<u>Theorem</u>: there exists a splitting exact sequence

$$0 \longrightarrow H_n^{\mathbb{Z}} \otimes R \xrightarrow{g_n} H_n^R \xrightarrow{f_n} \mathrm{Tor}(H_{n-1}^{\mathbb{Z}}, R) \longrightarrow 0 .$$

<u>Proof</u>: we first note that there is a canonical isomorphism

$$c_n : C_n^{\mathbb{Z}} \otimes R \to C_n^R$$
$$c_n(\sigma \otimes \lambda) = \lambda \cdot \sigma$$

and we claim that

$$(1) \qquad \partial_n^R \circ c_n = c_{n-1} \circ (\partial_n^{\mathbb{Z}} \otimes Id_R)$$

because

$$\partial_n^R(c_n(\sigma)) = \partial_n(\lambda \cdot \sigma) =$$
$$= \lambda \cdot \partial_n^{\mathbb{Z}} \sigma = c_{n-1}(\partial_n^{\mathbb{Z}} \sigma \otimes \lambda)$$
$$= c_{n-1}((\partial_n^{\mathbb{Z}} \otimes Id_R)(\sigma)) .$$

Now (1) easily implies that

$$c_n(Z_n^{\mathbb{Z}} \otimes R) \subset Z_n^R \quad , \quad c_n(B_n^{\mathbb{Z}} \otimes R) \subset B_n^R$$

because

$$\partial_m^{\mathbb{Z}} u = 0 \implies \partial_m^R(c_n(u \otimes \lambda)) = c_{n-1}(\partial_n^{\mathbb{Z}} u \otimes \lambda) = 0$$

$$u = \partial_{m+1}^{\mathbb{Z}}(w) \implies \partial_{m+1}^R(c_{n+1}(w \otimes \lambda)) = c_n(\partial_{n+1} w \otimes \lambda) = c_n(u \otimes \lambda)$$

therefore c_m induces well-defined

$$z_m : Z_m^{\mathbb{Z}} \otimes R \to Z_m^R \qquad b_m : B_m^{\mathbb{Z}} \otimes R \to B_m^R .$$

For $\Lambda = \mathbb{Z}, R$ we also fix the following notation

$$0 \to B_n^\Lambda \xhookrightarrow{j_n^\Lambda} Z_n^\Lambda \xrightarrow{p_n^\Lambda} H_n^\Lambda \to 0$$

$$Z_n^\Lambda \xhookrightarrow{i_n^\Lambda} C_n^\Lambda \quad , \quad B_n^\Lambda \xhookrightarrow{b_n^\Lambda} C_n^\Lambda$$

and we recall that by definition

$$(2) \quad \mathrm{Tor}(H_{n-1}^{\mathbb{Z}}, R) = \mathrm{Ker}\left(B_{n-1}^{\mathbb{Z}} \otimes R \xrightarrow{j_{n-1}^{\mathbb{Z}} \otimes Id_R} Z_{n-1}^{\mathbb{Z}} \otimes R\right).$$

We now define the maps g_n and f_n:

$$g_m(p_m^{\mathbb{Z}}(u) \otimes \lambda) = p_m^R(z_m(u \otimes \lambda)) \qquad \forall u \in Z_m^{\mathbb{Z}}$$

$$f_m(p_m^R(\textstyle\sum \lambda_i \sigma_i)) = \sum (\partial_m^{\mathbb{Z}} \sigma_i) \otimes \lambda_i \quad \forall \sum \lambda_i \sigma_i \in Z_n^R$$

There are many things we must prove:

- g_m <u>is well-defined</u>.

To this end we first note that for $u \in Z_m^{\mathbb{Z}}$ indeed $z_m(u \otimes \lambda) \in Z_m^R$, so $p_m(z_m(u \otimes \lambda)) \in H_n^R$ is well defined. For $u \in B_m^{\mathbb{Z}}$ we have $z_m(u \otimes \lambda) \in B_m^R$, so $p_m^R(z_m(u \otimes \lambda)) = 0$ and we are done.

- f_m <u>is well-defined</u>.

For $\sum \lambda_i \sigma_i \in C_n^R$ (even if not in Z_m^R) we have $\sum (\partial_m^{\mathbb{Z}} \sigma_i) \otimes \lambda_i \in B_{n-1}^{\mathbb{Z}} \otimes R$.

According to (2) we must show that for $\sum \lambda_i \sigma_i \in Z_m^R$ we have $(j_{n-1}^{\mathbb{Z}} \otimes Id_R)(\sum (\partial_n^{\mathbb{Z}} \sigma_i) \otimes \lambda_i) = 0$:

Indeed $z_m((j_{n-1}^{\mathbb{Z}} \otimes Id_R)(\sum (\partial_m^{\mathbb{Z}} \sigma_i) \otimes \lambda_i))$

$$= z_m(\sum (j_{n-1}^{\mathbb{Z}} (\partial_m^{\mathbb{Z}} \sigma_i)) \otimes \lambda_i)$$

(3) $$= z_m(\sum (\partial_m^{\mathbb{Z}} \sigma_i) \otimes \lambda_i)$$

$$= \sum \lambda_i \cdot \partial_m^{\mathbb{Z}} \sigma_i$$

$$= \partial_m^R (\sum \lambda_i \sigma_i)$$

and we only need to note that z_m is injective because $Z_m^{\mathbb{Z}} \oplus B_{m-1}^{\mathbb{Z}} \cong C_m^{\mathbb{Z}}$ $\Rightarrow (Z_m^{\mathbb{Z}} \otimes R) \oplus (B_{m-1}^{\mathbb{Z}} \otimes R) \cong C_m^{\mathbb{Z}} \otimes R \cong C_m^R$ so the inclusion $Z_m^{\mathbb{Z}} \subset C_m^{\mathbb{Z}}$ induces an injection $Z_m^{\mathbb{Z}} \otimes R \to C_m^R$ and z_m is its abbreviation to Z_m^R

To prove that f_m is well-defined we are left to show that $\sum \partial_m^{\mathbb{Z}} \sigma_i \otimes \lambda_i = 0$ for $\sum \lambda_i \sigma_i \in B_m^R$. In fact

$$\sum \lambda_i \sigma_i = \partial_{m+1}^R w, \quad w = \sum \mu_j \tau_j$$

$$\Rightarrow \sum \lambda_i \sigma_i = \sum \mu_j \partial_{m+1}^R \tau_j$$

$$\Rightarrow \sum \sigma_i \otimes \lambda_i = \sum (\partial_{m+1}^{\mathbb{Z}} \tau_j) \otimes \mu_j$$

$$\Rightarrow \sum (\partial_m^{\mathbb{Z}} \sigma_i) \otimes \lambda_i = (\partial_m^{\mathbb{Z}} \otimes id_R)(\sum \sigma_i \otimes \lambda_i)$$

$$= (\partial_m^{\mathbb{Z}} \otimes id_R)(\sum (\partial_{m+1}^{\mathbb{Z}} \tau_j) \otimes \mu_j)$$

$$= \sum (\partial_m^{\mathbb{Z}} \circ \partial_{m+1}^{\mathbb{Z}})(\tau_j) \otimes \mu_j = 0.$$

- g_n is injective

Suppose $g_n\left(\sum p_n^{\mathbb{Z}}(u_i)\otimes\lambda_i\right)=0$ i.e.

$$z_n\left(\sum u_i\otimes\lambda_i\right)=\partial^R_{n+1}\left(\sum \mu_j\sigma_j\right)$$
$$=\partial^R_{n+1}\left(c_{n+1}\left(\sum\sigma_j\otimes\mu_j\right)\right)$$
$$\underset{(1)}{=} c_n\left(\left(\partial^{\mathbb{Z}}_{n+1}\otimes Id_R\right)\left(\sum\sigma_j\otimes\mu_j\right)\right)$$
$$=c_n\left(\sum\left(\partial^{\mathbb{Z}}_{n+1}\sigma_j\right)\otimes\mu_j\right)$$

$$\Rightarrow \sum u_i\otimes\lambda_i=\sum\left(\partial^{\mathbb{Z}}_{n+1}\sigma_j\right)\otimes\mu_j$$

(because c_n is injective and z_n is its restriction) so

$$\sum p_n^{\mathbb{Z}}(u_i)\otimes\lambda_i=\left(p_n^{\mathbb{Z}}\otimes Id_R\right)\left(\sum u_i\otimes\lambda_i\right)$$
$$=\left(p_n^{\mathbb{Z}}\otimes Id_R\right)\left(\sum\left(\partial^{\mathbb{Z}}_{n+1}\sigma_j\right)\otimes\mu_j\right)$$
$$=\sum 0\otimes\mu_j=0.$$

- f_n is surjective

While proving (3) we have seen that if the generic element $\sum(\partial_n^{\mathbb{Z}}\sigma_i)\otimes\lambda_i$ of $B^{\mathbb{Z}}_{n-1}\otimes R$ belongs to $Ker\left(j^{\mathbb{Z}}_{n-1}\otimes Id_R\right)=Tor(H^{\mathbb{Z}}_{n-1},R)$ then $\partial_n^R\left(\sum\lambda_i\sigma_i\right)=0$, so it is the image under f_n of $p_n^R\left(\sum\lambda_i\sigma_i\right)$.

- $\mathrm{Im}\, g_m \subset \mathrm{Ker}\, f_m$

$$f_m\left(g_m\left(p_m^{\mathbb{Z}}(u)\otimes\lambda\right)\right)$$
$$= f_m\left(p_m^{R}\left(z_m(u\otimes\lambda)\right)\right)$$
$$= f_m\left(p_m^{R}(\lambda u)\right)$$
$$= (\partial_m^{\mathbb{Z}} u)\otimes\lambda = 0$$

- $\mathrm{Ker}\, f_m \subset \mathrm{Im}\, g_m$

To see this it's enough to exhibit an isomorphism

$$\varphi_m : \mathrm{Tor}\left(H_{m-1}^{\mathbb{Z}}, R\right) \to \frac{H_n^R}{\mathrm{Im}\,(g_n)}$$

whose inverse is induced by f_m.

For $\sum \partial_m^{\mathbb{Z}}\sigma_i \otimes \lambda_i \in \mathrm{Tor}\left(H_{m-1}^{\mathbb{Z}}, R\right) = \mathrm{Ker}\left(j_{m-1}^{\mathbb{Z}} \otimes Id_R\right)$

we define $\varphi_m\left(\sum \partial^{\mathbb{Z}}\sigma_i\otimes\lambda_i\right) = \left[\sum \lambda_i\sigma_i\right] + \mathrm{Im}(g_n)$

which is well defined because

$$\partial_m^R\left(\sum \lambda_i\sigma_i\right) = \sum \lambda_i \partial_m^{\mathbb{Z}}\sigma$$
$$= \left(j_{m-1}^{\mathbb{Z}} \otimes Id_R\right)\left(\sum \partial_m^{\mathbb{Z}}\sigma_i\otimes\lambda_i\right) = 0$$

Of course $\bar{f}_m : \dfrac{H_m^R}{\mathrm{Im}(g_m)} \to \mathrm{Tor}\left(H_{n-1}^{\mathbb{Z}}, R\right)$ is the inverse of φ_m.

- The sequence splits

We know $0 \to Z_n^{\mathbb{Z}} \to C_n^{\mathbb{Z}} \to B_{n-1}^{\mathbb{Z}} \to 0$ splits because $B_{n-1} \subset C_{n-1}$ is free, so there exists $q_n^{\mathbb{Z}} : C_n^{\mathbb{Z}} \to Z_n^{\mathbb{Z}}$ that is the identity on $Z_n^{\mathbb{Z}}$. We now define

$$q_n^R : H_n^R \to H_n^{\mathbb{Z}} \otimes R$$

$$\left[\sum \lambda_i \sigma_i\right] \longmapsto \sum \left[q_n^{\mathbb{Z}}(\sigma_i)\right] \otimes \lambda_i$$

This map is well-defined because

- $q_n^{\mathbb{Z}}(\sigma_i) \in Z_n$ so $\left[q_n^{\mathbb{Z}}(\sigma_i)\right] \in H_n$
- $\left[\sum \lambda_i \sigma_i\right] = \left[\sum \lambda_i' \sigma_i'\right]$

$$\Rightarrow \sum \lambda_i \sigma_i - \sum \lambda_i' \sigma_i' = \sum \mu_j \partial_{n+1}^{\mathbb{Z}} \tau_j$$

$$\Rightarrow \sum \left[q_n^{\mathbb{Z}}(\sigma_i)\right] \otimes \lambda_i - \sum \left[q_n^{\mathbb{Z}}(\sigma_i')\right] \otimes \lambda_i'$$

$$= \sum \left[q_n^{\mathbb{Z}}(\underbrace{\partial_{n+1} \tau_j}_{\in Z_n^{\mathbb{Z}}})\right] \otimes \mu_j$$

$$\Downarrow$$

$$\underbrace{}_{\left[\partial_{n+1}^{\mathbb{Z}} \tau_j\right] = 0}$$

Of course

$$(q_n^R \circ g_n)(\underbrace{[u]}_{u \in Z_n^{\mathbb{Z}}} \otimes \lambda) = q_n^R \left[\lambda \cdot u\right] = \left[\underbrace{q_n^{\mathbb{Z}}(u)}_{= u}\right] \otimes \lambda. \qquad ▨$$