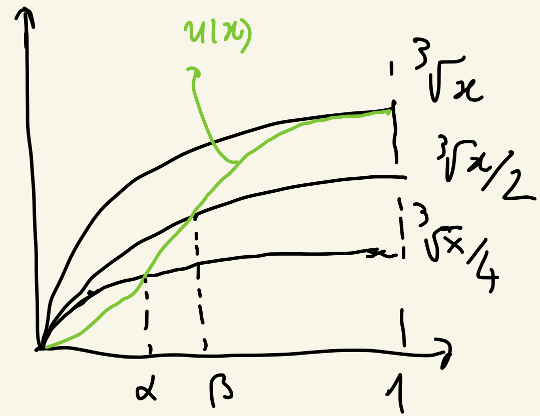
FENOMENO DI LAURENTIEV

$$\mathcal{L}(u) = \int_0^1 (u^3 - x)^2 |u'|^6 \, dx \quad \text{con} \quad u(0)=0 \quad u(1)=1$$



FISSIAMO $u(x) \in Lip([0,1])$

CON $u(0)=0$, $u(1)=1$.

PER CONTINUITÀ $\exists \; 0<\alpha<\beta<1$

T.C. $\begin{cases} u(x) \in \left[\frac{\sqrt[3]{x}}{4}, \frac{\sqrt[3]{x}}{2}\right] & \forall x \in [\alpha, \beta] \\ u(\alpha) = \sqrt[3]{\alpha}/4 \\ u(\beta) = \sqrt[3]{\beta}/2 \end{cases}$

IN PARTICOLARE

$$\frac{u^3(x)}{x} \leq \frac{1}{8} \quad \forall \; x \in [\alpha, \beta]$$

$$\mathcal{L}(u) = \int_0^1 x^2 \left(1 - \frac{u^3}{x}\right)^2 |u'|^6 \geq \left(\frac{7}{8}\right)^2 \int_\alpha^\beta x^2 |u'|^6 \, dx$$

$$\boxed{c = \frac{7^2 3^5}{8^2 5^5}}$$

$$\boxed{\begin{array}{ll} y = x^{3/5} & v(y) = u(x) \\ u'(x) = v'(y) \frac{3}{5} y^{-\frac{2}{3}} & \\ x = y^{\frac{5}{3}} & dx = \frac{5}{3} y^{\frac{2}{3}} dy \end{array}}$$

$$= \left(\frac{7}{8}\right)^2 \left(\frac{3}{5}\right)^5 \int_{\alpha^{\frac{3}{5}}}^{\beta^{3/5}} |v'(y)|^6 \, \cancel{y^{\frac{10}{3}} \left(y^{-\frac{2}{3}}\right)^6 y^{\frac{2}{3}}} \, dy$$

$$= \frac{7^2 3^5}{8^2 5^5} \int_{\alpha^{3/5}}^{\beta^{3/5}} |v'|^6 \, dy \geq c\left(\beta^{\frac{3}{5}} - \alpha^{\frac{3}{5}}\right) \left[\fint_{\alpha^{\frac{3}{5}}}^{\beta^{\frac{3}{5}}} v'(y)\right]^6$$

$$\boxed{\text{JENSEN: } \quad \fint_a^b f(u') \geq f\left(\fint_a^b u'\right), \quad f \text{ CONVESSA}}$$

$$= \frac{c\left[v\left(\beta^{\frac{3}{5}}\right) - v\left(\alpha^{\frac{3}{5}}\right)\right]^6}{\left[\beta^{\frac{3}{5}} - \alpha^{\frac{3}{5}}\right]^5}$$

$$= c \frac{\left(\frac{\sqrt[3]{\beta}}{2} - \frac{\sqrt[3]{\alpha}}{4}\right)^6}{\left(\beta^{\frac{3}{5}} - \alpha^{\frac{3}{5}}\right)^5} = c \frac{\left[\frac{\sqrt[3]{\beta}}{2}\left(1 - \frac{1}{2}\left(\frac{\alpha}{\beta}\right)^{\frac{1}{3}}\right)\right]^6}{\left[\beta^{\frac{3}{5}}\left(1 - \left(\frac{\alpha}{\beta}\right)^{\frac{3}{5}}\right)\right]^5}$$

$$= c\,\frac{\beta^2}{2^6\beta^3}\cdot\frac{\left[1-\frac{1}{2}\left(\frac{\alpha}{\beta}\right)^{\frac{1}{3}}\right]^6}{\left[1-\underbrace{\left(\frac{\alpha}{\beta}\right)^{\frac{3}{5}}}_{<1}\right]^5} \geq \frac{c}{2^6}\underbrace{\left(1-\frac{1}{2}\left(\frac{\alpha}{\beta}\right)^{\frac{1}{3}}\right)}_{>\frac{1}{2}}{}^6 \geq \frac{c}{2^{12}} > 0.$$

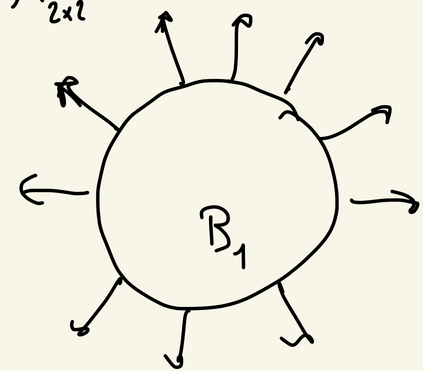
$$\left[\,0<\frac{\alpha}{\beta}<1\,\right] \Rightarrow \inf_{\mathrm{Lip}} \mathcal{L} \geq \frac{c}{2^{12}} > 0.$$

ALTRO ESEMPIO:

$$\int_{B_1} |Du|^2\,dx \qquad u: B_1 \to \mathbb{R}^2 \qquad u\big|_{\partial B_1} = x$$

$$Du \in M_{2\times 2}$$

$$B_1 = \{x \in \mathbb{R}^2 : |x| < 1\}$$



$$\mathcal{L}(u) = \int_{B_1} (1-|u|)^2 |Du|^2 \qquad u \in W^{1,p}(B_1) \qquad \boxed{u\big|_{\partial B_1} = x}$$

$$\bar{u}(x) = \frac{x}{|x|} \text{ è minimo} \qquad \mathcal{L}(u) = 0$$

$$Du = \frac{Id - \frac{x}{|x|} \otimes \frac{x}{|x|}}{|x|} \qquad |Du| = \frac{1}{|x|} \qquad |Du|^p = \frac{1}{|x|^p}$$

$$\int_{B_1} |Du|^p < +\infty \iff p < 2 \quad \text{cioè} \quad u \in W^{1,p} \iff p \in [1,2)$$

$$\Rightarrow \inf_{W^{1,p}} \mathcal{L}(u) = \mathcal{L}(\bar{u}) \quad \text{se } p < 2.$$

MA $\inf_{W^{1,2}} \mathcal{L}(u) = \inf_{C^1} \mathcal{L}(u) \geq c > 0$

SEMICONTINUITÀ

(X, d) SP. METRICO

$F : X \to \mathbb{R} \cup \{\pm\infty\}$ FUNZIONE

F È SEMICONTINUA INFERIORMENTE (RISP. SUPERIORMENTE) IN x SE

- $F(x) > -\infty$ (RISP. $F(x) < +\infty$) $\forall x$
- $F(x) \leq \liminf_n F(x_n)$ (RISP. $F(x) \geq \limsup_n F(x_n)$)

  $\forall$ SUCCESSIONE $x_n \to x$ (VA BENE ANCHE $x_n = x$)

OSS: F È CONT. $\iff$ F È SEMICONT. INFER. E SUPERIORMENTE

OSS: F è s.c.i. $\iff \{F \le c\}$ è CHIUSO IN X $\forall c \in \mathbb{R}$

F è s.c.s. $\iff \{F \ge c\}$ " " " "

ESEMPI:

$X = (L^p, \|\cdot\|_p)$ $\quad u_n \to u$ SE $\|u_n - u\|_{L^p} \to 0$

$X = (W^{1,p}, \|\cdot\|_{1,p})$ $\quad$ " $\quad \|u_n - u\|_{W^{1,p}} \to 0$

$X = \{u \in L^p : \|u\| \le R\}$ $\quad p \in (1, \infty)$ CON $u_n \xrightarrow[L^p]{} u$

$\circledast$ $X = \{u \in W^{1,p} : \|u\| \le R\}$ $\quad$ " $\quad u_n \xrightarrow[W^{1,p}]{} u$

OSS: F_i s.c.i. $\forall i \in I \Rightarrow \sup_i F_i$ è s.c.i.

F_i s.c.s. " $\Rightarrow \inf_i F_i$ è s.c.s.

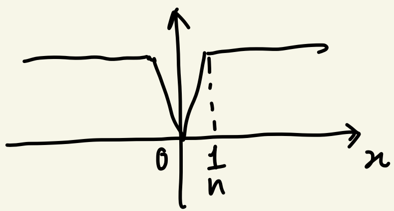
INFATTI, SE $F(x) = \sup_i F_i(x)$ E $x_n \to x \Rightarrow$

$$F_i(x) \leq \liminf_n F_i(x_n) \leq \liminf_n F(x_n) \quad \forall i$$

$$\Rightarrow F(x) = \sup_i F_i(x) \leq \liminf_n F(x_n)$$

IN GENERALE IL SUP DI FUNZIONI CONTINUE È S.C.I. MA NON CONTINUO.

$$f_n(x) = \min\left(n|x|, 1\right) \qquad f_n(x) \to f(x) = \begin{cases} 1 & x \neq 0 \\ 0 & x = 0 \end{cases}$$



f È S.C.I. MA NON CONTINUA

DEF: F COERCIVA SE $\overline{\{F \leq c\}}$ o EQUIV. $\overline{\{F < c\}}$ È COMPATTO IN X $\forall c \in \mathbb{R}$

TEO (WEIERSTRASS) X COMPATTO, F s.c.i. $\Rightarrow \exists \min_X F$

F COERCIVA E s.c.i. $\Rightarrow \exists \min_X F$

TEO

$$\mathcal{L}(u) = \int_a^b L(x, u, u'(x))\, dx$$

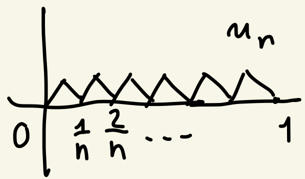
L DI CARATHEODORY, $L \geq c \in \mathbb{R}$, $\tilde{\forall} x$ e $\forall y$ $z \to L(x,y,z)$ CONVESSA

$\Rightarrow \mathcal{L}$ È DEBOLMENTE s.c.i. IN $W^{1,p}$ $\forall p \in (1, +\infty)$

CIOÈ $\mathcal{L}(u) \leq \liminf_n \mathcal{L}(u_n)$ $\forall u_n \xrightarrow[W^{1,p}]{} u$.

OSS: L'IPOTESI CRUCIALE È LA CONVESSITÀ DI $z \to L(x,y,z)$

ES (BOLZA): $\mathcal{L}(u) = \int_0^1 (1-|u'|^2)^2 \, dx$



$\exists\, u_n$ T.C. $|u_n'(x)| = 1$ $\tilde{\forall} x$, $u_n(x) \to 0$ unif. su $[0,1]$

E $u_n \underset{W^{1,p}}{\rightharpoonup} 0$ $\forall p$ MA $\mathcal{L}(0) = 1 > \lim_n \mathcal{L}(u_n) = 0$.

VALE ANCHE L'IMPLICAZIONE INVERSA:

TEO: $\mathcal{L}(u) = \int_a^b L(x, u(x), u'(x))\, dx$

L DI CARATH., $L \geq c \in \mathbb{R}$

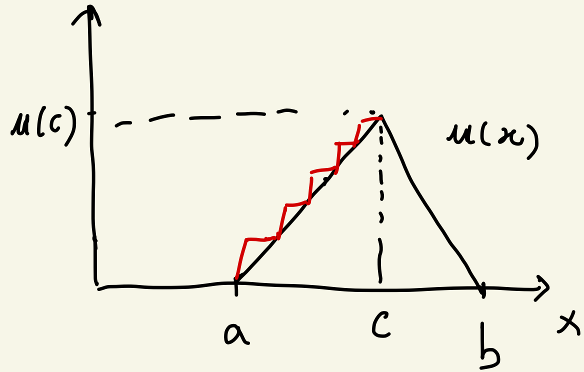
SONO EQUIVALENTI

① $\mathcal{L}$ S.C.I. IN $W^{1,p}$ PER $1 < p < +\infty$

② $z \to L(x, y, z)$ CONVESSA $\tilde{\forall} x$ E $\forall y$

IDEA DELLA DIM: SUPP. $L = L(z)$ CONTINUA MA NON CONVESSA

$\exists\, p_1, p_2$ T.C. $L\left(\frac{p_1 + p_2}{2}\right) > \frac{L(p_1) + L(p_2)}{2}$



c T.C. $\dfrac{u(c)}{c} = p$

CONSIDERIAMO UNA SUCC. $u_n \in Lip([a,b])$ T.C.

$u_n = u$ IN $[c,b]$ $\quad u_n' \in \{p_1, p_2\}$ $\forall x \in [a,c]$

$u_n \to u$ UNIF. IN $[a,b]$

$\Rightarrow u_n \rightharpoonup u$ in $W^{1,p}$ $\forall\, 1 < p < +\infty$

MA $\mathcal{L}(u_n) = (c-a)\,\dfrac{L(p_1)+L(p_2)}{2} + (b-c)\,L\left(-\dfrac{u(c)}{b-c}\right)$

$> (c-a)\,L(p) + (b-c)\,L\left(-\dfrac{u(c)}{b-c}\right) = \mathcal{L}(u) \Rightarrow \mathcal{L}$ NON È S.C.I.