Algebra Lineare 19/11/13

Sistema lineare: $A \cdot x = b \in \mathbb{R}^m$

$A \in M_{m \times n}$

m = num. equaz.

n = num. incognite

x = incognite in $\mathbb{R}^n$

Terminologia: omogeneo se $b=0$

sottodeterminato se $m<n$

quadrato se $m=n$

sovradeterminato se $m>n$

Lem: Se x_0 è una soluz. di $A\cdot x=b$ allora

$$\{x\in\mathbb{R}^n : Ax=b\}=\{x_0+y : y\in \operatorname{Ker} A\}$$

generica soluz. di $Ax=b$

una soluz particolare di $Ax=b$

generica soluz. di $A\cdot x=0$

Dim: $Ax = b \iff A \cdot x = A \cdot x_0 \iff A(x - x_0) = 0$
$\iff x - x_0 = y$ con $y \in \operatorname{Ker} A \iff x = x_0 + y$ con $y \in \operatorname{Ker} A$.

$\boxed{\square}$

Oss: Numero di soluzioni di un sistema:

	omogeneo	non omogeneo
quadrato o sovradet.	una o ∞	nessuna, una o ∞
sottodet.	∞	nessuna o ∞

Oss: Se fisso A e faccio variare b ho:

quadrato → sempre soluz unica (1)
quadrato → nessuna o infinite (2)

sovradet → una soluz o nessuna (3)
sovradet → nessuna o ∞ (4)

(1) matrice sistema invertibile; (2) non invertib

Riscrivo
$$\begin{cases} \text{\_\_\_\_\_\_} = \dots \\ \text{\_\_\_\_\_\_} \\ \dots \\ \text{\_\_\_\_\_\_} = \dots \\ \text{\_\_\_\_\_\_} = \dots \\ \text{\_\_\_\_\_\_} = \dots \end{cases}$$

k con parte omog. lin. indip.

dipendenti dalle prime k

(3) $k = n$

(4) $k < n$

Q : • Decidere se $A \in M_{n \times n}$ è invertibile

• In tal caso trovare A^{-1}

• Calcolare rank (B) $\quad B \in M_{m \times n}$

A : Determinante

$A = (a_{11}) \in M_{1 \times 1}$ invertibile se $a_{11} \neq 0$; in tal caso $A^{-1} = (a_{11}^{-1})$.

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$; cerco A^{-1}

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{cases} ax + cy = 1 \\ bx + dy = 0 \end{cases} \qquad \begin{cases} az + cw = 0 \\ bz + dw = 1 \end{cases}$$

Oss: se d e c sono entrambi nulli, A^{-1} non esiste

Analogamente se $a = b = 0$.

$d\cdot \mathrm{I} - c\cdot \mathrm{II}$ $\quad$ $-b\cdot \mathrm{I} + a\cdot \mathrm{II}$

$$\begin{cases} (ad-bc)x = d \\ (ad-bc)y = -b \end{cases} \qquad \begin{cases} (ad-bc)x = -c \\ (ad-bc)y = a \end{cases}$$

Oss: Se $ad = bc$ allora $\begin{pmatrix} a \\ c \end{pmatrix}$ $\begin{pmatrix} b \\ d \end{pmatrix}$ sono proporzionali (e viceversa) dunque non esiste A^{-1}!

Prop: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}$ esiste $\Longleftrightarrow$ $ad - bc \neq 0$ e in tal caso è $\dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

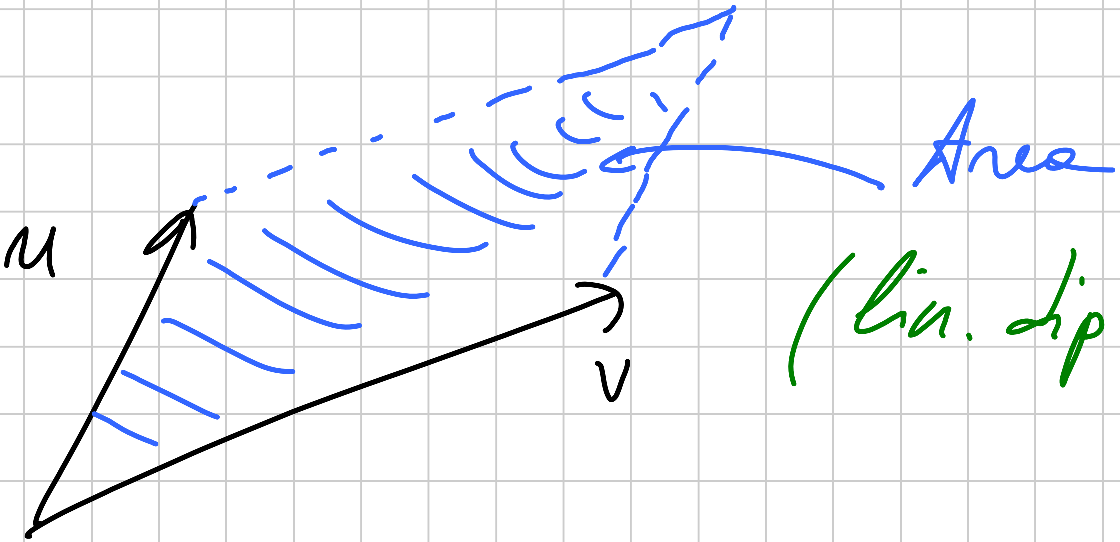
Es: $\begin{pmatrix} 3 & 7 \\ -4 & 2 \end{pmatrix}^{-1} = \frac{1}{34}\begin{pmatrix} 2 & -7 \\ 4 & 3 \end{pmatrix}$.

Def: pongo $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad-bc$.

Significato geometrico

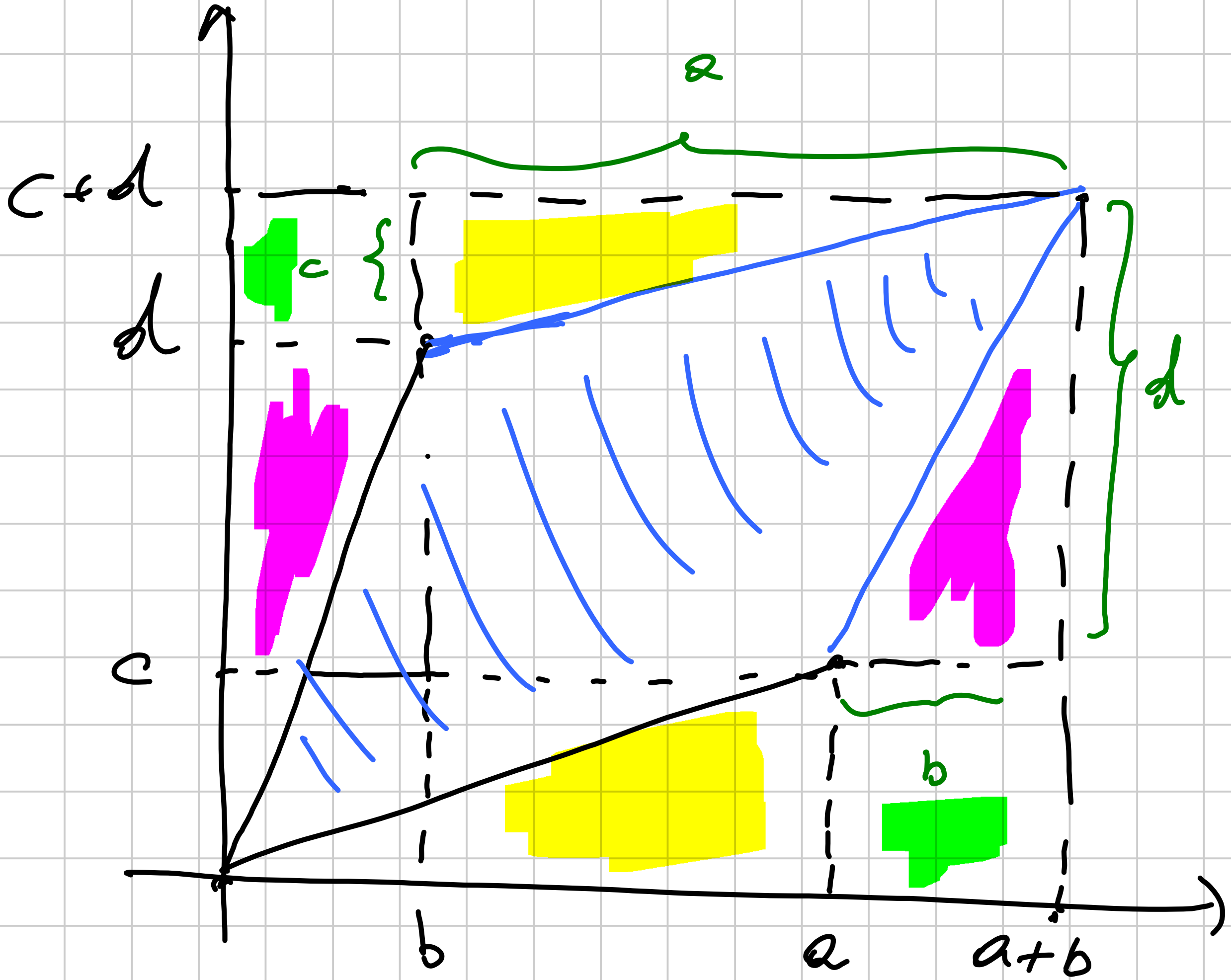
$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ invertibile $\Leftrightarrow$ $\begin{pmatrix} a \\ c \end{pmatrix} \begin{pmatrix} b \\ d \end{pmatrix}$ lin. indip

"Misura" della indipendenza lin. di u e v



(lin. dip. $\iff$ area $= 0$)

Calcolo per $\begin{pmatrix} a \\ c \end{pmatrix}$, $\begin{pmatrix} b \\ d \end{pmatrix}$:



Area ( ) = Area (quadrato) − Area (4 triang. e 2 quadratini)

$$= (a+b)(c+d) - 2\cdot\frac{1}{2}a\cdot c - 2\cdot\frac{1}{2}b\cdot d - 2\cdot bc$$

$$= \cancel{ac} + ad + bc + \cancel{bd} - \cancel{ac} - \cancel{bd} - 2bc = ad - bc$$

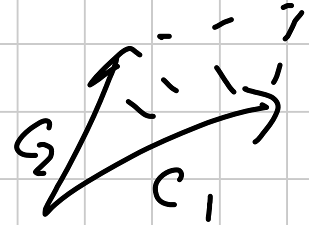
$\Rightarrow \quad ad - bc = \cancel{\text{Area}}$ (parallelogramma)

Area con segno

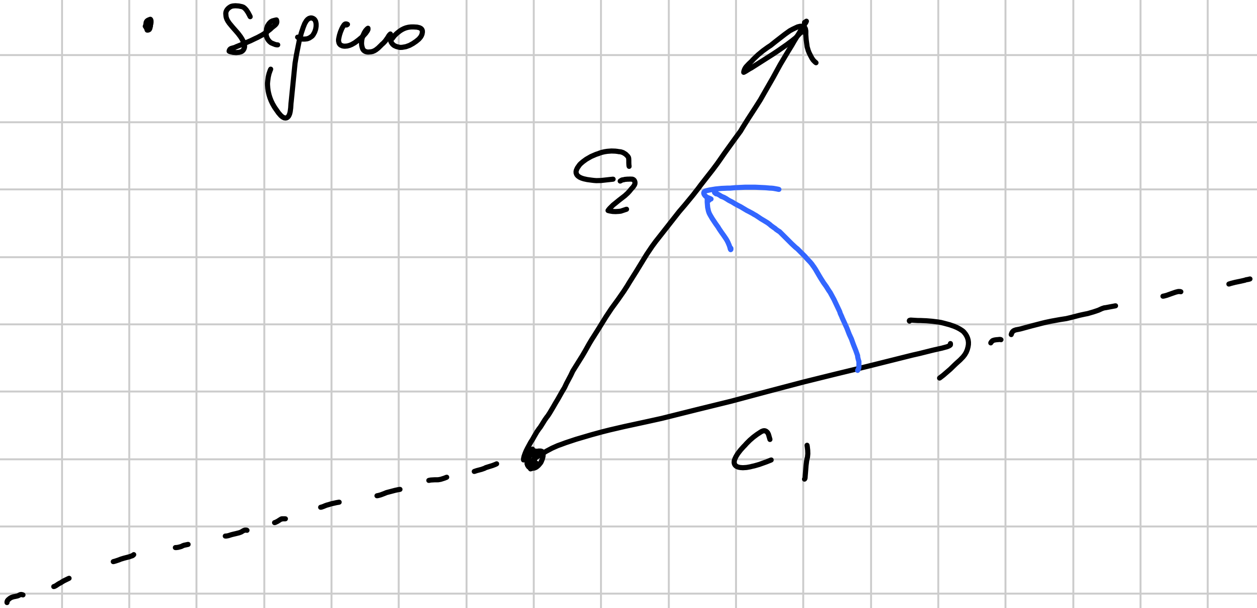
Es: $\begin{pmatrix} 4 & 7 \\ 3 & 5 \end{pmatrix} \quad \det = -1$

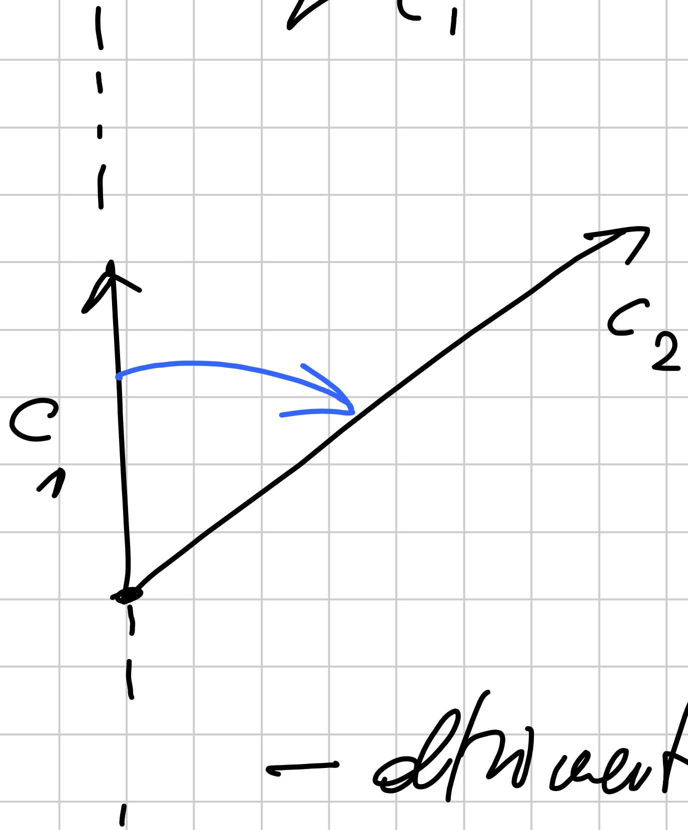
Dunque $\det(A)$ ha $\qquad A = (c_1, c_2) \in M_{2\times 2}$

- valore assoluto = area parallelogramma



- segno



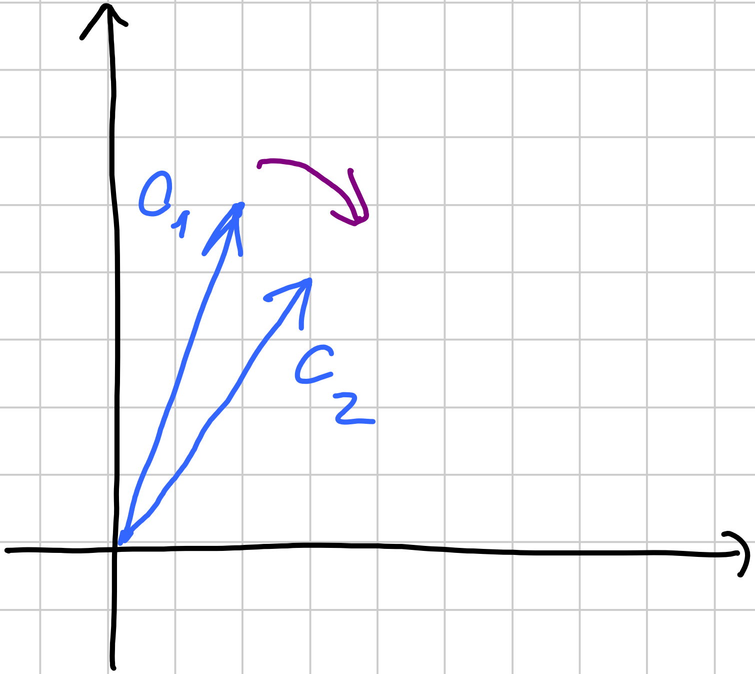


+ se la rotazione di $< 180°$

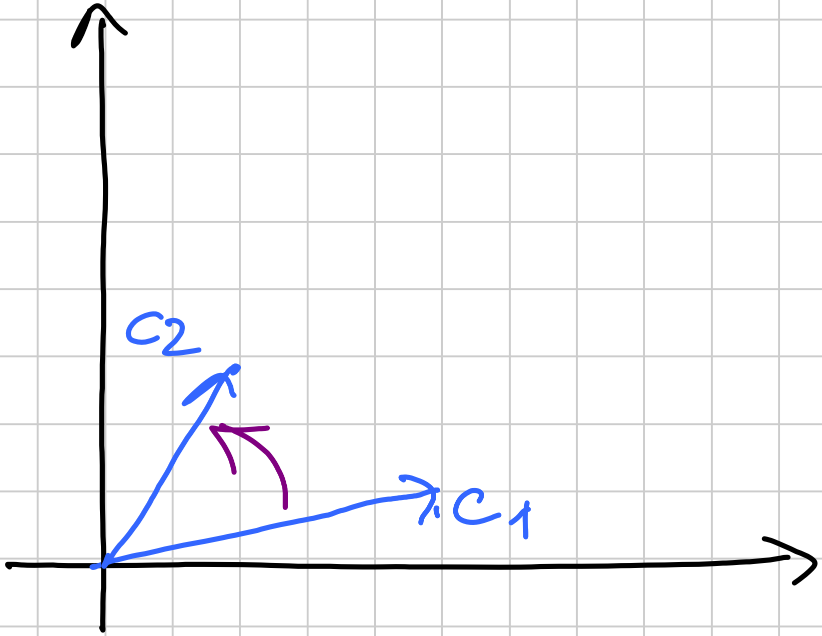
− altrimenti

che porta c_1 su c_2 è antioraria

Es: $\det\begin{pmatrix} 2 & 3 \\ 5 & 4 \end{pmatrix} = -7$



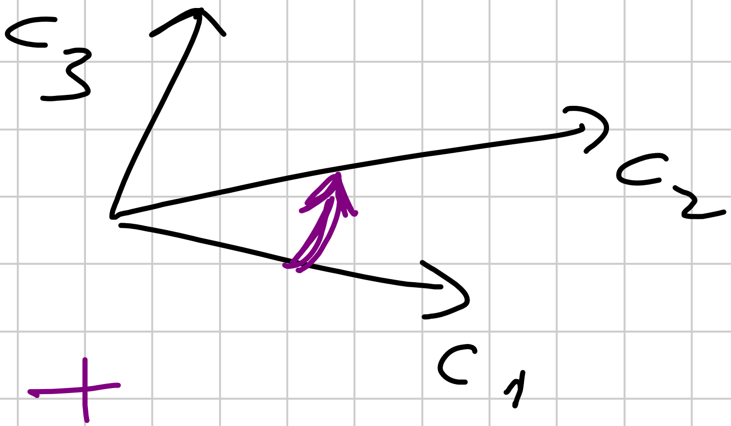
$\det\begin{pmatrix} 5 & 2 \\ 1 & 3 \end{pmatrix} = +13$

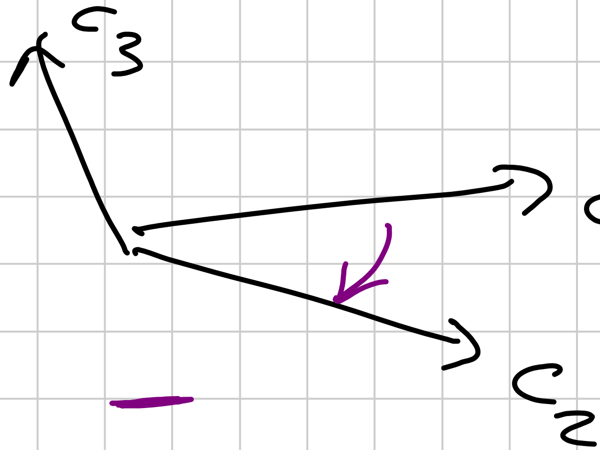


$A \in M_{3\times 3}$: $\det(A)$ = "Volume con segno" del parallelepipedo che ha per lati c_1, c_2, c_3

$A = (c_1, c_2, c_3)$

"segno" = "regola mano dx" (vite destrogira)





Con calcolo come sopra ;

Regola di Sarrus :

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$$

$$- \quad - \quad - \qquad + \quad + \quad +$$

$$= a_{11}\cdot a_{22}\cdot a_{33} + a_{12}\cdot a_{23}\cdot a_{31} + a_{13}\cdot a_{21}\cdot a_{32} - a_{13}\,a_{22}\,a_{31} - a_{11}\,a_{23}\,a_{32} - a_{12}\cdot a_{21}\cdot a_{33} -$$

Lo stesso si può fare per $A \in M_{n\times n}$ usando un "volume n-dim. con segno". Vediamo invece oltre due diverse costruzioni del det(A) con $A \in M_{n \times n}$.

25/10 3(b)

$V = \mathbb{R}^4 \quad W = \mathrm{Span}\left(\begin{pmatrix} 1 \\ 2 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ -4 \\ 5 \\ 1 \end{pmatrix}, \begin{pmatrix} 7 \\ 2 \\ -1 \\ -1 \end{pmatrix} \right) \quad \dim W = 3$

$Z: \begin{cases} 2x_1 + x_2 - 3x_3 + 5x_4 = 0 \\ 3x_1 - x_2 + 2x_3 + x_4 = 0 \end{cases} \qquad \dim Z = 4 - 2 = 2$

$$3+2 = \dim(Z+W) + \dim(Z\cap W)$$

due poss $\to$ 3   2   (se $W \subset Z$)

$\to$ 4   1   Quale?

$Z \cap W$: Cerco per quali $a, b, c \in \mathbb{R}$

$$a\begin{pmatrix}1\\2\\-1\\3\end{pmatrix} + b\begin{pmatrix}3\\-4\\5\\1\end{pmatrix} + c\begin{pmatrix}7\\2\\-1\\-1\end{pmatrix} \text{ è in } Z:$$

$$\begin{cases} 2\cdot(a+3b+7c) + (2a-4b+2c) - 3(-a+5b-c) + 5(3a+b-c) = 0 \\ 3\cdot(\quad " \quad) - (\quad " \quad) + 2(\quad " \quad) + 1(\quad " \quad) = 0 \end{cases}$$

$$\begin{cases} 22a - 8b + 14c = 0 \\ 2a + 24b + 18c = 0 \end{cases} \Longrightarrow \dim Z \cap W = 3 - 2 = 1.$$

$$(\Longrightarrow Z + W = \mathbb{R}^4).$$

1/11/13 (f)

$$f : \mathbb{R}^4 \to \mathbb{R}^3 \qquad f(x) = \begin{pmatrix} x_1 - 2x_2 + x_4 \\ x_2 + x_3 - x_4 \\ x_1 + 3x_3 + 2x_4 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & -2 & 0 & 1 \\ 0 & 1 & 1 & -1 \\ 1 & 0 & 3 & 2 \end{pmatrix}}_{A} \cdot x$$

$$\dim \mathbb{R}^4 = \dim \operatorname{Im} f + \dim \operatorname{Ker} f$$

4	=	0		4
		1		3
		2		2
		3		1

dim Im f : estraiamo base dalle colonne di A

$$\text{I}, \text{II}, \text{III}, \text{IV}$$

$\checkmark \quad \checkmark \quad \checkmark \quad \times$

Ker f : $\begin{cases} x_1 = 2x_2 - x_4 \\ x_3 = -x_2 + x_4 \\ 2x_2 - x_4 + 3(-x_2 + x_4) + 2x_4 = 0 \end{cases}$ $\begin{cases} x_1 = \text{—} \\ x_3 = \text{—} \\ +x_2 = 4x_4 \end{cases}$ $\underline{\dim = 1}$

(j) $f : M_{2\times 2} \to \mathbb{R}^3$ $\quad f(A) = \begin{pmatrix} a_{11} - 3a_{12} \\ 2a_{21} + a_{12} \\ -a_{11} + 5a_{22} \end{pmatrix}$

4 $\quad$ 3

$\operatorname{Im} f$:

$$\begin{array}{cccc}
\begin{pmatrix}1\\0\\-1\end{pmatrix} \text{\color{red}✗} & \begin{pmatrix}-3\\1\\0\end{pmatrix} \text{\color{green}✓} & \begin{pmatrix}0\\2\\0\end{pmatrix} \text{\color{green}✓} & \begin{pmatrix}0\\0\\5\end{pmatrix} \text{\color{green}✓} \\
\shortparallel & \shortparallel & \shortparallel & \shortparallel \\
f\begin{pmatrix}1&0\\0&0\end{pmatrix} & f\begin{pmatrix}0&1\\0&0\end{pmatrix} & f\begin{pmatrix}0&0\\1&0\end{pmatrix} & f\begin{pmatrix}0&0\\0&1\end{pmatrix}
\end{array}$$

$\Rightarrow \dim = 3$

$\Rightarrow \dim \operatorname{Ker} f = 4-3=1$

$\operatorname{Ker} f$:
$$\begin{cases} a_{11} = 3a_{12} \\ a_{21} = -\frac{1}{2} a_{12} \\ a_{22} = \frac{1}{5} a_{11} = \frac{3}{5} a_{12} \end{cases}$$

$\Rightarrow \dim = 1$

1/11/13 (m)

$f: \mathbb{R}_{\leq 3}[t] \to \mathbb{R}^2$, $f(p(t)) = \begin{pmatrix} p(2) - p'(-1) \\ p(1) + 2p''(2) \end{pmatrix}$

4   2

$\Rightarrow \dim \operatorname{Ker} f \geq 2$

$\operatorname{Im} f$: $f(1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ ✓ $f(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ ✗ $f(t^2) = \begin{pmatrix} 6 \\ 5 \end{pmatrix}$ ✓ $f(t^3) = \begin{pmatrix} 5 \\ 25 \end{pmatrix}$ ✗ $\dim = 2$

Ker f: $a_0 + a_1 t + a_2 t^2 + a_3 t^3 \in \operatorname{Ker} f \iff$

$a_0\begin{pmatrix}1\\1\end{pmatrix} + a_1\begin{pmatrix}1\\1\end{pmatrix} + a_2\begin{pmatrix}6\\5\end{pmatrix} + a_3\begin{pmatrix}5\\25\end{pmatrix} = 0 \iff$

$$\begin{cases} a_0 + a_1 + 6a_2 + 5a_3 = 0 \\ a_0 + a_1 + 5a_2 + 25a_3 = 0 \end{cases}$$

$\dim = 4 - 2 = 2.$

8/11 7 (d) Trovare p, q associate e $V = W \oplus Z$

$V = \mathbb{R}^3$ $\quad W = \operatorname{Span}\left(\begin{pmatrix}2\\1\\0\end{pmatrix}\begin{pmatrix}-1\\0\\3\end{pmatrix}\right)$ $\quad Z = \operatorname{Span}\begin{pmatrix}0\\1\\-1\end{pmatrix}.$

$$x = \underbrace{\left(a\begin{pmatrix}2\\1\\0\end{pmatrix} + b\begin{pmatrix}-1\\0\\3\end{pmatrix}\right)}_{p(x)} + \underbrace{\left(c \cdot \begin{pmatrix}0\\1\\-1\end{pmatrix}\right)}_{q(x)}$$

(devo trovare a, b, c)

il generico el. di W

il generico el. di Z.

$$\begin{cases} 2a - b = x_1 \\ a + c = x_2 \\ 3b - c = x_3 \end{cases} \qquad \begin{cases} b = -x_1 + 2a \\ c = x_2 - a \\ 7a = 3x_1 + x_2 + x_3 \end{cases}$$

$$a = \frac{1}{7}(3x_1 + x_2 + x_3) \qquad b = \frac{-x_1 + 2x_2 + 2x_3}{7} \qquad c = \frac{-3x_1 + 6x_2 - x_3}{7}$$

$$\Rightarrow p(x) = \frac{1}{7}\left(3x_1 + x_2 + x_3\right)\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \frac{1}{7}\left(-x_1 + 2x_2 + 2x_3\right)\begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}$$

$$= \underbrace{\frac{1}{7}\begin{pmatrix} 7 & 0 & 0 \\ 3 & 1 & 1 \\ -3 & 6 & 6 \end{pmatrix}}_{P} \cdot x$$

$$q(x) = \frac{-3x_1 + 6x_2 - x_3}{7} \cdot \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \underbrace{\frac{1}{7}\begin{pmatrix} 0 & 0 & 0 \\ -3 & 6 & -1 \\ 3 & -6 & 1 \end{pmatrix}}_{Q} \cdot x$$

$$p \circ p = p \quad \text{cioè} \quad P \cdot P = P$$

$$\frac{1}{7}\begin{pmatrix} 7 & 0 & 0 \\ 3 & 1 & 1 \\ -3 & 6 & 6 \end{pmatrix} \cdot \frac{1}{7}\begin{pmatrix} 7 & 0 & 0 \\ 3 & 1 & 1 \\ -3 & 6 & 6 \end{pmatrix}$$

$$= \frac{1}{49}\begin{pmatrix} 49 & 0 & 0 \\ 21 & 7 & 7 \\ -21 & 42 & 42 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 7 & 0 & 0 \\ 3 & 1 & 1 \\ -3 & 6 & 6 \end{pmatrix} = P$$

$p \circ q = 0$ cioè $P \cdot Q = 0$ facilissimo ...

(e) $V = \mathbb{R}^3$ $W : 5x_1 - 3x_2 + 2x_3 = 0$ $Z = \mathrm{Span}\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$

$$x = \underbrace{\underset{W}{\overset{\cap}{w}}}_{p(x)} + \underbrace{t \cdot \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}}_{q(x)}$$

Va bene se

$$x - t \cdot \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} \in W$$

cioè $(5, -3, 2) \cdot \left(x - t \cdot \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} \right) = 0$

cioè $\quad 5x_1 - 3x_2 + 2x_3 = t(15 - 3 - 4)$

cioè $\quad t = \frac{1}{8}(5x_1 - 3x_2 + 2x_3)$

$$\Rightarrow p(x) = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} - \frac{1}{8}(5x_1 - 3x_2 + 2x_3)\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$$

$$= \frac{1}{8}\begin{pmatrix} -7x_1 + 9x_2 - 6x_3 \\ -5x_1 + 11x_2 - 2x_3 \\ 10x_1 - 6x_2 + 12x_3 \end{pmatrix} = \underbrace{\frac{1}{8}\begin{pmatrix} -7 & 9 & -6 \\ -5 & 11 & -2 \\ 10 & -6 & 12 \end{pmatrix}}_{P} \cdot x$$

$p+q = id_{\mathbb{R}^3}$ cioè $P+Q = I_3$ $\quad Q = \frac{1}{8}\begin{pmatrix} 15 & -9 & 6 \\ 5 & -3 & 2 \\ -10 & 6 & -4 \end{pmatrix}$

$P\cdot P = P \qquad P\cdot Q = 0 \qquad Q\cdot P = 0 \qquad Q\cdot Q = 0$

$$\frac{1}{8}\begin{pmatrix} -7 & 9 & -6 \\ -5 & 11 & -2 \\ 10 & -6 & 12 \end{pmatrix} \cdot \frac{1}{8}\begin{pmatrix} -7 & 9 & -6 \\ -5 & 11 & -2 \\ 10 & -6 & 12 \end{pmatrix}$$

$$= \frac{1}{\cancel{64}\ 8}\begin{pmatrix} 49-45-60 = \cancel{-56}\ {-7} & -63+99+36 = \cancel{72}\ 9 & \ldots \\ \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \end{pmatrix}$$

(I prossimi due non sono stati svolti in aula.)

(h) Poiché $(1,1,1,1)\cdot\begin{pmatrix}2\\3\\-9\\4\end{pmatrix}=0$ abbiamo $W\subset V$.

Il generico elemento di V ha la forma $\begin{pmatrix}a\\b\\c\\-a-b-c\end{pmatrix}$ e vogliamo scriverlo come

$$\begin{pmatrix}a\\b\\c\\-a-b-c\end{pmatrix}=t\begin{pmatrix}2\\3\\-9\\4\end{pmatrix}+z$$ con $z\in Z$, dunque vogliamo che

$$(3,2,5,7)\cdot\left(\begin{pmatrix}a\\b\\c\\-a-b-c\end{pmatrix}-t\begin{pmatrix}2\\3\\-9\\4\end{pmatrix}\right)=0$$ cioè

$(3a+2b+5c-7a-7b-7c)=t\cdot(6+6-45+28)$, $t=\frac{1}{5}(4a+5b+2c)$

$$\Rightarrow p\begin{pmatrix} a \\ b \\ c \\ -a-b-c \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 8a+10b+4c \\ 12a+15b+6c \\ -36a-45b-18c \\ 16a+20b+8c \end{pmatrix} \quad ; \text{ se } \mathcal{B} = \left(\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}, \begin{pmatrix}0\\1\\0\\-1\end{pmatrix}, \begin{pmatrix}0\\0\\1\\-1\end{pmatrix}\right)$$

abbiamo allora

$$[p]^{\mathcal{B}}_{\mathcal{B}} = \frac{1}{5}\begin{pmatrix} 8 & 10 & 4 \\ 12 & 15 & 6 \\ -36 & -45 & -18 \end{pmatrix}$$

e tale matrice ha in effetti quadrato uguale a sé stessa.

(i) $a_0 + a_1 t + a_2 t^2 + a_3 t^3 \in Z$

$$\Longleftrightarrow \begin{cases} a_0 + a_1 + a_2 + a_3 + (a_1 - 2a_2 + 3a_3) = 0 \\ a_0 + 2a_1 + 4a_2 + 8a_3 - 2(2a_2 + 6a_3) = 0 \end{cases}$$

$$\Longleftrightarrow \begin{cases} a_0 + 2a_1 - a_2 + 4a_3 = 0 \\ a_0 + 2a_1 - 4a_3 = 0 \end{cases} \Longleftrightarrow \begin{cases} a_2 = 2a_0 + 4a_1 \\ a_3 = \frac{1}{4} a_0 + \frac{1}{2} a_1 \end{cases}$$

Dunque Z ha base $4 + 8t^2 + t^3$, $2t + 8t^2 + t^3$

Ora bisogna scrivere

$$a_0 + a_1 t + a_2 t^2 + a_3 t^3 = \alpha_1 \cdot (2 - t + t^3) + \alpha_2 \cdot (1 + t^2 - 4t^3) \Big\} \; p(a_0 + \dots)$$
$$+ \beta_1 \cdot (4 + 8t^2 + t^3) + \beta_2 \cdot (2t + 8t^2 + t^3) \Big\} \; q(a_0 + \dots)$$

Facendo i conti, rispetto alla base canonica $1, t, t^2, t^3$ si trova

$$[p] = \frac{1}{4} \begin{pmatrix} 74 & 148 & -38 & 8 \\ -33 & -66 & 17 & -4 \\ 8 & 16 & -4 & 0 \\ 1 & 2 & -1 & 4 \end{pmatrix}$$

che ha quadrato uguale a sé stessa.