Esercizi del 14/5/2014 (primo foglio)

13. Se $v = 0$ l'uguaglianza è ovvia.

Se $w = \lambda v$, si verifica immediatamente che ambo i membri sono nulli. Se $w \neq \lambda v$, la terna $(v, v \wedge w, (v \wedge w) \wedge v)$ è una base ortogonale, quindi

$$w = a\, v + b\, v \wedge w + c\,(v \wedge w) \wedge v\,, \quad a, b, c \in \mathbb{R}$$

Chiaramente:

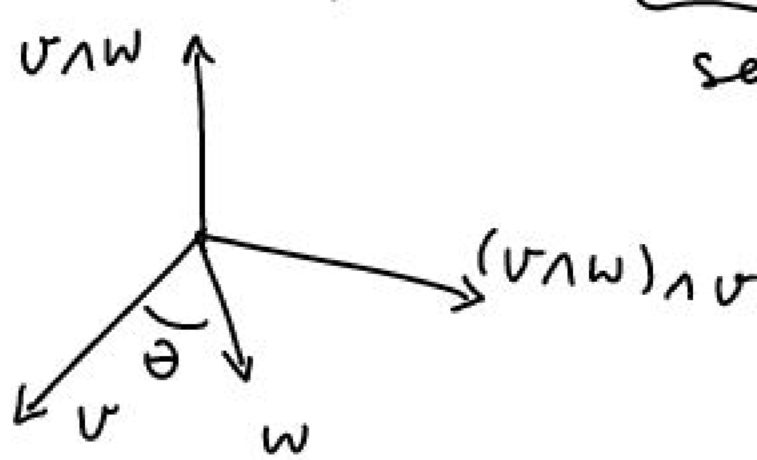
$$\langle w \mid v \rangle = a \cdot \|v\|^2,$$

$$0 = \langle w \mid v \wedge w \rangle = b\, \|v \wedge w\|^2,$$

$$\langle w, (v \wedge w) \wedge v \rangle = C \cdot \|(v \wedge w) \wedge v\|^{\cancel{2}}$$

$$\|$$

$$\|w\| \cdot \|\cancel{(v \wedge w) \wedge v}\| \underbrace{\cos(\pi/2 - \theta)}_{\text{sen}\,\theta} \quad \Updownarrow$$



$$\|\cancel{w}\| \cdot \cancel{\text{sen}\,\theta} =$$

$$= C \cdot \|v\|^2 \cdot \|\cancel{w}\| \cdot \cancel{\text{sen}\,\theta}$$

$$\Rightarrow \quad a = \langle w \mid v \rangle / \|v\|^2\,, \quad b = 0\,,$$

$$c = 1/\|v\|^2\,,$$ e l'uguaglianza cercata segue moltiplicando ambo i membri per $\|v\|^2$.

14.

$$\alpha'(t) \wedge \alpha''(t) = \begin{vmatrix} e_1 & e_2 & e_3 \\ -\operatorname{sen} t & -\cos t & -2\operatorname{sen}(2t) \\ -\cos t & \operatorname{sen} t & -4\cos(2t) \end{vmatrix} =$$

$$= \begin{pmatrix} 4\cos t(\cos^2 t - \operatorname{sen}^2 t) + 4\operatorname{sen}^2 t \cos t \\ -4\operatorname{sen} t(\cos^2 t - \operatorname{sen}^2 t) - 4\operatorname{sen} t \cos^2 t \\ -\operatorname{sen}^2 t - \cos^2 t \end{pmatrix} = \begin{pmatrix} 4\cos^3 t \\ 4\operatorname{sen}^3 t \\ -1 \end{pmatrix}$$

$$k(t) = \frac{\|\alpha'(t) \wedge \alpha''(t)\|}{\|\alpha'(t)\|^3} = \frac{(16(\cos^6 t + \operatorname{sen}^6 t) + 1)^{1/2}}{(1 + 4\operatorname{sen}^2(2t))^{3/2}}$$

$$\tau(s) = \frac{\langle \alpha'(t) \wedge \alpha''(t) \mid \alpha'''(t) \rangle}{\|\alpha'(t) \wedge \alpha''(t)\|^2} =$$

$$= \frac{\begin{vmatrix} \operatorname{sen} t & \cos t & 8\operatorname{sen} 2t \\ -\operatorname{sen} t & -\cos t & -2\operatorname{sen}(2t) \\ -\cos t & \operatorname{sen} t & -4\cos(2t) \end{vmatrix}}{16(\cos^6 t + \operatorname{sen}^6 t) + 1}$$

15. (a) $t = \alpha'(s)/\|\alpha'(s)\| =$

$$= \begin{pmatrix} \cos(s) - s\operatorname{sen}(s) \\ 2s \\ 1/(1+s) \end{pmatrix} \Big/ \|\alpha'(s)\| \qquad t(s_0 = 0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\alpha'(0) \wedge \alpha''(0) = \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{vmatrix} = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$$

$$\Rightarrow b(0) = \alpha'(0) \wedge \alpha''(0) / \|\alpha'(0) \wedge \alpha''(0)\| = \frac{1}{3} \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$$

$$n(0) = b(0) \wedge t(0) = \begin{vmatrix} e_1 & e_2 & e_3 \\ -2/3 & 1/3 & 2/3 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{vmatrix} = \frac{1}{3\sqrt{2}} \begin{pmatrix} 1 \\ 4 \\ -\underline{1} \end{pmatrix}$$

$$k(0) = \|\alpha'(0) \wedge \alpha''(0)\| / \|\alpha'(0)\|^3 = \frac{3}{2\sqrt{2}}$$

Calcolando: $\alpha'''(0) = \begin{pmatrix} -3 \\ 0 \\ 2 \end{pmatrix}$

$$\Rightarrow \tau(0) = \frac{\langle \alpha'(0) \wedge \alpha''(0) \mid \alpha'''(0) \rangle}{\|\alpha'(0) \wedge \alpha''(0)\|^2} =$$

$$= \begin{vmatrix} -3 & 0 & 2 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 2 & -1 \end{vmatrix} \cdot \frac{\sqrt{2}}{3} = \frac{\sqrt{2}}{3} \cdot (-2) \cdot \left(-3/\sqrt{2} - \sqrt{2}\right) = \frac{10}{9}$$

(b) $\alpha'(0) = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$ $\quad t(0) = \alpha'(0) / \|\alpha'(0)\| = \frac{1}{3} \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$

$$\alpha'(0) \wedge \alpha''(0) = \begin{vmatrix} e_1 & e_2 & e_3 \\ 2 & 2 & 1 \\ 2 & 4 & 0 \end{vmatrix} = \begin{pmatrix} -4 \\ 2 \\ 4 \end{pmatrix}$$

$$\Rightarrow b(0) = \frac{1}{3} \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}, \quad n(0) = b(0) \wedge t(0) = \frac{1}{3} \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix}$$

$$k(0) = \frac{\|\alpha'(0) \wedge \alpha''(0)\|}{\|\alpha'(0)\|^3} = \frac{2 \cdot 3}{3^3} = \frac{2}{9} \qquad \alpha'''(0) = \begin{pmatrix} -6 \\ 8 \\ -1 \end{pmatrix}$$

$$\tau(0) = \frac{\langle \alpha'(0) \wedge \alpha''(0) \mid \alpha'''(0) \rangle}{\|\alpha'(0) \wedge \alpha''(0)\|^2} = \frac{\begin{vmatrix} -6 & 8 & -1 \\ 2 & 2 & 1 \\ 2 & 4 & 0 \end{vmatrix}}{2^2 \cdot 3^2} = 1$$

Esercizi del 14/5/2014 (secondo foglio)

$\boxed{1.}$ $\displaystyle\int_\alpha 2xy^2\,dx + 3x^2y\,dy =$

$$= \int_0^1 2\cdot(1-2t)(1+t^2)^2(-2)\,dt + 3(1-2t)^2(1+t^2)\,2t\,dt =$$

$$= \int_0^1 (-4+14t-32t^2+46t^3-28t^4+32t^5)\,dt$$

$$= \left[-4t+7t^2-\frac{32}{3}t^3+\frac{23}{2}t^4-\frac{28}{5}t^5+\frac{16}{3}t^6\right]_0^1 =$$

$= 107/30$

$\boxed{2.}$ $\displaystyle\int_\alpha x\,dy - y\,dx = \int_0^\theta \cos^2 t\,dt + \operatorname{sen}^2 t\,dt = \theta$

$\boxed{3.}$ $$e^{x+\cos y}\cdot\left(\log\left(\frac{\cos x}{y}\right)+\frac{y}{\cos x}(-\operatorname{sen} x)\right)dx$$

$$+\,e^{x+\cos y}\cdot\left(-\operatorname{sen}(y)\cdot\log\left(\frac{\cos x}{y}\right)-\frac{\cos x}{y^{\cancel{2}}}\cdot\frac{\cancel{y}}{\cancel{\cos x}}\right)dy$$

$\boxed{4.}$ $\left[-x\,e^{xy}\cos(x)+e^{x-y}\left(\operatorname{sen}(xy)+x\cos(xy)\right)\right]dx\,dy$

$\boxed{5.}$ $\Omega = \mathbb{R}^2$, che è semplicemente connesso.

$d\omega = -5\cdot(-6x^2y+6x^2y)\,dx\,dy = 0$ ✓

$$\begin{cases} U_x = -15x^2y^2 \\ U_y = -10x^3y \end{cases} \Rightarrow U(x,y) = -5x^3y^2$$

(b) $\Omega = (0, +\infty) \times \mathbb{R}$ è s.c.

$$\begin{cases} U_x = e^y/x \\ U_y = e^y \log(x) \end{cases} \Rightarrow U(x,y) = e^y \log(x)$$

(c) $d\omega = \big(-\operatorname{sen}(xy) - xy\cos(xy) + \operatorname{sen}(xy) + {}$
$+ xy\cos(xy)\big)\, dx\, dy = 0$

$$\begin{cases} U_x = 2x + y\operatorname{sen}(xy) \\ U_y = x\operatorname{sen}(xy) - 1 \end{cases} \Rightarrow U(x,y) = x^2 - y - \cos(xy)$$

(d) $d\omega = \left(-\frac{2y}{(x-y^2)^2} + \frac{2y}{(x-y^2)^2}\right) dx\, dy = 0$

$$\begin{cases} U_x = \dfrac{1}{x-y^2} \\ U_y = \dfrac{-2y}{x-y^2} \end{cases} \quad U(x,y) = \log(x-y^2)$$

$\Omega = \{(x,y) \mid x \neq y^2\}$ è s.c.

(e) $\Omega = \mathbb{R}^2$ è s.c.

$$d\omega = (-\operatorname{sen} x + \operatorname{sen} x)\, dx\, dy = 0$$

$$\begin{cases} U_x = y \operatorname{sen} x \\ U_y = \cos y - \cos x \end{cases} \Rightarrow U(x,y) = -y\cos x + \operatorname{sen} y$$

$\boxed{6.}$ $d\omega = \partial f/\partial x \, dx\, dy = 0 \iff \dfrac{\partial f}{\partial x} \equiv 0$

$\iff f(x,y) = g(y)$

$\boxed{7.}$ Essendo definita su $\mathbb{R}^2$, che è sempl. connesso, la forma è esatta $\iff$ è chiusa, ovvero $\iff$

$$(-2ky + y)\, dx\, dy \equiv 0,$$

che chiaramente succede $\iff$ $k = 1/2$.

$\boxed{8.}$ $\text{Area} = \int_\alpha y\, dx = \int_0^{2\pi} (1-\cos t)(1-\cos t)\, dt =$

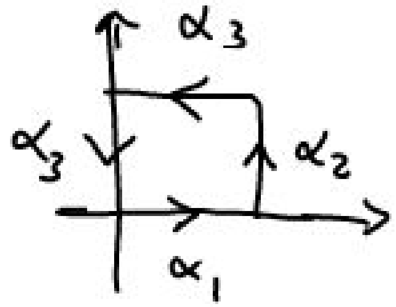
$$= \int_0^{2\pi} (1 + \cos^2 t - 2\cos t)\, dt = \int_0^{\pi} 1\, dt + \int_0^{2\pi} \cos^2 t\, dt$$

$$-2\int_0^{2\pi} \cos t\, dt = 2\pi + \pi + 0 = 3\pi$$

$\boxed{9.}$ ∂A = unione di curve chiuse, df esatta

$$\Rightarrow \int_{\partial A} df = 0$$

10.



$$\int_{\partial Q}\omega = \sum_{i=1}^{4} \int_{\alpha_i}\omega =$$

$$= \int_{\alpha_1} x^2\,dx + \int_{\alpha_2}(2y+e^y)\,dy + \int_{\alpha_3}(x^2+1)\,dx + \int_{\alpha_4} e^y\,dy$$

$$= \frac{1}{3}t^3\Big|_0^1 + t^2\Big|_0^1 + e^t\Big|_0^1 - 2t\Big|_0^1 + t^2\Big|_0^1 - \frac{1}{3}t^3\Big|_0^1 + e^{1-t}\Big|_0^1$$

$$= \cancel{\frac{1}{3}} + \cancel{1} + \cancel{e} - \cancel{1} - \cancel{2} + \cancel{1} - \cancel{\frac{1}{3}} + \cancel{1} - \cancel{e} = 0$$