{2}\arcsin y + \frac{1}{2}y\sqrt{1-y^2}\right]_{-1}^{1}$$

$$= 2R^2\left[\frac{1}{2}\cdot\frac{\pi}{2} - \frac{1}{2}\left(-\frac{\pi}{2}\right)\right] = 2R^2\,\frac{\pi}{2} = \pi R^2. \quad \square$$

INTEGRAZIONE PER PARTI

$$(F\cdot g)' = \underbrace{F'\cdot g} + \underbrace{F\cdot g'} \qquad F' = f$$

$$\boxed{\int f\cdot g = \underbrace{F\cdot g} - \int F\cdot g'}$$

$$\boxed{\int_a^b f \cdot g = \left[F \cdot g\right]_a^b - \int_a^b F \cdot g'}$$

Es. $\displaystyle \int x \cdot \cos x \, dx = x \cdot \sin x - \int 1 \cdot \sin x \, dx$

$$= x \sin x + \cos x$$

per parti

Es. $\displaystyle \int e^x \cdot \cos x \, dx = e^x \sin x - \int e^x \sin x$

$$= e^x \sin x - \left[e^x(-\cos x) - \int e^x \cdot (-\cos x) \right]$$

$$= e^x \sin x + e^x \cos x - \int e^x \cos x$$

$$2 \int e^x \cos x = e^x \sin x + e^x \cos x$$

$$\int e^x \cos x = \frac{1}{2} e^x (\sin x + \cos x).$$

Es $\displaystyle \int \cos^2 x =$

Es $\displaystyle \int \ln x \, dx = \int 1 \cdot \ln x \, dx = x \ln x - \int x \cdot \frac{1}{x} \, dx$

$$= x \ln x - x$$

Es $\displaystyle \int \operatorname{arctg} x \, dx = x \cdot \operatorname{arctg} x - \int x \cdot \frac{1}{1+x^2} \, dx$

$$= x \cdot \operatorname{arctg} x - \frac{1}{2} \int \frac{2x}{1+x^2} \, dx$$

$\begin{cases} t = x^2 \\ dt = 2x\,dx \end{cases}$

sostituzione inversa

$$= x \cdot \operatorname{arctg} x - \frac{1}{2} \ln(1+x^2)$$

ES $\int \operatorname{tg} x = \int \frac{\sin x}{\cos x}\, dx = -\ln \cos x$

$y = \cos x$
$dy = -\sin x\, dx$

ES $\int \arcsin x \cdot dx = x \cdot \arcsin x - \frac{1}{2}\int \frac{2 \cdot x}{\sqrt{1-x^2}}\, dx$

$t = 1-x^2$
$dt = -2x\, dx$

$= x \cdot \arcsin x + \frac{1}{2}\int \frac{1}{\sqrt{t}}\, dt$

$= x \cdot \arcsin x + \cancel{\frac{1}{2}}\, \frac{t^{\frac{1}{2}}}{\cancel{\frac{1}{2}}} = x \arcsin x + \sqrt{1-x^2}$.

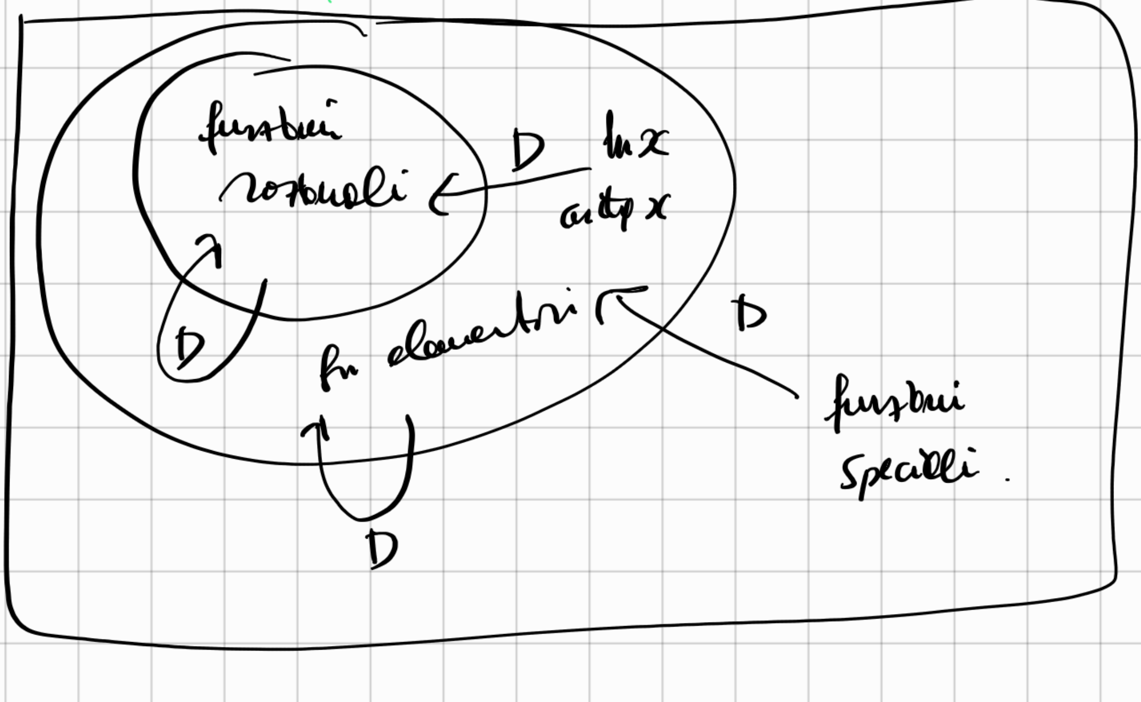
ES $\int \arccos x\, dx \overset{?}{=}$

INTEGRALI di FUNZIONI RAZIONALI

$\int \frac{P(x)}{Q(x)}\, dx$ P e Q polinomi

↑ si può scrivere tramite funzioni elementari.

PROSSIMA VOLTA.



FUNZIONI ELEMENTARI LA CUI PRIMITIVA NON E' ELEMENTARE

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int e^{-x^2} dx$$

$$Li(x) = \int \frac{1}{\ln x} dx$$

$$ei(x) = \int \frac{e^x}{x} dx$$

$$si(x) = \int \frac{\sin x}{x} dx$$

$$S(x) = \int \sin \frac{\pi x^2}{2} dx$$

Teorema di Liouville

[vedi: Canuillo De Lellis]