# Geometria 22/5/14

Foglio 7/5/14 (quadriche) —

(e) $x^2+y^2-4xz+6yz+2x+4z=0$

$$\begin{pmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 3 & 0 \\ -2 & 3 & 0 & 2 \\ 1 & 0 & 2 & 0 \end{pmatrix}$$

$d_1>0 \quad d_2>0$

$d_3<0 \quad d_4<0$

Autoval:

$(+\ +\ -)\ +$

$x^2 + y^2 - z^2 + 1 = 0$

$z^2 = 1 + x^2 + y^2$

iperboloide 2 falde (ellittico)

(7) $\begin{pmatrix} -1 & -2 & 0 & -1 \\ -2 & 1 & 3 & 0 \\ 0 & 3 & 0 & 2 \\ -1 & 0 & 2 & 0 \end{pmatrix}$

$(d_1 < 0)$ $d_2 < 0$ $d_3 > 0$ $d_4 > 0$

Autoval: $(+ - -) +$ $x^2 - y^2 - z^2 + 1 = 0$

$X^2+Y^2=1+Z^2$

ipab. 1 falda (ipab)

(g) $$\begin{pmatrix} 1 & 1 & 0 & -2 \\ 1 & 2 & -2 & 1 \\ 0 & -2 & 5 & 0 \\ -2 & 1 & 0 & 0 \end{pmatrix}$$

$d_1>0 \quad d_2>0 \quad d_3>0 \quad d_4<0$

Autoval: $(+\ +\ +)\ -$

$x^2+y^2+z^2-1=0$ ellissoide

(m) $x^2+5y^2+(k^2+9)z^2+2xy-2kxz+2kyz-1=0$

$$\begin{pmatrix} 1 & 1 & -k & 0 \\ 1 & 5 & k & 0 \\ -k & k & k^2+9 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$d_1>0$

$d_2>0$

$d_3 = 5(k^2+9)-k^2-k^2-5k^2-k^2-9-k^2$

$= 36-4k^2 = 4(9-k^2)$

$d_4 = 4(k^2-9)$

$|k|<3$ $\quad d_1>0 \quad d_2>0 \quad d_3>0 \quad d_4<0$

Antord $(+++)-$ ellissoide

$|k| > 3$ $\quad d_1 > 0 \quad d_2 > 0 \quad d_3 < 0 \quad d_4 > 0$

Autoval $(+ + -) -$

$$x^2 + y^2 - z^2 - 1 = 0$$

iperboloide a 1 falda (iperb)

$|k| = 3$ degenere $\quad d_1 > 0 \quad d_2 > 0 \quad d_3 = d_4 = 0$

Autoval $+ + 0 -$

$x^2 + y^2 = 1$ cilindro

(c) $x^2 + (k^2-1)y^2 - k^2 z^2 + 2kxy - 2kyz$

$-4x - 4ky - 2z + 4 = 0$

$$\begin{pmatrix} 1 & k & 0 & -2 \\ k & k^2-1 & -k & -2k \\ 0 & -k & -k^2 & -1 \\ -2 & -2k & -1 & 4 \end{pmatrix}$$

$d_1 > 0$

$d_2 = -1 < 0$

$d_3 = -k^4 + k^2 + k^4 - k^2 = 0$

Viene parab. iperb. (sella) purché $d_4 \neq 0$

Conti ... $d_4 = 1 \Rightarrow$ sempre parab. iperb.

(p) $$\begin{pmatrix} 1 & 1 & 0 & -1 \\ 1 & 2 & k & -1 \\ 0 & k & k^2-1 & 0 \\ -1 & -1 & 0 & 1-k \end{pmatrix}$$

$d_1 > 0 \quad d_2 > 0$

$d_3 = 2k^2 - 2 - k^2 + 1 - k^2 = \underline{-1} < 0$

$$d_4 = \det \begin{pmatrix} 0 & 0 & 0 & -k \\ 0 & 1 & k & -k \\ 0 & k & k^2-1 & 0 \\ -1 & -1 & 0 & 1-k \end{pmatrix} = -k \det \begin{pmatrix} 1 & k \\ k & k^2-1 \end{pmatrix} = k$$

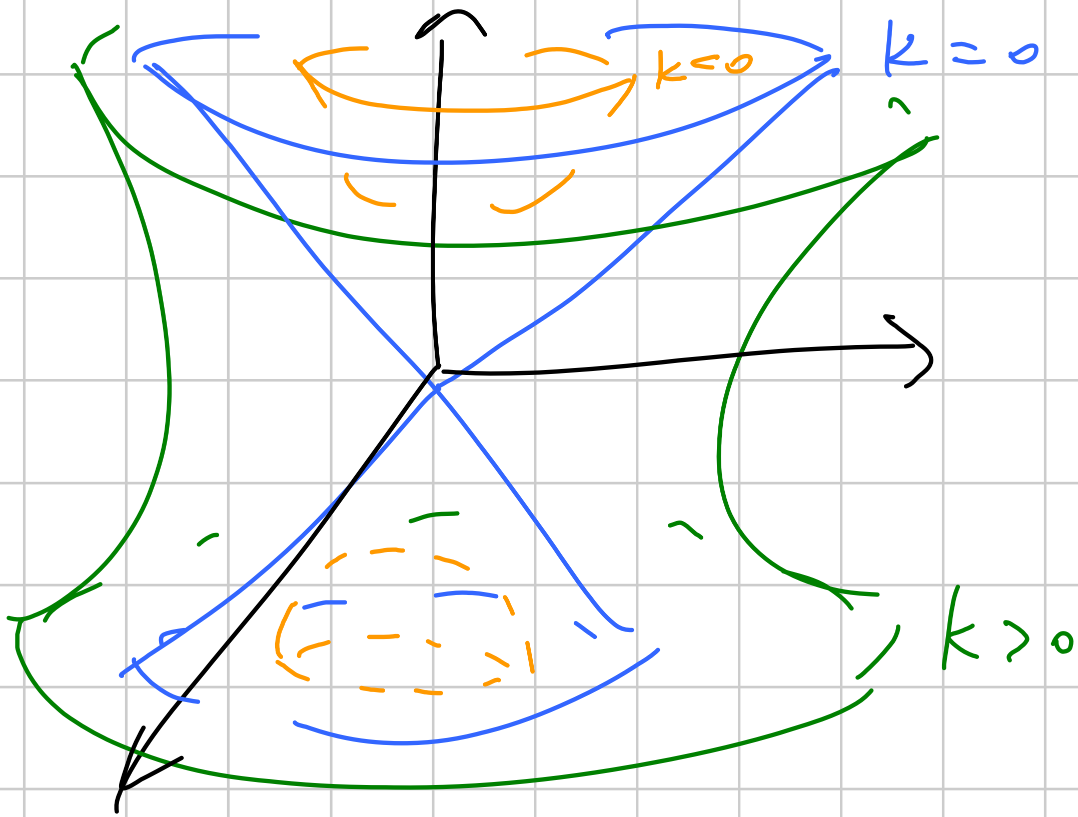
$k>0$ : Autoval : $+ + - -$ $\quad x^2+y^2=1+z^2$

iperb. 1 falda

$k<0$ Autoval $+ + - +$ $\quad x^2+y^2+1=z^2$

iperb. 2 falde

$k=0$ deg : Autoval $+ + - 0$ $\quad z^2 = x^2+y^2$ cono


k<0
k=0
k>0

(q) $d_1 > 0$ $d_2 = k$ $d_3 = k$ $d_4 = -k$

$k > 0$ $(+++)-$ ellissoide

$k < 0$ $(+-+)-$ $x^2 + z^2 = 1 + y^2$

iperb. 1 falda (iperb)

(r) $x^2 + (k^2+k)y^2 + z^2 + 2kxy + 2kyz$

$+4x + 4ky - 2z + 4 - 2k = 0$ —

$$\begin{pmatrix} \boxed{1} & k & \boxed{0} & 2 \\ k & k^2+k & k & 2k \\ \boxed{0} & k & \boxed{1} & -1 \\ 2 & 2k & -1 & 4-2k \end{pmatrix}$$

$d_1 > 0$

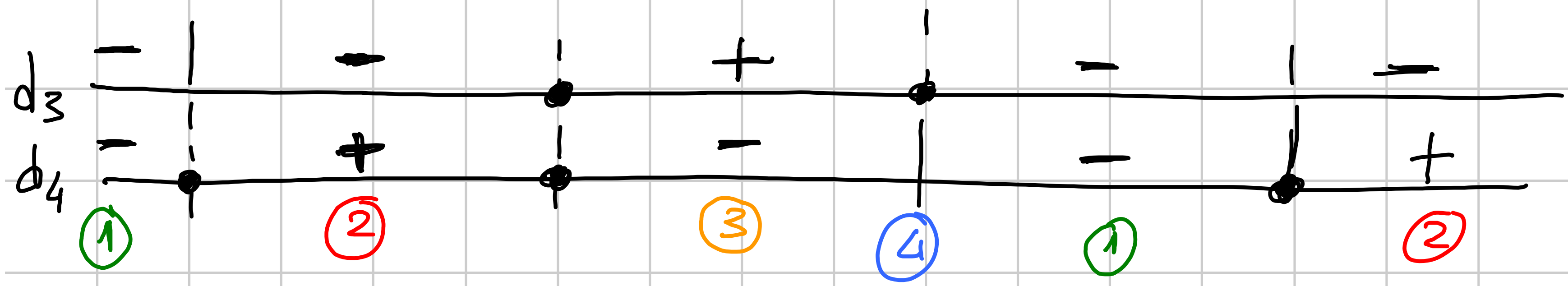
$d_2 > 0$ (scambio $y \leftrightarrow z$)

$d_3 = k(1-k) \qquad (k = 0, 1)$

$d_4 = k(2k^2 - 2k - 1)$

$\left( k = 0, \ k = \frac{1}{2}(1 \pm \sqrt{3}) \right)$

$\frac{1}{2}(1-\sqrt{3}) \qquad 0 \qquad 1 \qquad \frac{1}{2}(1+\sqrt{3})$




1. Autoval: $(+ + -)+$ $z^2 = 1 + x^2 + y^2$ iperb. 2 falde
2. Autoval: $(+ + -)-$ $x^2 + y^2 = 1 + z^2$ iperb. 1 falda
3. Autoval $(+ + +)-$ ellissoide
4. Parab. ell

Degenere per $\underbrace{k=0}_{\text{retta}}$, $\underbrace{k = \frac{1}{2}(1 \pm \sqrt{3})}_{\text{cono}}$

Foglio 14/5 primo (cap. 14) -

(1) $\alpha : [0, 2\pi] \to \mathbb{R}^3 \quad \alpha(t) = \begin{pmatrix} \sin(t) \\ \cos(t) \\ t^2 \end{pmatrix}$

$\beta : [0, \pi] \to \mathbb{R}^3 \quad \beta(t) = \begin{pmatrix} \sin(2t) \\ \cos(2t) \\ 4t^2 \end{pmatrix}$

Noto: $\beta(t) = \alpha(2t)$ cioe'

$\beta = \alpha \circ \sigma \qquad \sigma : [0, \pi] \to [0, 2\pi]$

$\sigma(t) = 2t$

$\beta$ ottenuta da $\alpha$ per cambio parametrizzazione

$\Longrightarrow L(\beta) = L(\alpha)$ _

$$\alpha(t) = \begin{pmatrix} \sin t \\ \cos t \\ t^2 \end{pmatrix}, \quad \alpha'(t) = \begin{pmatrix} -\cos t \\ \sin t \\ 2t \end{pmatrix}$$

$$L(\alpha) = \int_0^{2\pi} \|\alpha'(t)\| \, dt = \int_0^{2\pi} \sqrt{1+4t^2} \, dt$$

ATTENZIONE

$\int_\alpha ?$

$\rightarrow$ ? una funzione scalare (non ci sono $dx, dy \dots$)

$\Downarrow$

CI VUOLE $\|\alpha'(t)\|$

$\rightarrow$ ? una 1-forma (ci sono $dx, dy, \dots$)

CI VOGLIONO

$X'(t), Y'(t) \dots$

② $\int_\alpha \sqrt{1+x^2+y^2}$ $\quad \alpha(t) = \begin{pmatrix} t\cdot\cos t \\ t\cdot\sin t \end{pmatrix}$ $\quad t \in [0,1]$

$$\alpha'(t) = \begin{pmatrix} \cos t - t \sin t \\ \sin t + t \cos t \end{pmatrix} \quad \|\alpha'(t)\| = \sqrt{1+t^2}$$

$$\int_\alpha \sqrt{1+x^2+y^2} = \int_0^1 \underbrace{\sqrt{1+t^2}}_{\substack{\| \\ \sqrt{1+x^2+y^2}}} \cdot \underbrace{\sqrt{1+t^2}}_{\substack{\| \\ \|\alpha'(t)\|}} \, dt$$

$$= \int_0^1 (1+t^2)\, dt = t + \frac{1}{3} t^3 \Big|_0^1 = \frac{4}{3}.$$

③ $\int_\alpha xy^2 \qquad \alpha : [0, \frac{\pi}{2}] \to \mathbb{R}^2 \qquad \alpha(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$

$\|\alpha'(t)\| \equiv 1$

$$\int_0^{\pi/2} \cos t \cdot \sin^2 t \cdot 1 \cdot dt = \frac{1}{3} \sin^3 t \Big|_0^{\pi/2} = \frac{1}{3}$$

④ $\int_\alpha \sqrt{12 + x} \qquad \alpha(t) = \begin{pmatrix} 3t^2 \\ 1 + t^3 \end{pmatrix}$ su $[0,1]$ –

$$\alpha'(t) = \begin{pmatrix} 6t \\ 3t^2 \end{pmatrix} = 3t \begin{pmatrix} 2 \\ t \end{pmatrix}$$

$$\|\alpha'(t)\| = 3t \sqrt{4 + t^2}$$

$$\int_\alpha \sqrt{12+x} = \int_0^1 \sqrt{12+3t^2} \cdot 3t\sqrt{4+t^2}\, dt$$

$$= 3\sqrt{3} \int_0^1 t\,(4+t^2)\,dt = 3\sqrt{3}\left(2t^2+\frac{1}{4}t^4\right)\Big|_0^1$$

$$= 3\sqrt{3}\left(2+\frac{1}{4}\right) = \frac{27}{4}\sqrt{3}$$

5 $\int_\alpha \sqrt{1+y^2}$ $\quad \alpha(t) = \begin{pmatrix} t \\ e^t \end{pmatrix}$ $\quad t \in [0,1]$

$$= \int_0^1 \sqrt{1+e^{2t}} \cdot \sqrt{1+e^{2t}}\, dt$$

$$= \int_0^1 \left(1+e^{2t}\right) dt = \left(t + \frac{1}{2} e^{2t}\right)\Big|_0^1$$

$$= 1 + \frac{1}{2} e^2 - \frac{1}{2} = \frac{1}{2}\left(e^2 - 1\right)$$

⑥ $\int_\alpha x \qquad \alpha(t) = \begin{pmatrix} t \\ t^2 \end{pmatrix}$ on $[0,1]$

$$= \int_0^1 t \cdot \sqrt{1+4t^2}\, dt = \frac{2}{3} \cdot \frac{1}{8} \left(1+4t^2\right)^{\frac{3}{2}} \Big|_0^1$$

$$= \frac{1}{12} \left(5\sqrt{5} - 1\right)$$

⑦ $\int_\alpha xy \qquad \alpha(t) = \begin{pmatrix} \sin t \\ 2\cos t \end{pmatrix}$ an $[0, \pi/2]$.

$$= \int_0^{\pi/2} \sin t \cdot 2\cos t \cdot \sqrt{\cos^2 t + 4\sin^2 t}\, dt$$

$$= \int_0^{\pi/2} 2\sin t\cos t\sqrt{1+3\sin^2 t}\,dt$$

$$= \frac{2}{3}\cdot\frac{1}{3}\cdot\left(1+3\sin^2 t\right)^{3/2}\Big|_0^{\pi/2}$$

$$= \frac{2}{9}\cdot(8-1) = \frac{14}{9}$$

⑧ $C_1 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : \log(1 + x\cos y) + \cos x = 1 \right\}$

Provare che $\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in C_1$; $C_1$ curva vicino a $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ e trovare vettore tg

Dini: $C_1 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : f(x,y) = 0 \right\}$

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in C_1, \quad \operatorname{grad}_{\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}} f \neq 0$$

$\Longrightarrow C_1$ è curva vicino a $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ con tangente $\left( \operatorname{grad}_{\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}} f \right)^{\perp}$

Noi: $f(x,y) = \log(1 + x\cos y) + \cos x - 1$

$f(0,0) = \log(1+0) + \cos 0 - 1 = 0$ ✓

$$\frac{\partial f}{\partial x} = \frac{\cos y}{1 + x\cos y} - \sin x \qquad \frac{\partial f}{\partial x}\begin{pmatrix}0\\0\end{pmatrix} = 1$$

$$\frac{\partial f}{\partial y} = \frac{-x\sin y}{1 + x\cos y} \qquad \frac{\partial f}{\partial y}\begin{pmatrix}0\\0\end{pmatrix} = 0$$

Ok: è curva con // l'asse y.

9 $\quad C_1 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : 2xy^2 + 3x^2y = 1 \right\}$

Esistono punti in cui il vettore $\begin{pmatrix} -1 \\ 4 \end{pmatrix}$ è tg. a $C_1$?

$$\begin{cases} 2xy^2 + 3x^2y = 1 \\ \begin{pmatrix} -1 \\ 4 \end{pmatrix} \perp \begin{pmatrix} 2y^2 + 6xy \\ 4xy + 3x^2 \end{pmatrix} \end{cases}$$

$$\begin{cases} 2xy^2 + 3x^2y = 1 \\ -\cancel{2}(y^2 + 3xy) + \cancel{2} \cdot 2\,(4xy + 3x^2) = 0 \end{cases}$$

$$\begin{cases} 2xy^2+3x^2y=1 \\ 6x^2+5xy-y^2=0 \end{cases} \qquad \begin{cases} 2xy^2+3x^2y=1 \\ (6x-y)(x+y)=0 \end{cases}$$

$$\begin{cases} y=-x \\ 2x^3-3x^3=1 \end{cases} \quad \vee \quad \begin{cases} y=6x \\ 72x^3+18x^3=1 \end{cases}$$

$$\begin{cases} x=-1 \\ y=1 \end{cases} \quad \vee \quad \begin{cases} x=90^{-1/3} \\ y=6\cdot 90^{-1/3} \end{cases}$$

$$\begin{pmatrix} -1 \\ -1 \end{pmatrix} \qquad 90^{-1/3}\cdot\begin{pmatrix} 1 \\ 6 \end{pmatrix}$$

(10) $\alpha(t) = \left( \sin t ,\ \cos t + \log\left(\tan \frac{t}{2}\right) \right) \qquad t \in \left[\frac{\pi}{2}, \pi\right)$

Provare che ha lunghezza infinita e trovare $\tau : [0, +\infty) \to (\pi/2, \pi)$ t.c. $\alpha \circ \tau$ sia in p.d'a.

Lunghezza di $\alpha$ ristretta a $\left[\frac{\pi}{2}, t\right]$

è $\sigma(t) = \displaystyle\int_{\pi/2}^{t} \|\alpha'(u)\|\, du$

e il cambio di par $\tau$ cercato è l'inverso di $\sigma$.

$$\alpha(t) = \begin{pmatrix} \sin t \\ \cos t + \log\left(\tan \frac{t}{2}\right) \end{pmatrix}$$

$$\alpha'(t) = \begin{pmatrix} \cos t \\ -\sin t + \frac{1}{\tan \frac{t}{2}} \cdot \left(1 + \tan^2 \frac{t}{2}\right) \cdot \frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \cos t \\ \frac{1 + \tan^2 t/2}{2 \tan t/2} - \sin t \end{pmatrix} = \begin{pmatrix} \cos t \\ \frac{1}{\sin t} - \sin t \end{pmatrix}$$

$$\|\alpha'(t)\| = \sqrt{\cos^2 t + \frac{1}{\sin^2 t} - 2 + \sin^2 t}$$

$$= \sqrt{\frac{1}{\sin^2 t} - 1} = \sqrt{\frac{\cos^2 t}{\sin^2 t}} = -\frac{\cos t}{\sin t} -$$

$$\sigma(t) = \int_{\pi/2}^{t} \|\alpha'(u)\| du = \int_{\pi/2}^{t} -\frac{\cos u}{\sin u} du$$

$$= -\log \sin u \Big|_{u=\pi/2}^{u=t} = -\log \sin t$$

$\sigma(t) = -\log(\sin t)$ ; $\tau(s) = \sigma^{-1}(s)$

$s = -\log(\sin t)$ $\quad \sin t = e^{-s}$

$(\pi/2 \leq t < \pi)$ $\quad \tau(s) = \pi - \arcsin(e^{-s})$ _

Calcolare curvatura in ogni punto _

( In generale: $\quad \varkappa = \dfrac{\det(\alpha', \alpha'')}{\|\alpha'\|^3}$

$$\alpha'(t) = \begin{pmatrix} \cos t \\ \frac{1}{\sin t} - \sin t \end{pmatrix} \qquad \alpha'' = \begin{pmatrix} -\sin t \\ -\frac{\cos t}{\sin^2 t} - \cos t \end{pmatrix}$$

$$\mathcal{K}(t) = \frac{(\text{conti})}{-\cot g^3 t} = \frac{-\cot g^2 t}{-\cot g^3 t} = tg\, t.$$

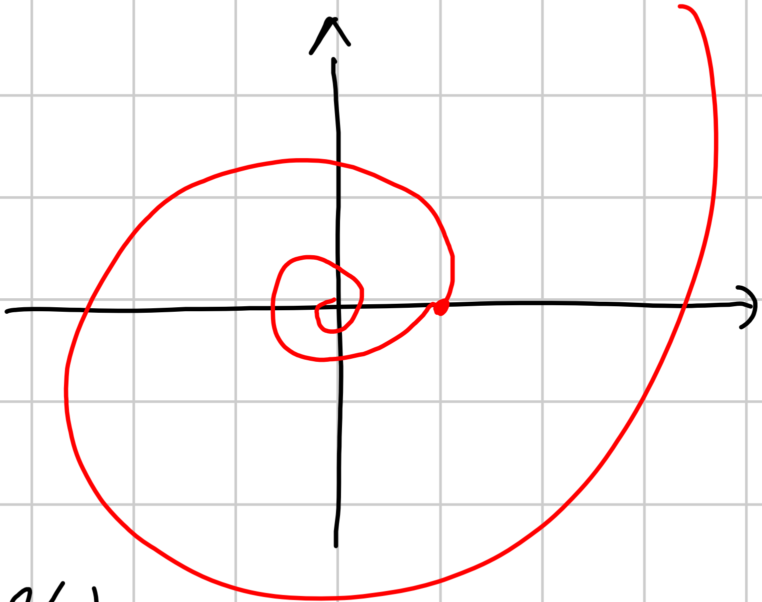
(11) Considerare il luogo in $\mathbb{R}^2$ di equaz.
$\rho = r \cdot e^{\lambda\vartheta}$ rispetto alle coord. polari $\rho, \vartheta$
($\lambda, r > 0$ fissati) _ [Spirale logaritmica]

•) Parametrizzarla con

$\alpha : (-\infty, \infty) \to \mathbb{R}^2$ , $\alpha(0) = \begin{pmatrix} \pi \\ 0 \end{pmatrix}$

$$\alpha(t) = \pi \cdot e^{\lambda t} \cdot \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$$

(ho usato $t = \vartheta$ come parametro, da cui $\rho = \pi \cdot e^{\lambda t}$).



•) trovare $\tau$ t.c. $\alpha \circ \tau$ sia in p.d'a.

$$\tau = \sigma^{-1} \quad \text{dove} \quad \sigma(t) = \int_0^t \| \alpha'(u) \| \, du$$

$$\alpha(t) = r \cdot \begin{pmatrix} e^{\lambda t} \cos t \\ e^{\lambda t} \sin t \end{pmatrix} \qquad \alpha'(t) = r \begin{pmatrix} \lambda e^{\lambda t} \cos t - e^{\lambda t} \sin t \\ \lambda e^{\lambda t} \sin t + e^{\lambda t} \cos t \end{pmatrix}$$

$$\|\alpha'(t)\| = r \cdot e^{\lambda t} \cdot \sqrt{1+\lambda^2}$$

$$\sigma(t) = \int_0^t r\sqrt{1+\lambda^2} \cdot e^{\lambda u}\, du = r\sqrt{1+\lambda^2} \cdot \frac{1}{\lambda} e^{\lambda u} \Big|_{u=0}^{u=t}$$

$$= \frac{r}{\lambda}\sqrt{1+\lambda^2} \cdot e^{(\lambda t - 1)}$$

$$\frac{r}{\lambda}\sqrt{1+\lambda^2} \cdot e^{\lambda t - 1} = 5$$

$$e^{\lambda t - 1} = \frac{\lambda s}{r\sqrt{1+\lambda^2}} \qquad \lambda t - 1 = \log \frac{\lambda s}{r\sqrt{1+\lambda^2}}$$

$$t = \underbrace{\frac{1}{\lambda}\left(1 + \log \frac{\lambda s}{r\sqrt{1+\lambda^2}}\right)}_{\tau(s)} .$$

•) Curvature

$$\alpha'(t) = r \begin{pmatrix} \lambda e^{\lambda t} \cos t - e^{\lambda t} \sin t \\ \lambda e^{\lambda t} \sin t + e^{\lambda t} \cos t \end{pmatrix}$$

$$\alpha''(t) = \pi e^{\lambda t} \begin{pmatrix} (\lambda^2 - 1)\cos t - 2\lambda \sin t \\ (\lambda^2 - 1)\sin t + 2\lambda \cos t \end{pmatrix}$$

CONTI

$$\kappa(t) = \frac{\det(\alpha', \alpha'')}{\|\alpha'\|^3} = \frac{1}{\pi\sqrt{1+\lambda^2}\cdot e^{\lambda t}}.$$