Titolo nota

Knot theory 21/3/2019

21/03/2019

$L \subseteq S^3$ link (with n-components)

$E = S^3 \setminus L$, $\tilde{E}_\infty$ = "canonical" cyclic covering of E

Alexander module of L is $A(L) = H_1(\tilde{E}_\infty)$,

which is a $\mathbb{Z}[t, t^{-1}]$-module.

Let S be a Seifert surface for L.

Seifert form: $\alpha : H_1(S) \times H_1(S) \to \mathbb{Z}$

$$(x, y) \mapsto \beta(x^-, y)$$

$$\beta : H_1(S^3 \setminus S) \times H_1(S) \to \mathbb{Z}$$

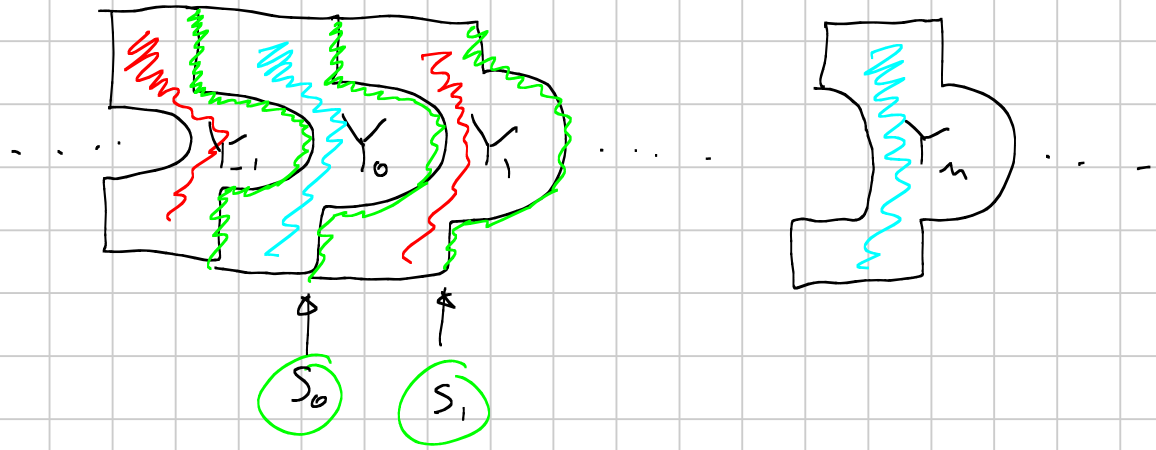
$$([c], [d]) \mapsto \operatorname{lk}(c, d)$$

Theorem: Let S be a connected Seifert surface for L, and let A be a matrix representing α w.r.t. some basis of $H_1(S)$. Then

$$tA - {}^{t}A \text{ presents } A(L) = H_1(\tilde{E}_\infty)$$

as a $\mathbb{Z}[t, t^{-1}]$-module.

Proof: Remember $\tilde{E}_\infty$ can be constructed starting from $Y = S^3 \setminus S$ by "gluing" countably many copies Y_i, $i \in \mathbb{Z}$ along countably many copies of S (say S_i, $i \in \mathbb{Z}$).



Today $Y_i = \overline{Y_i}$, so $Y_i \cap Y_{i+1} = S_i$.

$$Y^e = \coprod_{i \text{ even}} Y_i \quad , \quad Y^o = \coprod_{i \text{ odd}} Y_i$$

$$\tilde{E}_\infty = Y^o \cup Y^e \qquad Y^o \cap Y^e = \coprod_{i \in \mathbb{Z}} S_i$$

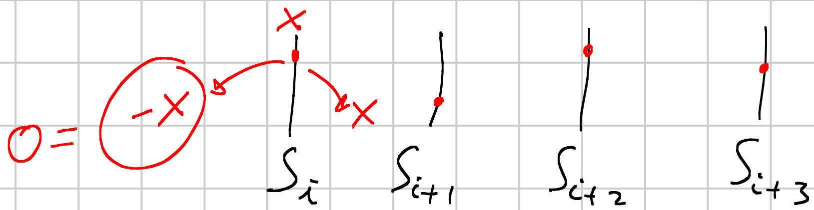
$$\begin{tikzcd}
H_1(Y^o \cap Y^e) \arrow[r] & H_1(Y^o) \oplus H_1(Y^e) \arrow[r] & H_1(\tilde{E}_\infty) \arrow[d] \\
& H_0(Y^o) \oplus H_0(Y^e) & H_0(Y^o \cap Y^e) \arrow[l]
\end{tikzcd}$$

coming from
$$C_i(Y^o \cap Y^e) \to C_i(Y^o) \oplus C_i(Y^e) \to C_i(\tilde{E}_\infty)$$
$$\| $$
$$C_i(\coprod S_j)$$
$$x \longmapsto (-x, x)$$
$$(y, z) \longmapsto x + z$$

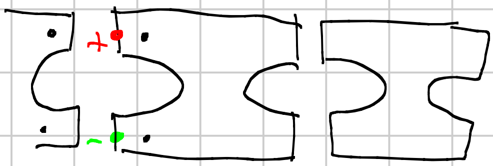
All the modules in the sequence are $\mathbb{Z}[t,t^{-1}]$-modules and the maps are $\mathbb{Z}[t,t^{-1}]$-linear.

$\left(t(C_i(Y^o)) \subseteq C_i(Y^e) \right.$ and viceversa $\left. \right)$.

The resulting map $H_0(\amalg S_i) \to H_0(Y^o) \oplus H_0(Y^e)$ is injective (thanks to the fact S is connected)



False if S is disconnected



Therefore, we have the exact sequence

$$
\begin{array}{ccccccc}
H_1(\amalg S_i) & \to & H_1(Y^o) \oplus H_1(Y^e) & \to & H_1(\tilde{E}_\infty) & \to & 0 \\
\Vert\wr & & \Vert & & & & \\
\bigoplus_{i \in \mathbb{Z}} H_1^i(S) & & \mathbb{Z}[t,t^{-1}] \otimes_{\mathbb{Z}} H_1(Y) & & & & \\
\Vert\wr & & & & & & \\
\mathbb{Z}[t,t^{-1}] \otimes_{\mathbb{Z}} H_1(S) \cong \left(\mathbb{Z}[t,t^{-1}]\right)^{2g+n-1} & & & & & &
\end{array}
$$

$$\mathbb{Z}[t,t^{-1}]\otimes H_1(S) \to \mathbb{Z}[t,t^{-1}]\otimes H_1(Y) \to H_1(\tilde{E}_\infty) \to 0$$
$$x \mapsto (-x, x)$$

If x is supported on S_i, the first element of $(-x,x)$ is supported in Y_i, the second in Y_{i+1}, that is that, as elements of $H_1(Y)$, $(-x,x) =$
$= (-x^-, x^+)$ (recall that if $x \in H_1(S)$ the $x^\pm \in H_1(S^3 \setminus S) = H_1(Y)$ is obtained by pushing x to the positive/negative side of S).
With our notation, $x \mapsto tx^+ - x^-$
(or $tx^- - x^+$)
(we are working with $\mathbb{Z}[t,t^{-1}]\otimes H_1(S)$, $\mathbb{Z}[t,t^{-1}]\otimes H_1(Y)$)
Let $f_1, \dots, f_{2g+n-1}$ be a basis of $H_1(S)$ with dual basis $e_1, \dots, e_{2g+n-1}$ of $H_1(Y)$.
Then the e_i's are a free basis of $\mathbb{Z}[t,t^{-1}]\otimes H_1(Y)$ over $\mathbb{Z}[t,t^{-1}]$, while the f_i's give a free basis of $\mathbb{Z}[t,t^{-1}]\otimes H_1(S)$.

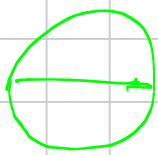
Therefore, a presentation for $H_1(\tilde{E}_\infty)$ has the e_i's as generators, and relations of the form

$$f_i \longrightarrow t f_i^- - f_i^+ = t(A_{ij} e_j) - (A_{ji} e_j) =$$
$$= (tA - {}^tA)_{ij} \cdot e_j$$

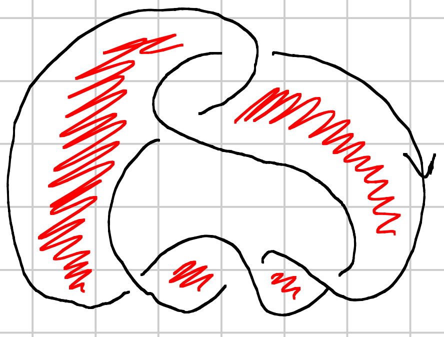
Then $tA - {}^tA$ is a presentation matrix for $H_1(\tilde{E}_\infty)$.

Remark: We discovered $H_1(\tilde{E}_\infty)$ has a square presentation matrix $\Rightarrow$ the first Alexander ideal is generated by $\det(tA - {}^tA)$, which is the Alexander polynomial of L (the fact that the ideal is principal was non-trivial).

Example



$\sim$ homotopy equivalent

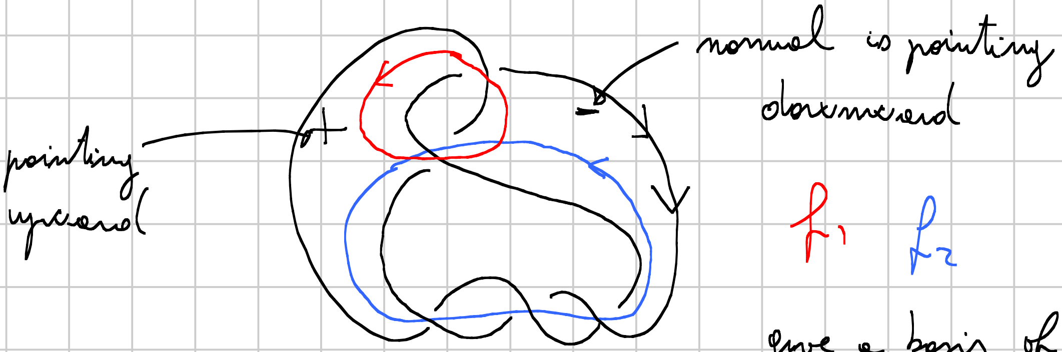


2n-1 crossings

red surface is orientable, hence it is a Seifert surface S

$S = \Sigma_{1,1}$

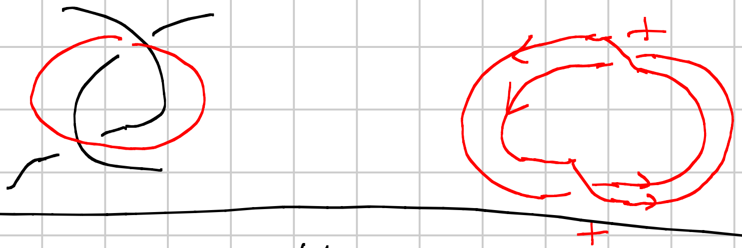
$H_1(S) = \mathbb{Z}^2$



f_1 f_2

give a basis of $H_1(S)$.

General strategy for $lk(f, f^+)$ (some sort of "auto"-linking number). Fact: f^+ is isotopic (hence, homologous) in $S^3 \setminus f$ to a parallel copy of f in S $\Rightarrow$ $lk(f, f^+) = lk(f,$ parallel copy on $S)$



$lk(f_1, f_1^+) = 1$

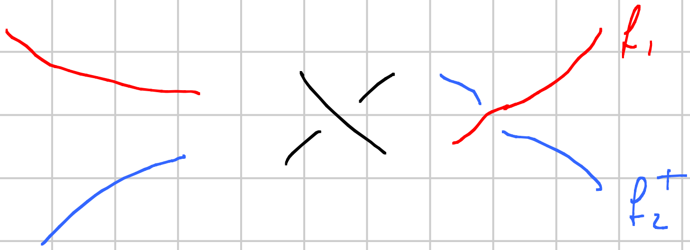
Doing the same with f_2, we have $2n-1$ negative crossings below, one positive crossing in the middle $\Rightarrow$ $lk(f_2, f_2^+) = -n+1$

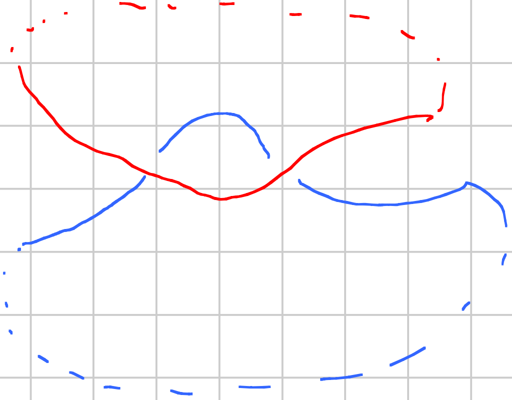
WRONG

For f_2, all the crossings are positive

$\Rightarrow lk(f_2, f_2^+) = n$

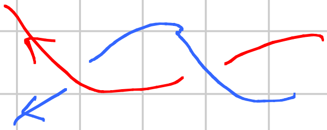
$lk(f_1, f_2^+)$





$$lk(f_1, f_2^+) = 0$$

$$lk(f_1, f_2^-) = 1$$

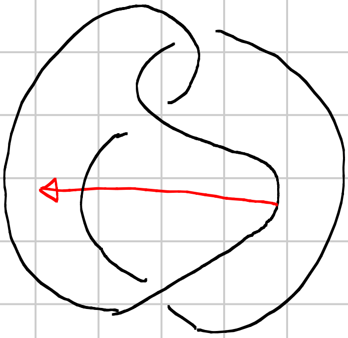


Hence $A = \begin{pmatrix} 1 & 0 \\ 1 & n \end{pmatrix}$

$$tA - {}^tA = \begin{pmatrix} t-1 & -1 \\ t & nt-n \end{pmatrix}$$

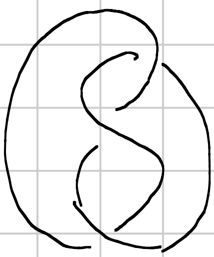
$$\Delta_K(t) = \det(tA - {}^tA) = n(t-1)^2 + t = nt^2 + (1-2n)t + n$$

If $n=0$ (which means 1 crossing with opposite sign w.r.t. the picture)



$=$ unknot $\quad \Delta_K(t) = t \doteq 1$

For $n=1$



$=$ trefoil

$$\Delta_K(t) = t^2 - t + 1$$

Theorem: Let L be a link. Then:

1. $\Delta_L(t) \doteq \Delta_L(t^{-1})$
2. If K is a knot, $\Delta_K(1) = \pm 1$

   (Δ_K is well-defined only up to $\pm t^n$

   $\Delta_K(2)$ does not make much sense).
3. If L has at least 2 components, $\Delta_K(1) = 0$
4. If $\overline{L}$ is the mirror of L, then

   $\Delta_L \doteq \Delta_{\overline{L}}$ (Alexander pol. does not detect chirality).
5. If $-L$ is the reverse of L, then

   $\Delta_L \doteq \Delta_{-L}$

Proof: 1. $\det(tA - {}^tA) = t^n \det(A - t^{-1}\,{}^tA) =$

$= (-1)^n t^n \det(t^{-1}\,{}^tA - A) =$

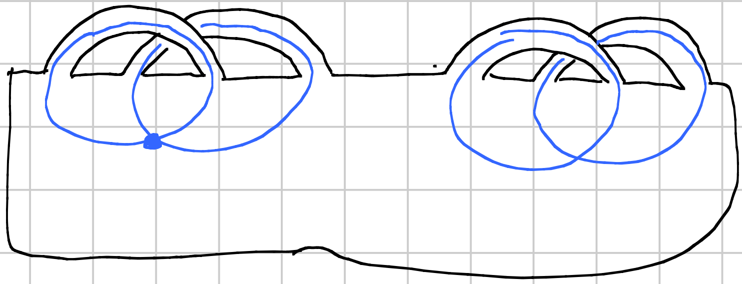
$= (-1)^n t^n \det(t^{-1}A - {}^tA) = (-1)^n t^n \Delta_L(t^{-1})$.

2. Work with a "standard" basis

$f_1, \dots, f_{2g}$ for $H_1(S)$ s.t. f_{2i} intersects

transversely f_{2i+1} in one point, and it is disjoint

from the other generators.



$\Delta_K(1) = \det(A - {}^tA)$ and

$(A - {}^tA)_{ij} = \mathrm{lk}(f_i, f_j^+) - \mathrm{lk}(f_i, f_j^-)$

Hence

$$A - {}^tA = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & 0 \\ & & 0 & 1 & \\ & & -1 & 0 & \\ & 0 & & & \ddots \end{pmatrix}$$

(with some $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ maybe).

Thus $\det(A - {}^tA) = \pm 1$

3. Same argument, but now we have at least one zero column (and line) in $A - {}^tA$.

$S =$ 

4. If $\bar{L} = r(L)$, $r : S^3 \to S^3$ reflection, just take $\bar{S} = r(S)$. The Seifert matrix associated to $\bar{S}$ is the opposite of the one associated

to S.

(5) For $-L$, just take the same Seifert surface with opposite orientation. In this way A changes into ${}^tA$.

Remark: (T) is equivalent to (5).