ANALISI MATEMATICA B
LEZIONE 47 - 31.1.2024

Test settimanale

$$\frac{1}{h}\int_0^h \cos(x^2)\,dx = *$$

Teor. media integrale: $\cos(t^2)$ con $t\in[0,h]$ $\to 1$

$*$ è compreso tra:

$$\cos h = \frac{1}{h}\int_0^h \cos h \le * \le \frac{1}{h}\int_0^h 1 = 1 \qquad 0<h<\frac{\pi}{2}$$

$\cos h \to 1$

[$\cos(x^2) = 1 + o(1)$ $\qquad \int_0^h o(1) \overset{?}{=} o(h)$ lo vedremo più avanti]

Cambio di variabile

$$\big(F(g(x))\big)' = f(g(x))\cdot g'(x)$$

- sostituzione inversa:

Se $\int f = F$

$$\int f(g(x))\,g'(x)\,dx \underset{\begin{cases} y = g(x) \\ dy = g'(x)\,dx \end{cases}}{=} \int f(y)\,dy$$

FORMALMENTE:
$$\frac{dy}{dx} = \frac{dg(x)}{dx} = g'(x)$$

Es

$$\int x\cdot e^{x^2}\,dx = \frac{1}{2}\int e^{x^2}\,2x\,dx \underset{\begin{cases} y = x^2 \\ dy = 2x\,dx \end{cases}}{=} \frac{1}{2}\int e^y\,dy = \frac{1}{2}e^y = \frac{1}{2}e^{x^2}.$$

$\int e^{x^2} = ?$

$\int e^y\,dy$

Es

$$\int \frac{1}{x\ln x}\,dx \underset{\begin{cases} y = \ln x \\ dy = \frac{1}{x}\,dx \end{cases}}{=} \int \frac{1}{y}\,dy = \ln|y| = \ln|\ln x|$$

Sostituzione diretta:

- $$\int f(x)\,dx = \left[\int f(h(y))\cdot h'(y)\,dy\right]_{y=h^{-1}(x)}$$

$$\cancel{\begin{cases} y = g(x) \\ x = g^{-1}(y) \\ dx = (g^{-1})'(y)\,dy \end{cases}} \qquad \begin{cases} x = h(y) \\ dx = h'(y)\,dy \\ y = h^{-1}(x) \end{cases}$$

Per fare la dimostrazione usiamo:

- $$\int_{g(a)}^{g(b)} f(x)\,dx = \int_a^b f(g(y))\,g'(y)\,dy$$

$$\begin{cases} x = g(y) \\ dx = g'(y)\,dy \end{cases}$$

<u>Esempio</u>

$$\int \cos^2 x\,dx \qquad \begin{cases} dy = 2dx \quad dx = \frac{1}{2}dy \\ y = 2x \end{cases}$$

<u>idea</u> $\int (e^x)^2 dx = \int e^{2x}\,dx = \frac{1}{2}e^{2x}$

$$\cos(2x) = \cos(x+x) = \cos^2 x - \sin^2 x = \cos^2 x - (1-\cos^2 x)$$
$$= 2\cos^2 x - 1 \qquad\qquad = 1 - 2\sin^2 x$$

$$\cos^2 x = \frac{1+\cos(2x)}{2} \qquad \left(\sin^2 x = \frac{1-\cos(2x)}{2}\right)$$

$$\int \cos^2 x\,dx = \int\left(\frac{1}{2} + \frac{1}{2}\cos 2x\right)dx = \frac{1}{2}\int 1 + \frac{1}{2}\int \cos 2x\,dx$$

$$= \frac{x}{2} + \frac{1}{4}\sin(2x) \qquad \begin{cases} y = 2x \\ x = \frac{1}{2}y \\ dx = \frac{1}{2}dy \end{cases}$$

Esempio

$$\int \sqrt{1-x^2}\cdot dx = -\int \sqrt{1-\cos^2 t}\,\sin t\,dt =$$

$$\left[\begin{array}{lll} x^2 \le 1 & x = \cos t & \sqrt{\sin^2 t} = |\sin t| \\ -1 \le x \le 1 & dx = -\sin t\,dt & \\ & 0 \le t \le \pi \longrightarrow & \sin t \ge 0 \\ & t = \arccos x & \end{array}\right]$$

$$= -\int \sin^2 t\,dt = -\int \frac{1-\cos 2t}{2}\,dt$$

$$= -\frac{1}{2}t + \frac{1}{4}\sin(2t) = \frac{1}{2}\sin t\cos t - \frac{1}{2}t$$

$$= \frac{1}{2}\sqrt{1-\cos^2 t}\cdot\cos t - \frac{1}{2}t$$

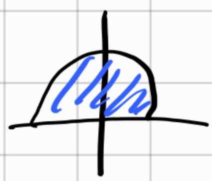
$$= \underbrace{\frac{1}{2}\sqrt{1-x^2}\cdot x - \frac{1}{2}\arccos x}_{} = F(x)$$

VERIFICA:

$$F'(x) = \frac{1}{2}\cdot\frac{1}{\cancel{2}}\cdot\frac{-\cancel{2}x}{\sqrt{1-x^2}}\cdot x + \frac{1}{2}\sqrt{1-x^2} + - \frac{1}{2}\frac{1}{-\sqrt{1-x^2}}$$

$$= \frac{1}{2}\frac{-x^2+1}{\sqrt{1-x^2}} + \frac{1}{2}\sqrt{1-x^2} = \frac{1}{2}\sqrt{1-x^2} + \frac{1}{2}\sqrt{1-x^2} = \sqrt{1-x^2}.$$

VERIFICA PARZIALE: $\boxed{\int_{-1}^{1}\sqrt{1-x^2}\,dx} \overset{!}{=} \frac{\pi}{2}$



$$\int_{-1}^{1}\sqrt{1-x^2}\,dx = \left[\cancel{\frac{1}{2}\sqrt{1-x^2}\cdot x} - \frac{1}{2}\arccos x\right]_{-1}^{1}$$

$$= \cancel{-\frac{1}{2}\arccos 1} - \left(-\frac{1}{2}\arccos(-1)\right)$$

$$= \frac{1}{2}\pi.$$

ALTERNATIVA

RIFACCIAMO TUTTO IL CONTO CON L'INTEGRALE DEFINITO

$$\int_{x=-1}^{1} \sqrt{1-x^2}\, dx = -\int_{\pi}^{0} \sqrt{1-\cos^2 t}\, \sin t \, dt =$$

$$\begin{cases} x = \cos t \\ dx = -\sin t\, dt \\ t = \arccos x \end{cases}$$

$$= \int_0^{\pi} \sin^2 t \, dt = \int_0^{\pi} \left(\frac{1-\cos 2t}{2}\right) dt = \frac{\pi}{2} - \cancel{\left[\frac{\sin 2t}{4}\right]_0^{\pi}} = \frac{\pi}{2}.$$

INTEGRAZIONE PER PARTI

$$(F \cdot g)' = F' \cdot g + F \cdot g' = f \cdot g + F \cdot g'$$

$$F \cdot g = \int f\, g + \int F g'$$

$$F' = f$$
$$F = \int f$$

$$\boxed{\int f \cdot g = F \cdot g - \int F \cdot g'}$$

$$\boxed{\int_a^b f \cdot g = [F \cdot g]_a^b - \int_a^b F \cdot g'}$$

Esempio

$$\int \underbrace{x \cdot \cos x}\, dx = x \cdot \sin x - \int 1 \cdot \sin x \, dx$$

derivo integro

$$= x \sin x + \cos x$$

Esempio

$$\int_0^{\pi} x \cos x \, dx = \cancel{[x \sin x]_0^{\pi}} - \int_0^{\pi} 1 \cdot \sin x \, dx$$

$$= \left[\cos x\right]_0^{\pi} = \cos\pi - \cos 0 = -2$$

Esempio (strano fenomeno)

$$\int \underset{\substack{\uparrow \\ \text{derivo}}}{e^x} \cdot \underset{\substack{\uparrow \\ \text{integro}}}{\cos x}\, dx = e^x \sin x - \int \underset{\substack{\uparrow \\ \text{derivo}}}{e^x}\, \underset{\substack{\uparrow \\ \text{integro}}}{\sin x}\, dx =$$

$$= e^x \sin x - \left(e^x \cdot (-\cos x) - \int e^x \cdot (-\cos x)\, dx\right)$$

$$= e^x \sin x + e^x \cos x - \int e^x \cos x\, dx$$

$$2\int e^x \cos x\, dx = e^x \sin x + e^x \cos x$$

$$\int e^x \cos x\, dx = \frac{1}{2} e^x (\sin x + \cos x) .$$

Integrali delle funzioni elementari

$$\int \ln x\, dx = \int \underset{\substack{\uparrow \\ \text{integro}}}{1} \cdot \underset{\substack{\uparrow \\ \text{derivo}}}{\ln x}\, dx = x \ln x - \int x \cdot \frac{1}{x}\, dx$$

$$= x \ln x - x$$

$$= x(\ln x - 1).$$

$$\int \arctan x\, dx = \int \underset{\substack{\uparrow \\ \text{integro}}}{1} \cdot \underset{\substack{\uparrow \\ \text{derivo}}}{\arctan x}\, dx = x \arctan x - \int x \frac{1}{1+x^2}\, dx$$

al modo dei fisici

$$= x \arctan x - \frac{1}{2}\int \frac{2x\, dx}{1+x^2} = x \arctan x - \frac{1}{2}\int \frac{d(1+x^2)}{1+x^2} =$$

$= x \arctan x - \frac{1}{2} \ln(1+x^2)$

Per casa

$\int \arcsin x \, dx \qquad \int \arccos x \, dx$