Geometria 19/3/19

In $\mathbb{R}^3$:

$$\text{rette (per } 0) \xleftrightarrow{\perp} \text{piani (per } 0)$$

$$r \longmapsto r^\perp$$

$$P^\perp \longleftarrow\!\shortmid P$$

Visto: eq. cart. per piano $P \iff$ eq. param. retta $P^\perp$

(eq. param. retta $r \iff$ eq. cart per piano $r^\perp$)

Eq. cart. retta r:

$$r = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : \begin{cases} a_1 x + b_1 y + c_1 z = 0 \\ a_2 x + b_2 y + c_2 z = 0 \end{cases} \right\}$$

$$= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : \begin{array}{l} \left\langle \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} \middle| \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right\rangle = 0 \\ \left\langle \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} \middle| \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right\rangle = 0 \end{array} \right\}$$

$$= \left\{ v : v \perp \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix}, \ v \perp \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} \right\}$$

$$= \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix}^\perp \cap \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}^\perp$$

$$= \mathrm{Span}\left(\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix}, \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} \right)^\perp$$

cioè $\quad r^\perp = \mathrm{Span}\left(\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} \right)$. Allora:

eq. cart. retta r $\longleftrightarrow$ eq. par. piano $r^{\perp}$

(eq. par. piano $P \longleftrightarrow$ eq. cart. retta $P^{\perp}$)

Fisso piano P retta r ortog. $P = r^{\perp}$, $r = P^{\perp}$:

$$\begin{tikzcd} \text{eq. par. piano } P \arrow[r, squiggly, "\text{det } 2\times 2"] \arrow[d, leftrightarrow] \arrow[dr] & \text{eq. cart. piano } P \arrow[d, leftrightarrow] \\ \text{eq. cart. retta } r \arrow[r, squiggly, "\text{det } 2\times 2"'] & \text{eq. param. retta } r \end{tikzcd}$$

regola dei det 2×2 :

dati due vettori v, w fornisce un generatore della retta ortogonale al piano generato da v e w

$v = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ $w = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$; def: lo chiamo prodotto vettoriale

$$v \wedge w = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \wedge \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} b\gamma - c\beta \\ -(a\gamma - c\alpha) \\ a\beta - b\alpha \end{pmatrix}$$

anche $\times$

Oss: se v, w sono proporzionali non generano un piano e $v \wedge w = 0$.

Altrimenti :

- $v \wedge w$ genera $(\mathrm{Span}(v,w))^\perp$ — direzione
- $\|v \wedge w\|$ = area parallelogramma di lati v, w — intensità
- $v, w, v \wedge w$ soddisfano regola mano destra — verso

v: pollice, w: indice, $v \wedge w$: medio

<u>Oss</u>: grazie all'ultima proprietà ho $\det(v, w, v\wedge w) > 0$
($v, w, v\wedge w$ terna levogira)

<u>Es</u>:
$$\begin{pmatrix} 7 \\ 2 \\ -4 \end{pmatrix} \wedge \begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2\cdot 3 - (-4)\cdot(-1) \\ -(7\cdot 3 - (-4)\cdot 5) \\ 7\cdot(-1) - 2\cdot 5 \end{pmatrix} = \begin{pmatrix} 2 \\ -41 \\ -17 \end{pmatrix}$$

verifica:
$$\left\langle \begin{pmatrix} 7 \\ 2 \\ -4 \end{pmatrix} \middle| \begin{pmatrix} 2 \\ -41 \\ -17 \end{pmatrix} \right\rangle = 14 - 82 + 68 = 0 \quad \checkmark$$
$$\left\langle \begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix} \middle| \begin{pmatrix} 2 \\ -41 \\ -17 \end{pmatrix} \right\rangle = 10 + 41 - 51 = 0 \quad \checkmark$$

$$\det \begin{pmatrix} 7 & 5 & 2 \\ 2 & -1 & -41 \\ -4 & 3 & -17 \end{pmatrix} = \dots > 0.$$

<u>Esercizi</u>:

$\boxed{9.2.1}$ Data $f: V \times V \to \mathbb{R}$ bil. simm. sia
$q(v) = f(v,v)$ (forma quadratica).
Nota q si ricostruisca f.

(b) $V = \mathbb{R}^2 \qquad q(x) = 3x_1^2 + 2x_1x_2 - x_2^2$

pol. omog. di grado 2 nelle coord.

$$f(x,y) = 3x_1y_1 + \cancel{2x_1y_2} - x_2y_2$$

$$x_1y_2 + x_2y_1$$

$$f = \langle \cdot | \cdot \rangle_A \qquad A = \begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix}$$

(c) $\mathbb{R}^3 \qquad q(x) = -2x_1^2 + x_2^2 - 5x_3^2 - x_1x_2 + 3x_1x_3 + 2x_2x_3$

$$f = \langle \cdot | \cdot \rangle_A \qquad A = \begin{pmatrix} -2 & -1/2 & 3/2 \\ -1/2 & 1 & 1 \\ 3/2 & 1 & -5 \end{pmatrix}.$$

(e) $V = M_{3\times 3}(\mathbb{R}) \qquad q(A) = \sum_{j=1}^{3} (A^2)_{j,4-j}$

$$A \cdot A = \begin{pmatrix} \cdot & \cdot & \odot \\ \cdot & \odot + & \cdot \\ \odot + & \cdot & \cdot \end{pmatrix}$$

$$f(A,B) = \sum_{j=1}^{3} \left(\frac{1}{2}(A \cdot B + B \cdot A) \right)_{j,4-j}$$

(f) $V = \mathbb{R}_{\leq 2}[t] \qquad q(p(t)) = p(1) \cdot p(-2)$

$$(a_0 + a_1 + a_2) \cdot (a_0 - 2a_1 + 4a_2)$$

$$f(p(t), r(t)) = \frac{1}{2}\left(p(1)\cdot r(-2) + p(-2)\cdot r(1)\right)$$

$\boxed{9.2.3}$ $\mathbb{R}_{\leq 2}[t]$ $\langle p(t) | q(t)\rangle = p(0)\cdot q(0) + p(1)\cdot q(1) + p(2)\cdot q(2)$

Trovare vettore $\perp$ $\underline{\underline{1+t}}$ e $\underline{\underline{1+t^2}}$ con $\|\cdot\| = \sqrt{5}$.

Perché prod. scal? bil. simm.: ovvio.

Def. pos? $\langle p(t) | p(t)\rangle = p(0)^2 + p(1)^2 + p(2)^2$

certamente sempre ≥ 0; nullo solo se $p(0) = p(1) = p(2) = 0$

$\Longrightarrow$ dato che $\deg(p(t)) \leq 2$ ho $p(t) = 0$.

Cerchiamo $p(t) = a_0 + a_1 t + a_2 t^2$: deve essere

$$\begin{cases} \underbrace{a_0\cdot 1 + (a_0+a_1+a_2)\cdot 2}_{} + \underbrace{(a_0+2a_1+4a_2)}_{}\cdot 3 = 0 \\ \underbrace{a_0\cdot 1 + (a_0+a_1+a_2)\cdot 2}_{} + \underbrace{(a_0+2a_1+4a_2)}_{}\cdot 5 = 0 \end{cases}$$

$$\begin{cases} 3a_0 + 2a_1 + 2a_2 = 0 \\ a_0 + 2a_1 + 4a_2 = 0 \end{cases}$$

$$\begin{cases} a_0 = -2a_1 - 4a_2 \\ -6a_1 - 12a_2 + 2a_1 + 2a_2 = 0 \end{cases}$$

$$\begin{cases} -4a_1 - 10a_2 = 0 \\ a_0 = \dots \end{cases}$$

$a_1 = -5$ $a_2 = +2$

$a_0 = +10 - 8 = 2$

Osservo che $\dim(\mathbb{R}_{\leq 2}[t]) = 3$;
$1+t, 1+t^2$ lin. indip.
$\Longrightarrow$ generano un piano
$\Longrightarrow$ ortog. a loro è retta (dim. 1)
$\Longrightarrow$ basta esibire un generatore di tale retta e poi aggiustare la lunghezza.

$$\Rightarrow \quad (1+t,\, 1+t^2)^{\perp} = \mathrm{Span}\,(2-5t+2t^2)$$

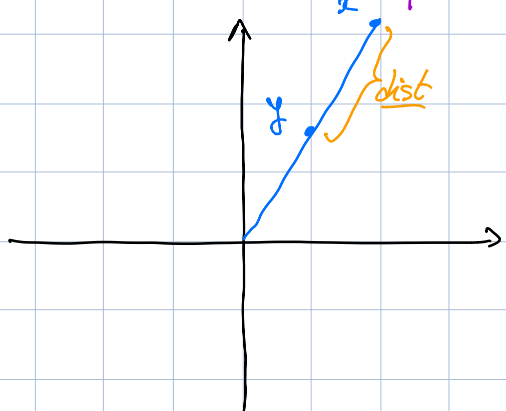
$$\|2-5t+2t^2\|^2 = 2^2 + (-1)^2 + 0^2 = 5$$

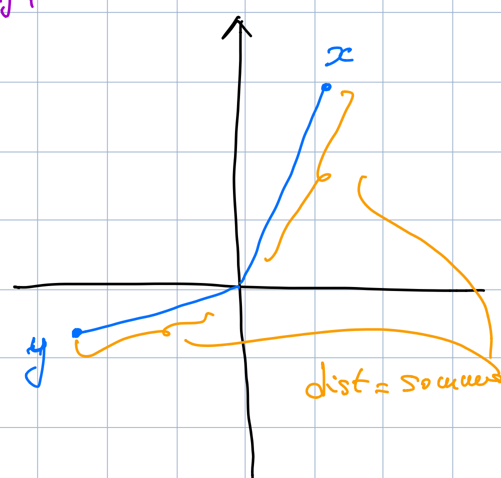
$\Rightarrow$ va bene $2-5t+2t^2$

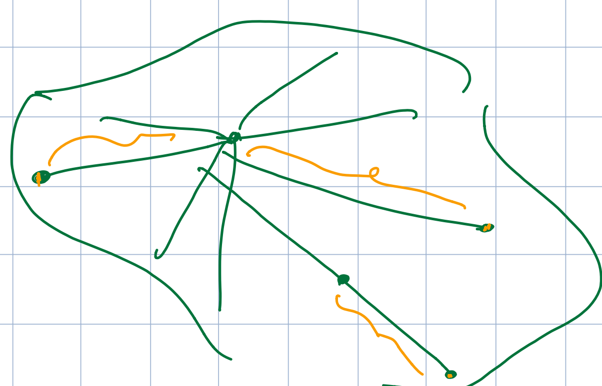
$\boxed{9.2.4}$ Provare che sono distanze su $\mathbb{R}^n$:

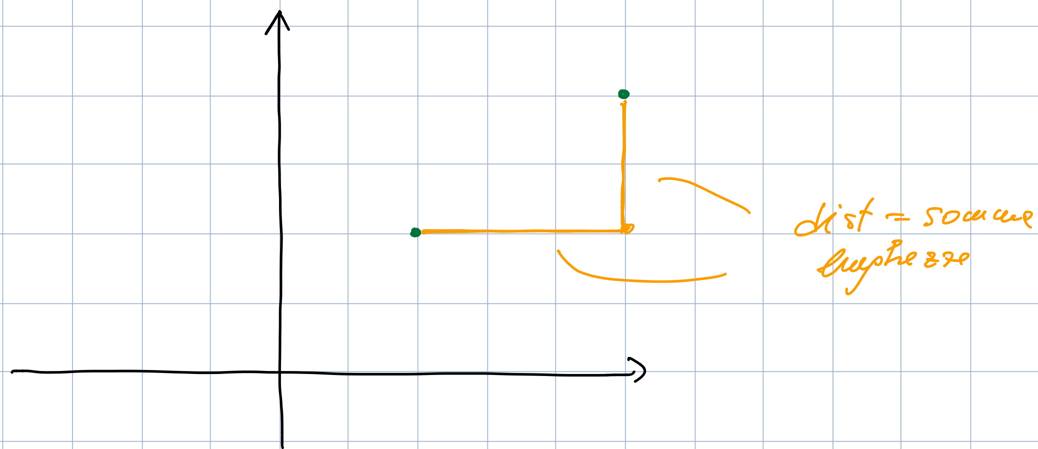
(a) $$SNCF(x,y) = \begin{cases} \|x-y\| & x,y \text{ lin. dip} \\ \|x\|+\|y\| & \text{altrimenti} \end{cases}$$

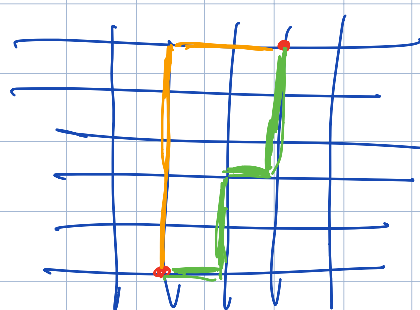
(b) $$NYC(x,y) = \sum_{j=1}^{n} |x_j - y_j|$$











$\Rightarrow$ è una distanza; algebricamente: $NYC(x,y) = \sum_{j=1}^{n} |x_j - y_j|$

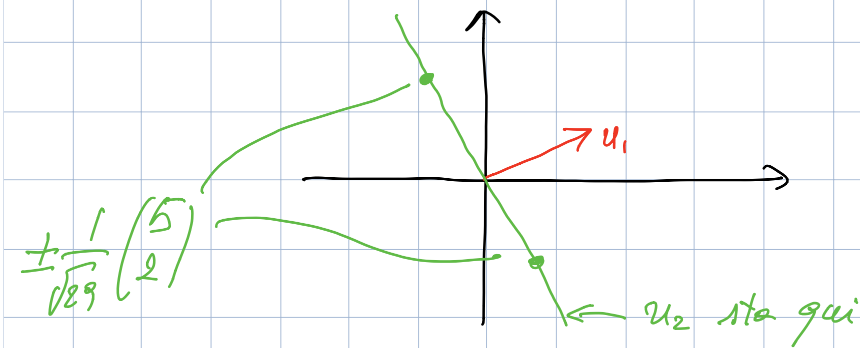
- $NYC(x,y) \geq 0$ nulla solo se $y = x$ ✓
- $NYC(x,y) = NYC(y,x)$ ✓
- disug. triang. : segue da $|x_j - y_j| \leq |x_j - z_j| + |z_j - y_j|$

$\boxed{9.2.5}$ In V con $\langle \cdot | \cdot \rangle$ ortonormalizzare $v_1, \dots, v_k$

(a) $\mathbb{R}^2$, $\langle \cdot | \cdot \rangle_{\mathbb{R}^2}$ $\begin{pmatrix} -2 \\ 5 \end{pmatrix}$ $\begin{pmatrix} -e \\ 1 \end{pmatrix}$

G-S : $v_1,\dots,v_k \rightsquigarrow u_1,\dots,u_k$ ortonormali t.c.

- $\mathrm{Span}(u_1,\dots,u_h) = \mathrm{Span}(v_1,\dots,v_h)$ $h=1\dots k$
- la matrice di cambio da $v_1,\dots,v_h$ a $u_1,\dots,u_h$ ha det > 0.



$u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{29}}\begin{pmatrix}-2\\5\end{pmatrix}$

so che det. matr. cambio A da $\underset{\mathcal{B}}{v_1, v_2}$ a $\underset{\mathcal{B}'}{u_1, u_2}$ è > 0

$\Rightarrow \mathcal{B}' = \mathcal{B}\cdot A \Rightarrow \det(\mathcal{B}') = \det(\mathcal{B})\cdot \det(A)$

$\Rightarrow \det(\mathcal{B}')$ concorde con $\det(\mathcal{B})$.

$$\det(\mathcal{B}) = \det\begin{pmatrix}-2 & -e^2\\ 5 & 1\end{pmatrix} = -2+5e^2 > 0$$

$$u_2 \overset{?}{=} -\frac{1}{\sqrt{29}}\begin{pmatrix}5\\2\end{pmatrix} : \text{sì se } \underbrace{\det\begin{pmatrix}-2 & -5\\ 5 & -2\end{pmatrix}}_{4+25} > 0$$

OK

Per l'ultimo vettore non ho bisogno di fare i calcoli di G-S ma bastano altre considerazioni.

(g) $\mathbb{R}^3$, $\langle \cdot | \cdot \rangle_{\mathbb{R}^3}$ $\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} \begin{pmatrix} -3e \\ 0 \\ 0 \end{pmatrix}$

$$u_1 = \frac{1}{\sqrt{14}} \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$$

$$z_2 = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} - \frac{\left\langle \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} \middle| \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} \right\rangle}{14} \cdot \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -1 \\ 4 \\ 5 \end{pmatrix}$$

è $\perp$ a u_1?

$3 \cdot (-1) + 2 \cdot 4 + (-1) \cdot 5 = 0$ ✓

$$u_2 = \frac{1}{\sqrt{42}} \begin{pmatrix} -1 \\ 4 \\ 5 \end{pmatrix}$$

$$z_3 = \begin{pmatrix} -3e \\ 0 \\ 0 \end{pmatrix} - \frac{1}{14} \left\langle \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} \middle| \begin{pmatrix} -3e \\ 0 \\ 0 \end{pmatrix} \right\rangle \cdot \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$$

$$- \frac{1}{42} \cdot \left\langle \begin{pmatrix} -1 \\ 4 \\ 5 \end{pmatrix} \middle| \begin{pmatrix} -3e \\ 0 \\ 0 \end{pmatrix} \right\rangle \cdot \begin{pmatrix} -1 \\ 4 \\ 5 \end{pmatrix}$$

$= \dots$ calcoli (poi $u_3 = z_3 / \| z_3 \|$).

Invece: voglio u_3 che sia $\perp$ a u_1, u_2 cioè $\perp$ $\mathrm{Span}(u_1, u_2) = \mathrm{Span}(v_1, v_2)$ cioè multiplo di $v_1 \wedge v_2$.

$$v_1 \wedge v_2 = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} \wedge \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 \\ -7 \\ 7 \end{pmatrix} = 7 \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

verifico

$3 - 2 - 1 = 0$ ✓

$-2 - 1 + 3 = 0$ ✓

$$u_3 = \pm \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$ segno t.c. $\det(u_1, u_2, u_3)$ concorde con $\det(v_1, v_2, v_3)$

pos.

anzi segno t.c. $\det(v_1, v_2, u_3)$ concorde con $\det(v_1, v_2, v_3)$.

$$u_3 = +\frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \text{ se } \det \begin{pmatrix} 3 & -2 & 1 \\ 2 & 1 & -1 \\ -1 & 3 & 1 \end{pmatrix}$$

$3-2+6$
$+1+4+9>0$

$$\text{è concorde con } \det \begin{pmatrix} 3 & -2 & -3e \\ 2 & 1 & 0 \\ -1 & 3 & 0 \end{pmatrix} \leftarrow -21e<0$$

prima e unica volta che uso v_3 per trovare u_3

No: quello giusto è $u_3 = -\frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.

(d) $V = \mathbb{R}^2$ $\quad \langle . | . \rangle_{\begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix}}$ $\quad \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$

ortonormale rispetto a $\langle . | . \rangle_{\mathbb{R}^2}$ ma non rispetto a $\langle . | . \rangle_{\begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix}}$

$$\|v_1\|^2 = (1 \ 0) \cdot \begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 5$$

$$u_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Due metodi per trovare u_2:

I) <u>Pochi conti</u> : cerco un generatore unitario di $\begin{pmatrix}1\\0\end{pmatrix}^{\perp}$ t.c. la matr. (v_1, u_2) abbia det concorde con $\det\begin{pmatrix}1&0\\0&1\end{pmatrix} > 0$.

$$\begin{pmatrix}1\\0\end{pmatrix}^{\perp} = \left\{ \begin{pmatrix}x\\y\end{pmatrix} : (x, y)\begin{pmatrix}5 & -2\\-2 & 1\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix} = 0 \right\}$$

$$= \left\{ \begin{pmatrix}x\\y\end{pmatrix} : 5x - 2y = 0 \right\} = \mathrm{Span}\begin{pmatrix}2\\5\end{pmatrix}.$$

$$\left\| \begin{pmatrix}2\\5\end{pmatrix} \right\|^2 = (2\ 5)\underline{\underline{\begin{pmatrix}5 & -2\\-2 & 1\end{pmatrix}}}\begin{pmatrix}2\\5\end{pmatrix}$$

$$\left\| \begin{pmatrix}x\\y\end{pmatrix} \right\|^2 = 5x^2 - 4xy + y^2$$

$$= 5\cdot 25 - 4\cdot 10 + 1 = 86$$

$$\Rightarrow \quad u_2 = \pm \frac{1}{\sqrt{86}}\begin{pmatrix}2\\5\end{pmatrix}$$

$$u_2 = + \frac{1}{\sqrt{86}}\begin{pmatrix}2\\5\end{pmatrix} \quad \text{se} \quad \det\begin{pmatrix}1 & 2\\0 & 5\end{pmatrix} > 0 \; : \; \underline{\text{ok}}$$

II) <u>Più conti</u> :

$$\begin{pmatrix}5 & -2\\-2 & 1\end{pmatrix} \quad u_1 = \frac{1}{\sqrt{5}}\begin{pmatrix}1\\0\end{pmatrix}$$

$$v_2 = \begin{pmatrix}0\\1\end{pmatrix}$$

$$z_2 = \begin{pmatrix}0\\1\end{pmatrix} - \frac{1}{5}\cdot\left(\begin{pmatrix}0\\1\end{pmatrix}\cdot\begin{pmatrix}5 & -2\\-2 & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\right)\cdot\begin{pmatrix}1\\0\end{pmatrix}$$

$$= \begin{pmatrix}0\\1\end{pmatrix} - \frac{1}{5}\cdot(-2)\cdot\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix} + \frac{2}{5}\begin{pmatrix}1\\0\end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix}2\\5\end{pmatrix}$$

$$u_2 = \frac{\binom{2}{5}}{\left\|\binom{2}{5}\right\|} = \frac{1}{\sqrt{86}}\binom{2}{5}.$$

(k) $V = M_{2\times 2}(\mathbb{R})$ $\quad \langle A|B\rangle = tr\left({}^tA \cdot \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix} \cdot B\right)$

$\begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}$ $\begin{pmatrix} -1 & 5 \\ 1 & 2 \end{pmatrix}$

Perché prod. scal.? $\quad$ bil: facile

simm

simm: $\langle B|A\rangle = tr\left({}^tB \cdot \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix} \cdot A\right)$

$= tr\left({}^t\left({}^tB \cdot \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix} \cdot A\right)\right)$

$= tr\left({}^tA \cdot \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix} \cdot B\right)$

$= \langle A|B\rangle$

def. pos: $A = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$

$$\langle A|A\rangle = tr\left(\begin{pmatrix} x & z \\ y & w \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} x & y \\ z & w \end{pmatrix}\right)$$

$$= tr\left(\begin{pmatrix} x & z \\ y & w \end{pmatrix} \cdot \begin{pmatrix} 2x-z & 2y-w \\ -x+3z & -y+3w \end{pmatrix}\right)$$

$$= 2x^2 - xz - zx + 3z^2 + 2y^2 - yw - wy + 3w^2$$

$$= 2x^2 - 2xz + 3z^2 + 2y^2 - 2yw + 3w^2$$

$$= (x-z)^2 + (y-w)^2 + x^2 + 2z^2 + y^2 + 2w^2$$

sempre ≥ 0; nullo solo se $x=y=z=w=0$, cioè $A = 0$.