Esercitazione 26-03

9.3.16 Nello spazio V con il prodotto scalare $\langle \cdot | \cdot \rangle$ assegnato, esibire la proiezione ortogonale p. sul sottospazio W indicato.

$V = \mathbb{R}_{\leq 2}[t]$,

$$\langle p(t) | q(t) \rangle = p(-1) \cdot q(-1) + p(1) \cdot q(1) + p(2) q(2)$$

$W = \mathrm{Span}(t, t^2)$

Osservazione: $\langle \cdot | \cdot \rangle$ è un prodotto scalare. È bilineare e simmetrica.

$\langle p(t) | p(t) \rangle = p(-1)^2 + p(1)^2 + p(2)^2 \geq 0$

$\langle p(t) | p(t) \rangle = 0 \iff p(-1) = p(1) = p(2) = 0 . \; *$

Fatto generale: un polinomio $\underset{\not\equiv 0}{p(t)} \in \mathbb{R}[t]$ di grado n ha al più n radici reali.

$p(t) \in \mathbb{R}_{\leq 2}[t]$, quindi se vale $*$, allora $p(t) = 0$. Quindi $\langle \cdot | \cdot \rangle$ è prodotto scalare.

(Oss., prendendo $V = \mathbb{R}_{\leq 3}[t]$, non era vero che $\langle \cdot | \cdot \rangle$ era prodotto scalare.

$p(t) = (x+1)\cdot(x-1)\cdot(x-2)$, $\underset{\not\equiv 0}{p(t)} \in V$ $\langle p(t), p(t) \rangle = 0$)

Scriviamo $P_W(p(t))$, $W = \mathrm{Span}(t, t^2)$.

2 modi per scrivere P_W:

1 Estraiamo una base ortogonale per W con Gram-Schmidt

$B = \{t, t^2 - \frac{4}{3}t\}$.

$$P_W(p(t)) = \frac{\langle p(t), t \rangle}{\|t\|^2} \cdot t + \frac{\langle p(t), t^2 - \frac{4}{3}t \rangle}{\|t^2 - \frac{4}{3}t\|} (t^2 - \frac{4}{3}t).$$

Scriviamo $p(t) = a + bt + ct^2$, sostituire questa espressione qui

Modo 2). $p(t) = a + bt + ct^2$, $\dim W = 2$ $\dim V = 3 \Rightarrow \dim W^\perp = 1$

Cercare un $\underset{0}{q(t)} \neq \in W^\perp$.

$$q(t) \in W^\perp \Leftrightarrow \begin{cases} \langle q(t), t \rangle = 0 \\ \langle q(t), t^2 \rangle = 0 \end{cases} \Leftrightarrow \begin{cases} -q(-1) + q(1) + 2q(2) = 0 \\ q(-1) + q(1) + 4q(2) = 0 \end{cases} \Leftrightarrow q(t) = a + bt + ct^2$$

$$\Leftrightarrow \begin{cases} a + 3b + 4c = 0 \\ 3a + 4b + 9c \end{cases}$$, soluzione non nulla $a = -11$ $b = -3$ $c = 5$

Troviamo $q(t) = -11 - 3t + 5t^2 \in W^\perp$.

$$\pi_W(p(t)) = p(t) - \frac{\langle p(t), q(t)\rangle}{\|q(t)\|^2} \cdot q(t)$$

Verifichiamo che $p(t) - \pi_W(p(t)) \in W^\perp$

$$\underset{\displaystyle \frac{\langle p(t), q(t)\rangle}{\|q(t)\|} \cdot \underset{W^\perp}{\underset{\cup}{q(t)}} \in \mathrm{Span}(q(t)) = W^\perp .}{\parallel}$$

$\pi_W(p(t)) \in W$

$$\langle \pi_W(p(t)), q(t)\rangle = \langle p(t), q(t)\rangle - \frac{\langle p(t), q(t)\rangle}{\cancel{\|q(t)\|^2}} \cdot \underset{0}{\underset{\cup}{\cancel{\|q(t)\|^2}}}$$

Ponendo $p(t) = a + bt + ct^2$ e sostituendo nell'espressione per $\pi_W(p(t))$ otteniamo:

$$(q(1) = -9,\ q(-1) = -3,\ q(2) = 3)$$

$$P_W(p(t)) = a + bt + ct^2 - \frac{-9 \cdot p(1) - 3 \cdot p(-1) + 3 \cdot p(2)}{99} \cdot (-11 - 3t + 5t^2) =$$

$\|q(t)\|^2$

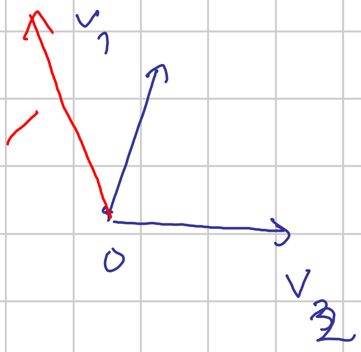
$$= \left(b - \frac{3}{11}a\right) \cdot t + \left(\frac{5a + 11c}{11}\right) \cdot t^2.$$

9.3.17. Esibire un'equazione parametrica della retta r ortogonale al piano V assegnato.

(b) $V = \text{Span}\left(\begin{pmatrix}-2\\4\\-3\end{pmatrix}, \begin{pmatrix}-2\\1\\5\end{pmatrix}\right)$

$\begin{pmatrix}-2\\4\\-3\end{pmatrix} = v_1$, $\begin{pmatrix}-2\\1\\5\end{pmatrix} = v_2$

$v_1 \wedge v_2 = v_3$



$v_3 \perp v_1$

$\wedge$

$v_3 \perp v_2$

$\Rightarrow v_3$ genera $V^\perp = r$

Calcoliamo

$$v_1 \wedge v_2 = \begin{pmatrix}-2\\4\\-3\end{pmatrix} \wedge \begin{pmatrix}-2\\1\\5\end{pmatrix} = \begin{pmatrix}23\\16\\6\end{pmatrix}$$

$23 \to \det\begin{pmatrix}4 & 1\\-3 & 5\end{pmatrix}$

$16 \to -\det\begin{pmatrix}-2 & -2\\-3 & 5\end{pmatrix}$

$6 \to \det\begin{pmatrix}-2 & -2\\4 & 1\end{pmatrix}$

$$V^{\perp} = r = \left\{ \begin{array}{l} x = 23t \\ y = 16t \\ z = 6t \end{array} \;\middle|\; t \in \mathbb{R} \right\}$$

(c) $V = \mathrm{Span}\left(\begin{pmatrix} 7 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} -6 \\ 5 \\ 1 \end{pmatrix} \right)$

$$\underbrace{\begin{pmatrix} 7 \\ -1 \\ 2 \end{pmatrix}}_{= v_1} \quad \underbrace{\begin{pmatrix} -6 \\ 5 \\ 1 \end{pmatrix}}_{= v_2}$$

$$v_1 \wedge v_2 = \begin{pmatrix} 7 \\ -1 \\ 2 \end{pmatrix} \wedge \begin{pmatrix} -6 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} -11 \\ -19 \\ 29 \end{pmatrix}$$

$$V^{\perp} = r = \left\{ \begin{array}{l} x = -11t \\ y = -19t \\ z = 29t \end{array} \;\middle|\; t \in \mathbb{R} \right\}$$

9.3.18. Esibire un'equazione cartesiana del piano V ortogonale alla retta r assegnata.

(b) $r = \begin{cases} 3x - 4y + 2z = 0 \\ 7x + 3y + 2z = 0 \end{cases} = \begin{pmatrix} 3 \\ -4 \\ 2 \end{pmatrix}^{\perp} \cap \begin{pmatrix} 7 \\ 3 \\ 2 \end{pmatrix}^{\perp} \Rightarrow V = \operatorname{Span}\left\{ \underbrace{\begin{pmatrix} 3 \\ -4 \\ 2 \end{pmatrix}}_{v_1}, \underbrace{\begin{pmatrix} 7 \\ 3 \\ 2 \end{pmatrix}}_{v_2} \right\}$

$\downarrow$

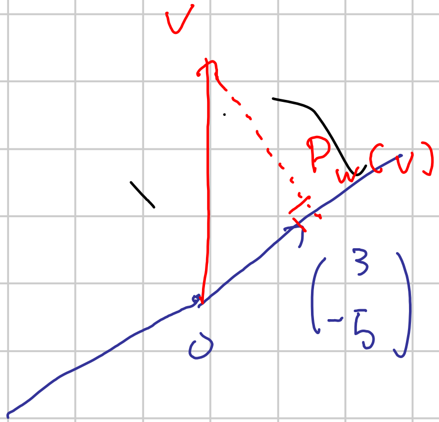
r in forma cartesiana

$$v_1 \wedge v_2 = \begin{pmatrix} 3 \\ -4 \\ 2 \end{pmatrix} \wedge \begin{pmatrix} 7 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} -14 \\ 8 \\ 37 \end{pmatrix} \qquad V = \left\{ -14x + 8y + 37z = 0 \right\} \qquad V = (v_1 \wedge v_2)^{\perp}$$

(c) $r: \begin{cases} -4x+7y-5z=0 \\ 2x-5y-4z=0 \end{cases}$ $\begin{pmatrix} -4 \\ 7 \\ -5 \end{pmatrix} \wedge \begin{pmatrix} 2 \\ -5 \\ -4 \end{pmatrix} = \begin{pmatrix} -53 \\ -26 \\ 6 \end{pmatrix}$ $V = \{-53x-26y+6z=0\}$

9.4.1. Determinare la matrice $A \in M_{n\times n}(\mathbb{R})$ associata (rispetto alla base canonica) alla proiezione ortogonale p di $\mathbb{R}^n$ sul sottospazio W assegnato, verificando che ${}^tA = A^2 = A$ $\quad (p^2 = p)$

(a) $n=2$ $\quad W = \operatorname{Span}\underbrace{\begin{pmatrix} 3 \\ -5 \end{pmatrix}}_{W}$



Dato $v = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$. $P_w(v) = \dfrac{\langle v, w \rangle}{\|w\|^2} \cdot w = \dfrac{3x - 5y}{34} \cdot \begin{pmatrix} 3 \\ -5 \end{pmatrix}$ $*$

Calcoliamo $P_w\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \dfrac{3}{34} \begin{pmatrix} 3 \\ -5 \end{pmatrix} = \begin{pmatrix} \frac{9}{34} \\ -\frac{15}{34} \end{pmatrix} = \dfrac{9}{34} \cdot e_1 + \left(-\dfrac{15}{34}\right) e_2$

sostituire

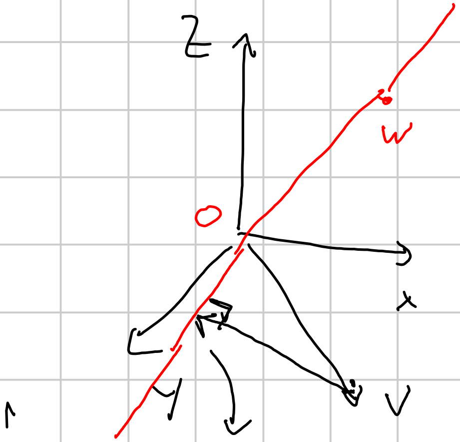
$x = 1, y = 0$ in $*$

$P_w\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} -\frac{15}{34} \\ \frac{25}{34} \end{pmatrix} = -\dfrac{15}{34} e_1 + \dfrac{25}{34} e_2$

base canonica

Pertanto $[P_w]^{B}_{B} = \begin{pmatrix} \frac{9}{34} & -\frac{15}{34} \\ -\frac{15}{34} & \frac{25}{34} \end{pmatrix} = A$ $\quad A^6 = A, \; A^2 = A$

(b) $n=3$ $\quad W = \mathrm{Span}\underbrace{\begin{pmatrix}4\\7\\-3\end{pmatrix}}_{= w}$



$P_w(v)$

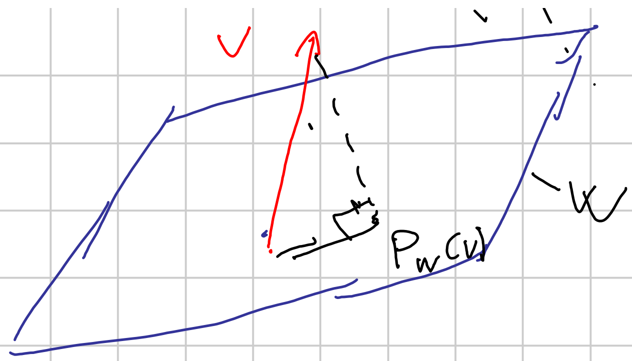
$$v = \begin{pmatrix}x\\y\\z\end{pmatrix} \in \mathbb{R}^3 \qquad P_w(v) = \frac{\langle v | w\rangle}{\|w\|^2}\cdot w = \frac{4x+7y-3z}{74}\cdot\begin{pmatrix}4\\7\\-3\end{pmatrix}$$

$$[P_w]^B_B = \begin{pmatrix} \frac{16}{74} & \frac{28}{74} & -\frac{12}{74} \\ \frac{28}{74} & \frac{49}{74} & -\frac{21}{74} \\ -\frac{12}{74} & -\frac{21}{74} & \frac{9}{74} \end{pmatrix} = A \qquad A^2 = A \qquad \mathrm{rk}(A) = 1$$

$${}^tA = A$$

(d) $n=3$

$$W = \mathrm{Span}\left(\underbrace{\begin{pmatrix} -3 \\ 2 \\ 7 \end{pmatrix}}_{= w_1}, \underbrace{\begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix}}_{= w_2} \right)$$



Cerchiamo $v_1 \in W^{\perp}$:

Prendiamo $v_1 = w_1 \wedge w_2 = \begin{pmatrix} -3 \\ 2 \\ 7 \end{pmatrix} \wedge \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix} = \begin{pmatrix} -37 \\ 11 \\ -19 \end{pmatrix}$

Se $v = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$.

$$P_W(v) = v - \frac{\langle v | v_1 \rangle}{\|v_1\|^2} \cdot v_1 = \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \frac{\overset{\langle v | v_1 \rangle}{-37x + 11y - 19z}}{\underset{\|v_1\|^2}{1851}} \cdot \begin{pmatrix} -37 \\ 11 \\ 19 \end{pmatrix}.$$

$$[P_W]_B^B = \begin{pmatrix} \frac{482}{1851} & \frac{407}{1851} & \frac{-703}{1851} \\ \frac{407}{1851} & \frac{1730}{1851} & \frac{209}{1851} \\ \frac{-703}{1851} & \frac{209}{1851} & \frac{1490}{1851} \end{pmatrix} = A \qquad {}^tA = A. \qquad A^2 = A$$

$$\frac{1851}{1851} - \frac{(-37)^2}{1851} = \frac{1851 - 37^2}{1851}$$

(e) $n=4$ $\quad W = \mathrm{Span}\left(\begin{pmatrix} 2 \\ 3 \\ -1 \\ 5 \end{pmatrix}\right)$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad =: w$

$$v = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}, \quad P_W(v) = \frac{\langle v | w \rangle}{\|w\|^2} \cdot w = \frac{2x + 3y - z + 5w}{39} \cdot \begin{pmatrix} 2 \\ 3 \\ -1 \\ 5 \end{pmatrix}$$

$$[P_W]^B_B = \begin{pmatrix} \frac{4}{39} & \frac{2}{13} & -\frac{2}{39} & \frac{10}{39} \\ \frac{2}{13} & \frac{3}{13} & -\frac{1}{13} & \frac{5}{13} \\ -\frac{2}{39} & -\frac{1}{13} & \frac{1}{39} & -\frac{5}{39} \\ \frac{10}{39} & \frac{5}{13} & -\frac{5}{39} & \frac{25}{34} \end{pmatrix} = A$$

${}^tA = A$

$A^2 = A$

$rk(A) = 1$