ALCUNI ESEMPI

$$F_n(u) = \begin{cases} \int_0^1 n u'^2 + (u - f(x))^2 \, dx & u \in H^1((0,1)) \\ +\infty & u \in L^2((0,1)) \setminus H^1((0,1)) \end{cases}$$

$$\boxed{\begin{array}{l} n \in \mathbb{N} \\ f \in L^2((0,1)) \end{array}}$$

CI POSSIAMO ASPETTARE CHE

$$\Gamma\text{-}\lim_n F_n(u) = F_\infty(u) = \begin{cases} \int_0^1 (c - f(x))^2 \, dx & \text{se } u \equiv c \\ +\infty & \text{ALTRIMENTI} \end{cases}$$

IL Γ-LIMITE È IN $L^2((0,1))$.

VERIFICA:

① (limsup) PER LA RECOVERY SEQUENCE BASTA PRENDERE $u_n = u$.

② (liminf)
SE $u \equiv c$

$u_n \xrightarrow[H^1]{} u \Rightarrow$ $u_n \to u$ IN L^2

$$F_\infty(u) = \int_0^1 (c - f(x))^2 = \lim_n \int_0^1 (u_n - f)^2 \leq \liminf_n \int_0^1 n u_n'^2 + (u_n - f)^2$$

SE $u \not\equiv c$ $\quad u_n \xrightarrow[H^1]{} u$ E $u_n \xrightarrow[L^2]{} u$. SE $F_n(u_n) \to +\infty \Rightarrow F_\infty(u) = +\infty \leq \liminf_n F_n(u_n)$

POSSO SUPPORRE $F_n(u_n) \leq C$.

IN PART. $\displaystyle\int_0^1 n u_n'^2 \leq F_n(u_n) \leq C \Rightarrow \int_0^1 u_n'^2 \leq \frac{C}{n} \xrightarrow[n]{} 0$

$$|u_n(x)-u_n(y)| \leq \int_x^y |u_n'| \underset{\substack{\uparrow \\ \text{C.S.}}}{\leq} \left(\int_0^1 {u_n'}^2\right)^{\frac{1}{2}} \sqrt{|x-y|} \leq \sqrt{\frac{C}{n}}\sqrt{|x-y|}$$

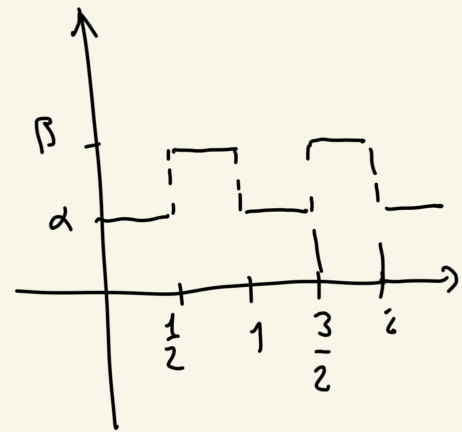
$$\Rightarrow \quad |u(x)-u(y)| \leq \lim_k |u_{n_k}(x)-u_{n_k}(y)| = 0 \qquad \tilde{\forall}\, x,y \in (0,1)$$

$\Rightarrow$ u costante [ASSURDO].

PROBLEMI DI OMOGENEIZZAZIONE

SIA $a(x) = \begin{cases} \alpha & \lfloor x \rfloor \in [0, \frac{1}{2}) \\ \beta & \lfloor x \rfloor \in [\frac{1}{2}, 1) \end{cases}$ $\alpha, \beta \in (0, +\infty)$

$a : \mathbb{R} \to \mathbb{R}$ 1- PERIODICA



$a_n(x) = a(nx)$

$a_n(x) \xrightarrow[L^2(I)]{} \frac{\alpha+\beta}{2}$

$\forall I = (a, b)$ INTERVALLO

DEFINIAMO $$F_n(u) = \int_a^b u'^2 + a_n u^2 \qquad u \in H^1((a,b))$$

$$G_n(u) = \int_a^b a_n u'^2 + u^2$$

ESTENDO I FUNZIONALI A $L^2((a,b))$ PONENDOLI $+\infty$ IN $L^2 \setminus H^1$.

CERCO IL Γ-LIMITE DI F_n E G_n IN $L^2((a,b))$.

$\boxed{F_n}$

MI POSSO ASPETTARE CHE

$$\Gamma\text{-}\lim_n F_n = F_\infty(u) = \int_a^b u'^2 + \frac{\alpha+\beta}{2} u^2 \qquad u \in H^1((a,b)).$$

① (limsup) BASTA PRENDERE $u_n = u$

SE $u \notin H^1 \Rightarrow F_\infty(u) = F_n(u) = +\infty \quad \forall n$

ALTRIMENTI $\displaystyle\int a_n u^2 \xrightarrow[n]{} \int \frac{\alpha+\beta}{2} u^2 \Rightarrow \lim_n F_n(u) = F_\infty(u)$

② (liminf) FISSO $u \in H^1$ E SIA $u_n \xrightarrow[H^1]{} u$ E $u_n \xrightarrow[L^2]{} u$.

COME SOPRA, POSSO SUPPORRE $F_n(u_n) \leq C \quad \forall n$.

$\|u_n\|_{H^1} \leq C \Rightarrow \|u_n\|_\infty \leq C \Rightarrow u_n^2 \to u^2$ IN L^2

$\displaystyle\int a_n u_n^2 \xrightarrow[n]{} \int \frac{\alpha+\beta}{2} u^2 \Rightarrow F_\infty(u) \leq \liminf_n F_n(u_n)$.

$\boxed{G_n}$

$$G_n(u) = \begin{cases} \int_a^b a_n u'^2 + u^2 \, dx & u \in H^1((a,b)) \\ +\infty & u \in L^2 \setminus H^1 \end{cases}$$

DICO CHE

$$\boxed{\Gamma\text{-}\lim_n G_n = G_\infty(u) = \begin{cases} \int_a^b c u'^2 + u^2 \, dx & u \in H^1 \\ +\infty & u \in L^2 \setminus H^1 \end{cases}}$$

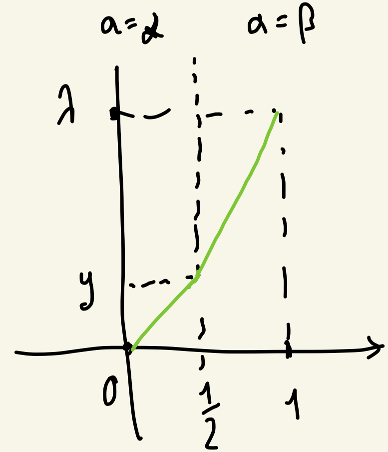
$$c = \frac{2\alpha\beta}{\alpha+\beta} = \frac{2}{\frac{1}{\alpha} + \frac{1}{\beta}}$$ MEDIA ARMONICA

OSS: $u \in H^1 \Rightarrow \lim_n G_n(u) = \int_a^b \frac{\alpha+\beta}{2} u'^2 + u^2 \geq G_\infty(u)$

OSS: È SUFF. MOSTRARE $\Gamma\text{-}\lim_n \int_a^b a_n u'^2 = \int c u'^2$.

POSSO SUPPORRE $G_n(u) = \int_a^b a_n u'^2$ E $G_\infty(u) = \int c u'^2$.

PROBLEMA DI CELLA



FISSATO $\lambda \in \mathbb{R}$ CERCO IL MINIMO DI

$$\min \left\{ \int_0^1 a(x)\, u'^2(x)\, dx \; : \; \begin{array}{l} u(0)=0 \\ u(1)=\lambda \end{array} \right\}$$

$a(x)$ COST. IN $[0,\frac{1}{2})$ E IN $[\frac{1}{2},1)$ $\Rightarrow$

LA u MINIMA È LINEARE A TRATTI)

$$u_y(x) = \begin{cases} 2yx & x \in [0,\frac{1}{2}) \\ 2(\lambda - y)x + 2y - \lambda & x \in [\frac{1}{2},1] \end{cases}$$

PER UN CERTO $y \in \mathbb{R}$

$$\Rightarrow \int_0^1 a(x)\, u_y'^2 = \frac{\alpha\, 4y^2}{2} + \frac{\beta}{2}\, 4(\lambda - y)^2 = \boxed{2\alpha y^2 + 2\beta(\lambda - y)^2 = f(y)}$$

CERCO IL MINIMO DI f.

$$f'(y) = 4\alpha y - 4\beta(\lambda - y) = 0 \iff \boxed{y = \frac{\beta\lambda}{\alpha+\beta}, \quad \lambda - y = \frac{\alpha\lambda}{\alpha+\beta}}$$

$$\Rightarrow \min\left\{\int_0^1 a(x)u'^2 : u(0)=0,\ u(1)=\lambda\right\} \geq f(y) = c\lambda^2 = c\left(u(1)-u(0)\right)^2$$

1 ($\liminf$) COME PRIMA POSSO SUPPORRE $u \in H^1$

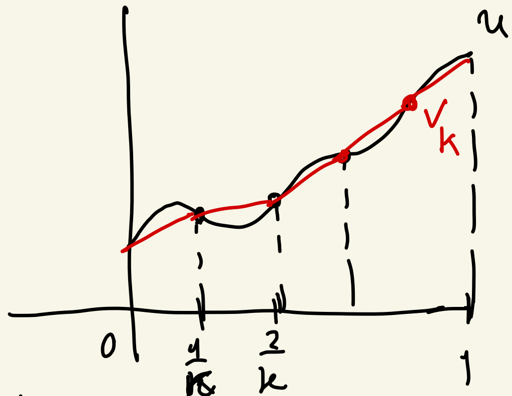
E $u_n \xrightarrow[H^1]{} u$, $u_n \to u$ UNIF., $G_n(u_n) = \int_a^b a_n u'^2 \leq C$.

FISSIAMO PER SEMPLICITÀ $(a,b) = (0,1)$ E SCRIVIAMO

$$G_n(u_n) = \int_0^1 a(nx)\, u'^2 = \sum_{k=0}^{n-1} \int_{\frac{k}{n}}^{\frac{k+1}{n}} a(nx)\, u_n'^2 \geq \sum_{k=0}^{n-1} cn\left(u_n\left(\frac{k+1}{n}\right) - u_n\left(\frac{k}{n}\right)\right)^2$$

$$\underset{\text{C.S.}}{\geq} c\left(\sum_k u_n\left(\frac{k+1}{n}\right) - u_n\left(\frac{k}{n}\right)\right)^2 = c\left(u_n(1) - u_n(0)\right)^2$$

DEFINISCO ORA UNA PARTIZIONE DI $(0,1)$ IN K INTERVALLI



$$G_n(u_n) = \sum_{i=0}^{K-1} \int_{\frac{i}{K}}^{\frac{i+1}{K}} a(nx)\, u_n'^2 \geq \sum_{i=0}^{K} c K \left(u_n\left(\frac{i+1}{K}\right) - u_n\left(\frac{i}{K}\right)\right)^2$$

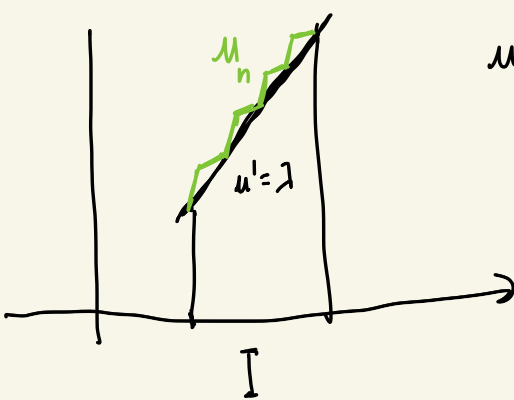
$$\Rightarrow \liminf_n G_n(u_n) \geq \sum_{i=0}^{K} c K \left(u\left(\frac{i+1}{K}\right) - u\left(\frac{i}{K}\right)\right)^2 = \int_0^1 c\, V_K'^2 \qquad \forall K$$

$$V_K \xrightarrow[K\to\infty]{} u \text{ IN } L^2 \Rightarrow \liminf_n G_n(u_n) \geq \liminf_K c \int V_K'^2 \geq c \int u'^2 .$$

(IN REALTÀ L'ULTIMO $\geq$ È UN $=$)

② (limsup) POICHÉ LE LINEARI A TRATTI SONO DENSE IN H^1, QUINDI IN ENERGIA, PER IL Γ-limsup POSSO SUPPORRE u LINEARE A TRATTI.

IN UN TRATTO DI PENDENZA λ APPROSSIMO u COME NEL PROBLEMA DI CELLA.



$$u'=\lambda \Rightarrow u_n' \in \{\ell y, \ell(\lambda-y)\} \qquad y=\frac{\beta\lambda}{\alpha+\beta}$$

$$\int_I a_n u_n^2 \xrightarrow[n]{} \int_I c\lambda^2 = \int_I c\, u'^2$$

SOMMANDO SUGLI INTERVALLI OTTENIAMO

$$\int_0^1 a_n u_n^2 \xrightarrow[n]{} \int_0^1 c\, u'^2 \qquad \text{RECOVERY SEQUENCE}$$