Knot theory 26/3/2019

Def: $p(t) \in \mathbb{Z}[t, t^{-1}]$, $p(t) = \sum_{n=-\infty}^{\infty} a_n t^n$

breadth(p) = br(p) =

$= \max\{n \mid a_n \neq 0\} - \min\{n \mid a_n \neq 0\}$.

If $p \doteq q$, then $br(p) = br(q)$.

Hence, the breadth of the Alexander polynomial(s) of a link is well-defined.

Theorem: Let L be a link, and $g = \text{genus}(L)$.

Then $2g + n - 1 \geq br(\Delta_L(t))$

where n = # components of the link.

Hence, if L is a knot, then

$$g \geq \frac{br(\Delta_L(t))}{2}$$

Proof: If S is a Seifert surface of L of genus g, then the Seifert form of S is represented by a $(2g+n-1) \times (2g+n-1)$ matrix A, and the Alexander module

$H_1(\widetilde{E(L)}_\infty)$ is presented over $\mathbb{Z}[t,t^{-1}]$

by $tA - {}^tA$, hence

$\Delta_L(t) = \det(tA - {}^tA)$ has breadth

at most $2g + n - 1$

Next goal: Compute the Alexander polynomial of torus knots.

To this aim, we will introduce a new tool: Fox calculus on free groups.
Before doing that, let's go for another application of the Alexander polynomial.

Definition: Let L be a link $L = K_1 \cup \dots \cup K_n$ with n components.
Then L is

① SPLIT if there exist disjoint balls $B_1 \dots B_m$ in S^3 with $K_i \subseteq B_i$ $\forall i = 1, \dots, m$. Equivalently, there exists a diagram for L s.t. K_i and K_j do not cross for $i \neq j$.

② L is a boundary link if there exist connected Seifert surfaces $S_1, \dots, S_m$ with $\partial S_i = K_i$ and $S_i \cap S_j = \emptyset$ for $i \neq j$.

③ L is algebraically split if $lk(K_i, K_j) = 0$ $\forall i \neq j$.

Proposition: Let L be a link. Then

L split $\Longrightarrow$ L is boundary link

L boundary link $\Longrightarrow$ L is algebraically split

Proof: L split $\Rightarrow$ take a split diagram and the Seifert algorithm produces disjoint Seifert surfaces for the components of L

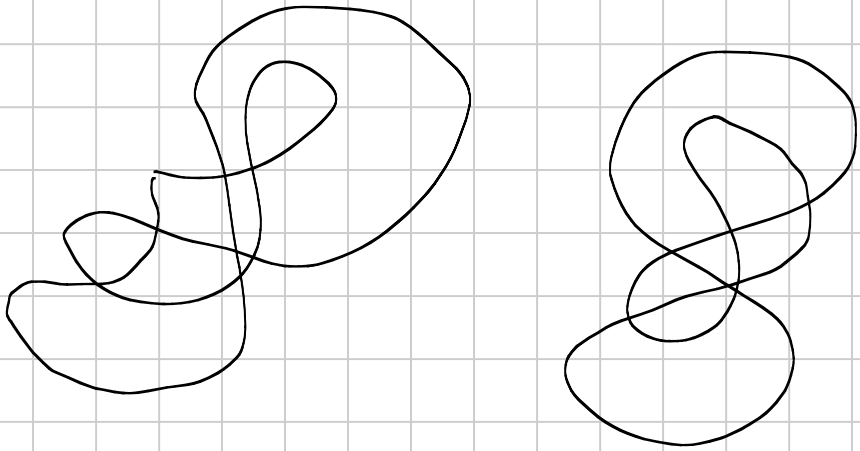
$\Rightarrow L$ is boundary.

If L is boundary and the S_i are the disjoint Seifert surfaces of its components, then $S_i \cap S_j = \emptyset \Rightarrow K_i \cap S_j = \emptyset \Rightarrow$

$\Rightarrow lk(K_i, K_j) = 0$.

Next: Produce examples that show these conditions to be non-equivalent.

Split link:



Boundary link = ?

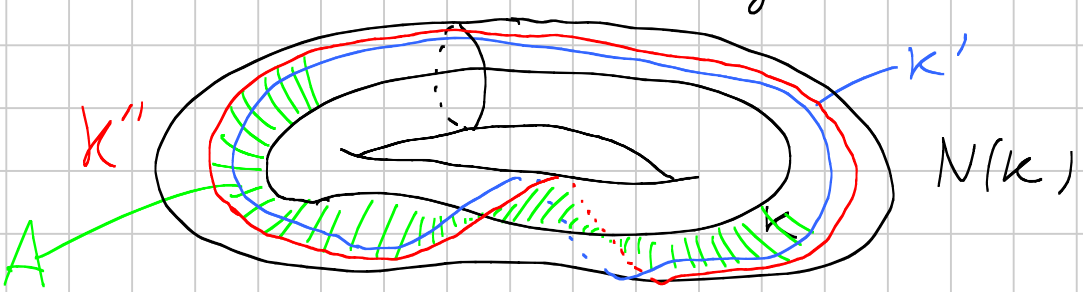
Elementary construction of boundary links.

K knot. Let K' be a longitude of K, i.e. $K' \subseteq \partial N(K)$ and K' bounds a surface in $S^3 \setminus N(K)$

(the longitude is the unique - up to isotopy - non-trivial curve on $\partial N(K)$ which is 0 in $H_1(S^3 \setminus N(K))$, i.e. it bounds a surface there).

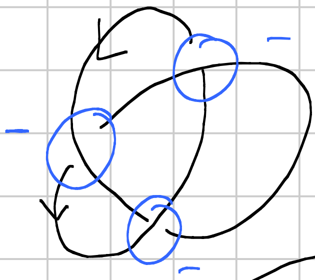
Let now K'' be a parallel copy of K' on $\partial N(K)$. Then we can take parallel surfaces $S', S'' \subseteq S^3 \setminus N(K)$ with $\partial S' = K'$, $\partial S'' = K''$.

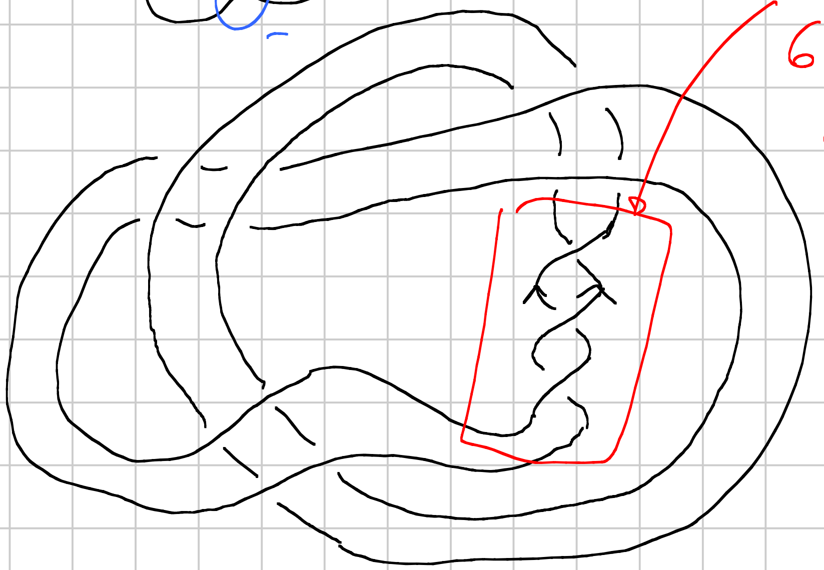
Glue S'' to an annulus $A \subseteq N(K)$ with $\partial A = K \cup K''$ to get $S = S'' \cup A$



$S \cap S' = \emptyset \implies K \cup K'$ is boundary.

Now I can construct a boundary link which is not split.

$K =$ 

$K \cup K' =$ 

WRONG: 6 crossings are needed.

This is a boundary link. Is it SPLIT?

It is NOT SPLIT.

In fact, suppose $L = K \cup K'$ is split. Then we have a 3-coloring on L with a constant colour on K and a constant different colour on K' on a split diagram. Then we perform Reidemeister moves to

get the diagram above. Let us study the so-obtained coloring of the diagram above.
If the same colour appears on two parallel arcs, then the coloring is constant, hence it was constant since the beginning, contradiction.

DUE TO THE ABOVE MISTAKE, THE PROOF THAT $K \cup K'$ IS NOT SPLIT IS POSTPONED.

Prop.: The Whitehead link is algebraically split but it is NOT a boundary link.

Theorem: Let L be a boundary link. Then $\Delta_L(t) = 0$.

Proof: For simplicity, let $L = K_1 \cup K_2$.

$K_1 = \partial S_1$, $K_2 = \partial S_2$, S_i connected

Seifert surface for K_i, $S_1 \cap S_2 = \emptyset$.

I can build a connected Seifert surface for

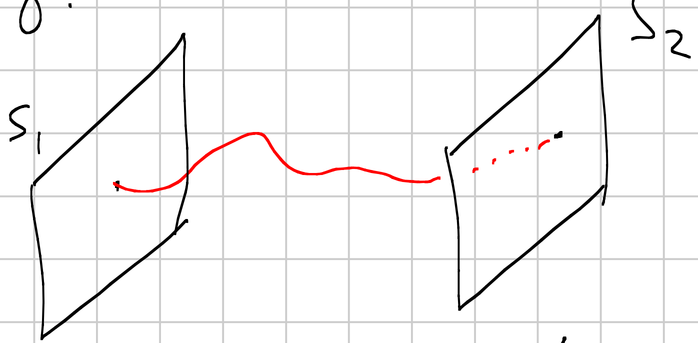
L by taking an arc γ joining one

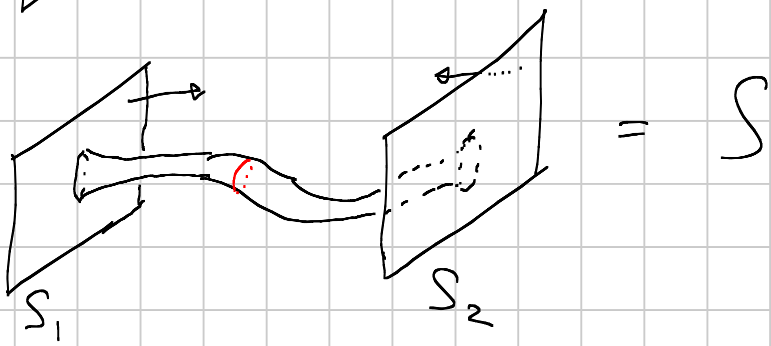
point $p_1 \in S_1$ with one point $p_2 \in S_2$, and

setting $S = (S_1 \setminus B_1) \cup (S_2 \setminus B_2) \cup C$

$B_i \subseteq S_i$ small ball around p_i, C cylinder

along γ.



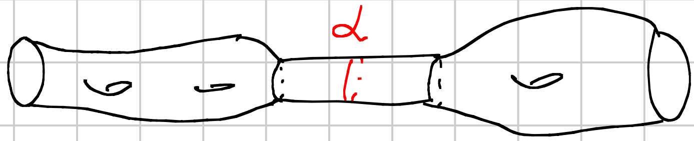


Exercise: ① $S^3 \setminus (S_1 \cup S_2)$ is connected

② Using ①, choose γ so that S is oriented

with the same orientation of the S_i's.

We obtain a basis of $H_1(S)$ as the union of a basis of $H_1(S_1)$, a basis of $H_1(S_2)$, and a loop bounding a meridian of the added cylinder (exercise)

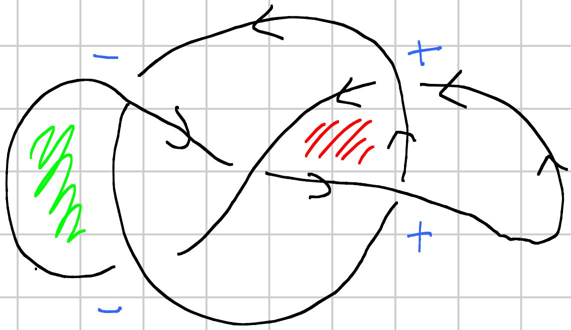
$S =$ 

If A is the matrix representing the Seifert form of $H_1(S)$ w.r.t. this basis, then the row and the column corresponding to α are null (α bound a disk disjoint from $f^{\pm}$ $\forall$ element f of the basis).

$\Longrightarrow$ $\det(tA - {}^{t}A) = 0$

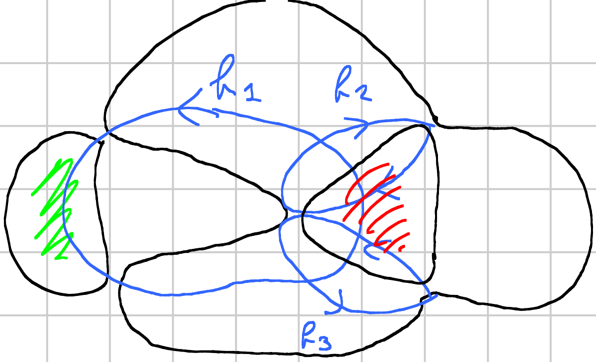
WHITEHEAD

LINK



SEIFERT

ALGORITHM



With these choices, $A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & -1 & -1 \\ -1 & 0 & -1 \end{pmatrix}$

$$\Rightarrow \Delta_L(t) = \det(tA - {}^tA) = -t^3 + 3t^2 - 3t + 1$$

Corollary: The Whitehead link is NOT a boundary link.

It is immediate to check it is algebraically split.

A MORE ALGEBRAIC APPROACH TO $H_1(\widetilde{E(K)}_\infty)$.

Henceforth, K is a knot (no links with more components).

Theorem (Hurewicz): let M be a

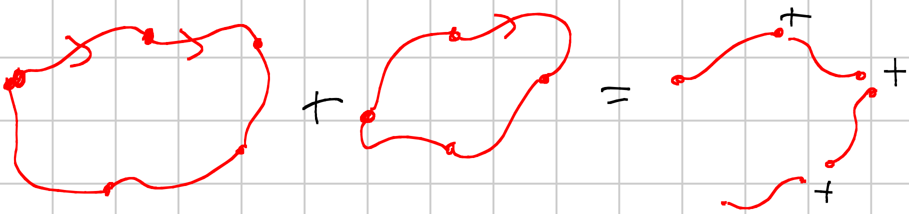
path-connected space, $x_0 \in M$. Then the map

$$\pi_1(M, x_0) \xrightarrow{\psi} H_1(M; \mathbb{Z})$$
$$[\gamma] \overset{\psi}{\longmapsto} [\gamma]$$

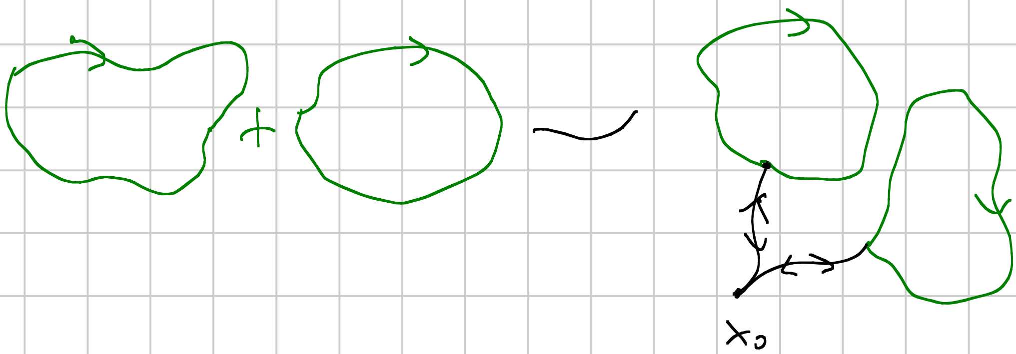
is a surjective group homomorphism with $\operatorname{Ker} \psi = [\pi_1(M, x_0), \pi_1(M, x_0)]$.

Sketch of proof:

- Well-defined and homomorphism: OK.
- Surjectivity: a cycle is something like

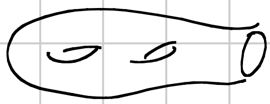


$\wr$ homologous



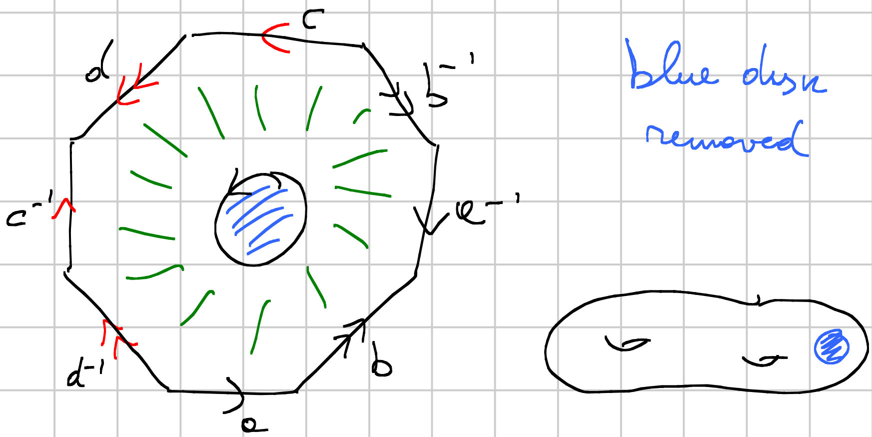
$\Longrightarrow$ every class is represented by a loop based at x_0.

$\mathrm{Ker}(\psi)$: If $[\gamma] \in \mathrm{Ker}\,\psi$, then γ (as a 1-simplex) bounds a 2-chain.

$\Longrightarrow \exists\, f:$  $\longrightarrow M$

S

s.t. $f_{|\partial S}$ "$=$" γ. But ∂S is a commutator in $\pi_1(S)$, whence the conclusion.



The boundary of the disk is homotopic (in $S \setminus$ disk) to $aba^{-1}b^{-1}cdc^{-1}d^{-1}$.

This shows $\mathrm{Ker}\,\psi \subseteq [\pi_1(M,x_0), \pi_1(M,x_0)]$. The other inclusion is obvious since $H_1(M)$ is abelian.

Henceforth, let us fix a path connected space M with $H_1(M) \cong \mathbb{Z}$

(with a preferred generator). (In our application, $M = E(K)$).

$G = \pi_1(M, x_0)$, $G' = [G,G]$, $G'' = [G', G']$.

Let $\varphi : \pi_1(M, x_0) \to H_1(M) \cong \mathbb{Z}$

Denote by $\tilde{M}_\infty$ the covering of M associated to $\operatorname{Ker} \varphi$.

By Hurewicz, $H_1(\tilde{M}_\infty) = \dfrac{\pi_1(\tilde{M}_\infty)}{[\pi_1(\tilde{M}_\infty), \pi_1(\tilde{M}_\infty)]} =$

$= \dfrac{G'}{G''}$.

Let $\bar{t} \in \pi_1(M)$ be an element projecting to the preferred generator of $H_1(M)$.

Since G', G'' are normal in G, conjugation by $\bar{t}$ acts on $G'/G'' \cong H_1(\tilde{M}_\infty)$.

Proposition: Under the identification $G'/G'' = H_1(\tilde{M}_\infty)$, the action of

$\overline{E}$ coincides with the action of $\tilde{\tau}_*$ on $H_1(\hat{M}_\infty)$, where $\tilde{\tau}$ is the positive generator of $\mathrm{Aut}(\hat{M}_\infty)$.

Corollary: The structure of $H_1(\hat{M}_\infty)$ as a $\mathbb{Z}[t,t^{-1}]$-module only depends on $\pi_1(M)$ (and a preferred generator of $H_1(M)$).