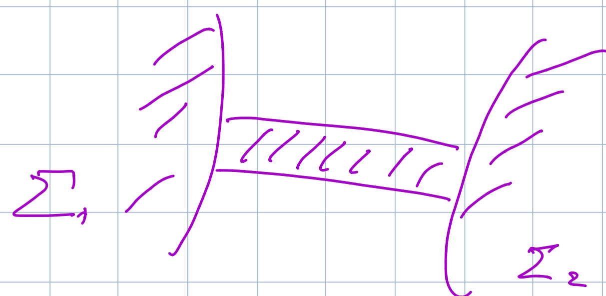
Teoria dei nodi
16/4/19

Thm.: $g(K_1 \# K_2) = g(K_1) + g(K_2)$

$\boxed{\leq}$:



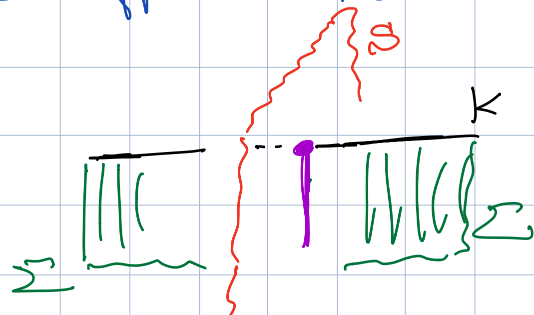
$g(\Sigma_1 \# \Sigma_2) = g(\Sigma_1) + g(\Sigma_2)$.

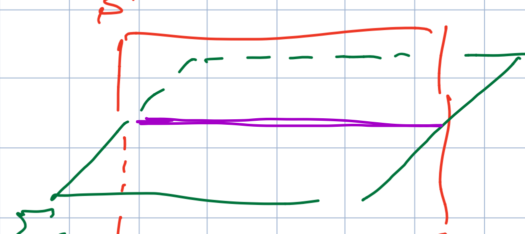
$\boxed{\geq}$ Take Σ that realizes $g(K_1 \# K_2)$.

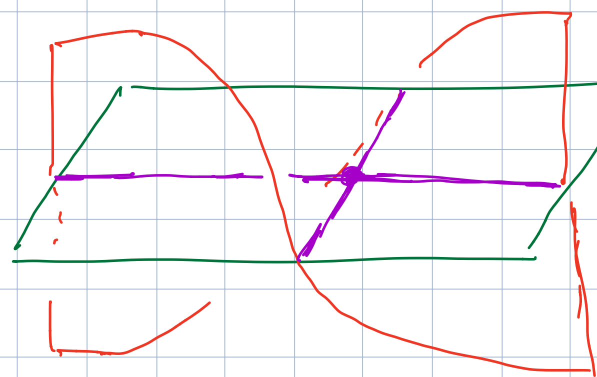
Take $S \subset S^3$, $S \cong S^2$ s.t. $S \cap (K_1 \# K_2) = 2$ pts

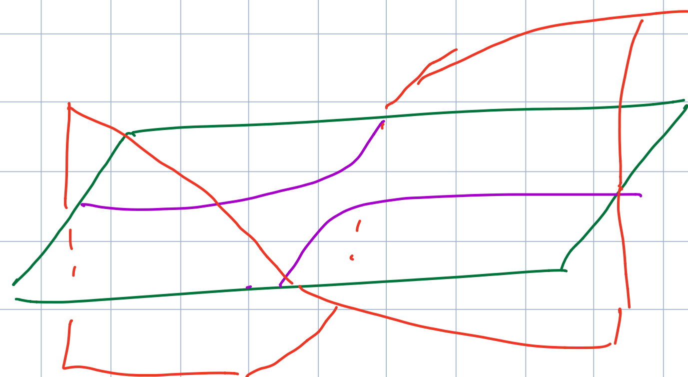
$S = \partial D_1 = \partial D_2$, $D_i \cong D^3$, $K_i = (K_1 \# K_2) \cap D_i$
$\cup$ an arc on ∂D_i.

I suppose $S \cap \Sigma$:









$\Rightarrow$ $\Sigma \cap S$ is a collection of circles + 1 arc. (total of 2 endpoints).

If no circle: $\Sigma = \Sigma_1 \# \Sigma_2$, Σ_i Seifert of K_i

$\Rightarrow g(\Sigma) = g(\Sigma_1) + g(\Sigma_2)$.

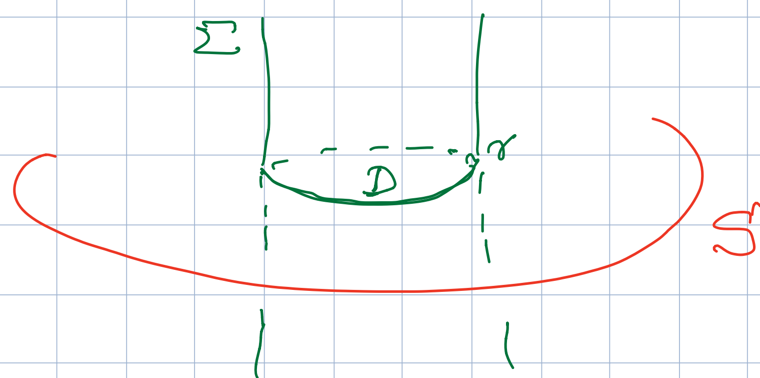
Aim: remove intersection circles without increasing genus of Σ (cannot decrease).

On S' there are ≥ 2 innermost circles

↳ bounds disc not containing other circles

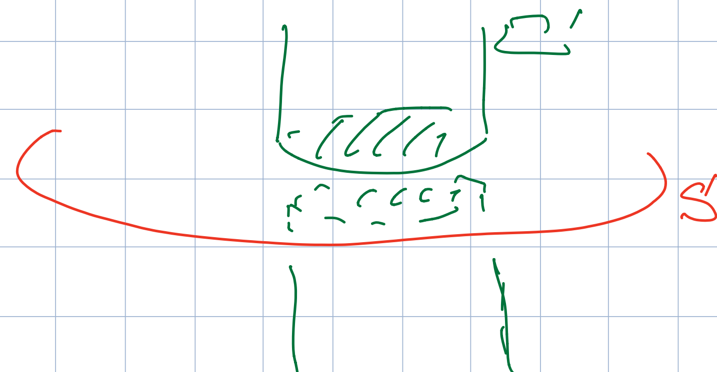
(perhaps one to both sides)

$\Rightarrow$ $\exists$ innermost circle γ bounding $D \subset S$ s.t.
D does not contain arc $\Rightarrow D \cap K = \emptyset$.



$\Rightarrow D \cap \Sigma = \gamma$

surger Σ



Σ' still an oriented surface bounded by $K_1 \# K_2$.

Three cases:

1) Σ' connected
2) $\Sigma' = \underset{\partial = K}{\Sigma''} \sqcup \underset{\partial = \emptyset}{\Sigma'''}$ 2.1) $\Sigma''' \not\cong S^2$

   2.2) $\Sigma''' \cong S^2$

1) $\chi(\Sigma') = \chi(\Sigma) + 2 \Rightarrow g(\Sigma') = g(\Sigma) - 1$
absurd

2) $g(\Sigma'') < g(\Sigma)$ absurd

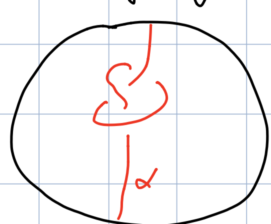
3) $g(\Sigma'') = g(\Sigma)$: proceed with Σ'' until all $\underbrace{\text{intersections}}_{\text{circles}}$ disappear.

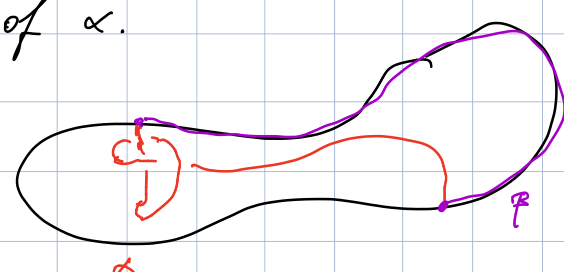
$\boxed{}$

Today: "knots" = "oriented knots"/isotopy

<u>Def</u>: K is <u>prime</u> if non-trivial and $K = K_0 \# K_1$
$\Rightarrow$ one of K_j is trivial

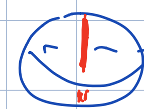
<u>Def</u>: if $D \cong D^3$ we say $\alpha \subset D$ is <u>tangle</u> if it is a properly embedded arc in D; (<u>oriented</u>)
$T = (D, \alpha)$; define $\hat{T}$ = knot in S^3 = $\alpha \cup \beta$ given by any embedding of D in S^3, $\beta \subset \partial D$ any arc joining the ends of α.



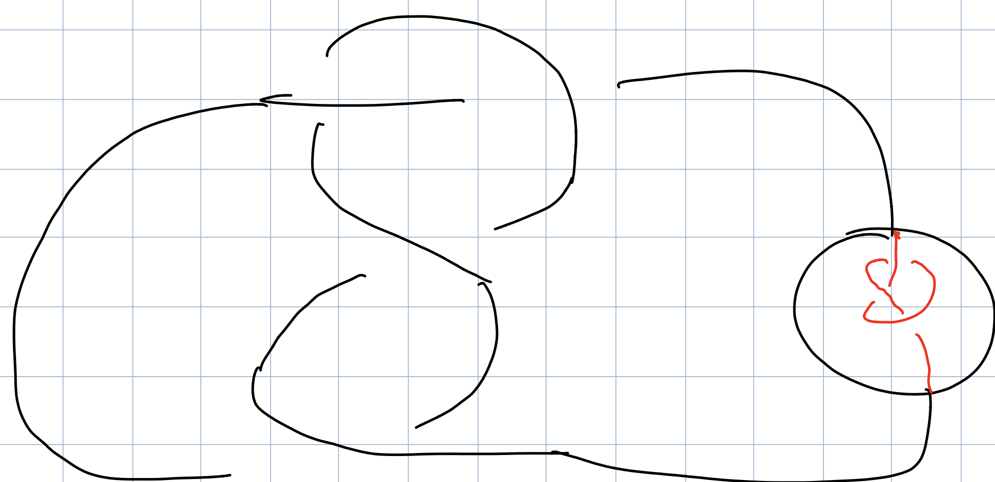


Fact: well-def up to isotopy.

$K_0 \# K_1$ defined as:

→ find D s.t. $(D, D\cap K_0) =$ 

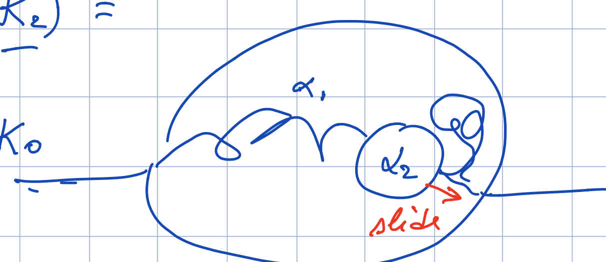
→ replace $(D, D\cap K_0)$ by $(D,\alpha) = T$
s.t. $\widehat{T} = K_1$

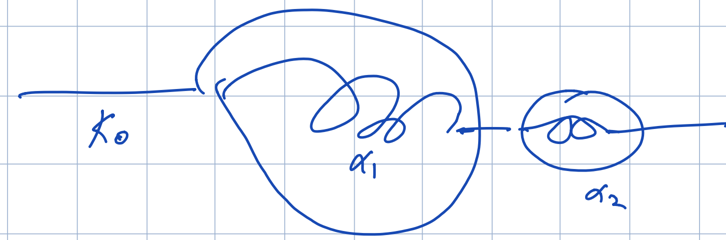


Prop: $K_0 \# K_1 = K_1 \# K_0$

Prop: $(K_0 \# K_1) \# K_2 \cong K_0 \# (K_1 \# K_2)$

Pf: $K_0 \# (\underline{K_1} \# \underline{K_2}) =$





$$= (K_0 \# K_1) \# K_2 \qquad \blacksquare$$

<u>About satellites:</u>

$P \subset D^2 \times S^1$ circle that cannot be isotoped to be disjoint from $D^2 \times \{*\}$

$f : D^2 \times S^1 \longrightarrow U(K)$

$\Rightarrow$ $J = f(P)$ <u>satellite</u> with companion K and pattern P.

True satellite if P is not core of $D^2 \times S^1$

<u>Fact</u>: true satellite of non-trivial is not-trivial.

Proof says: satellite of non-triv. is non-trivial

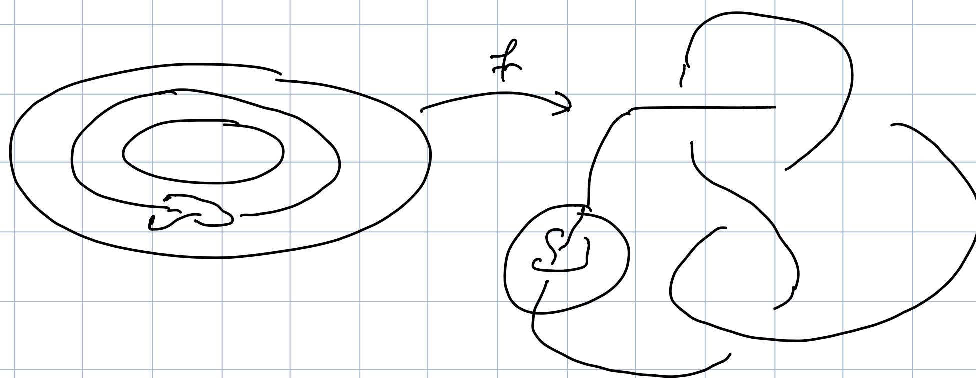
<u>or</u>

Trivial knot is satellite of itself only.

Prop: if $K_0 \# K_1$ trivial $\Rightarrow$ both K_0, K_1 trivial.

Pf 1: $g(K_0 \# K_1) = 0 \Rightarrow g(K_0) = g(K_1) = 0 \Rightarrow$ $\blacksquare$

Pf 2: $K_0 \# K_1$ is satellite of both



$\Rightarrow$ $\blacksquare$.

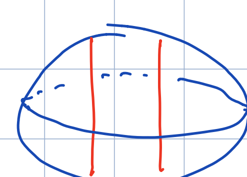
Prop: $K_0 \# K_1 \cong K_1 \Rightarrow K_0$ trivial (via genus).

———— o ————

Prop: genus-1 knots are prime. (via genus).

Prop: $b(K) = 2$ (bridge index)
$\Rightarrow K$ prime.

Pf: $b(K) = 2 \Rightarrow \exists\, \Sigma \cong S^2 \quad \Sigma = \partial D_0 = \partial D_1$

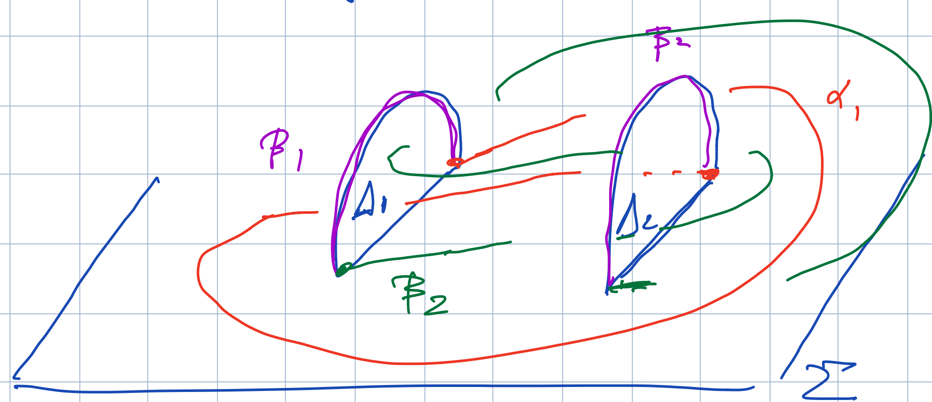
$(D_j, D_j \cap K) \cong$ 

I take Σ = horizontal plane

D_1 = upper half-plane with

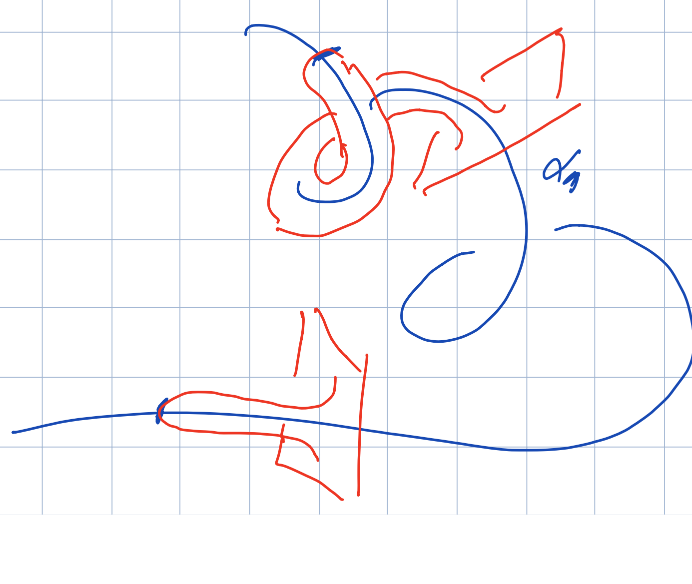
$K \cap D_1 = \beta_1, \beta_2$

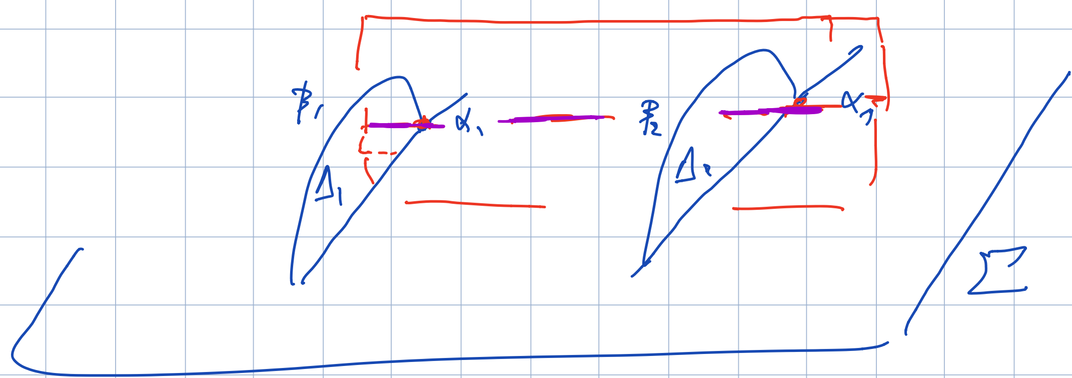
$\beta_j \subset \partial \Delta_j$, $\Delta_j \subset D_1$, $\partial\Delta_j \setminus \beta_j \subset \Sigma$



and $K \cap D_0 \subset \Sigma$, $K \cap D_0 = \alpha_1 \cup \alpha_2$.

Take S s.t. $S \pitchfork K$ = 2 pts; can choose these 2 pts to be the ends of α_1:

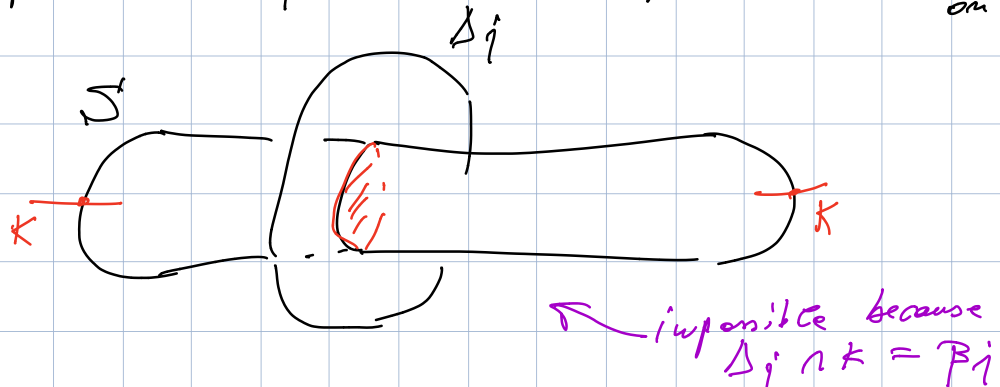


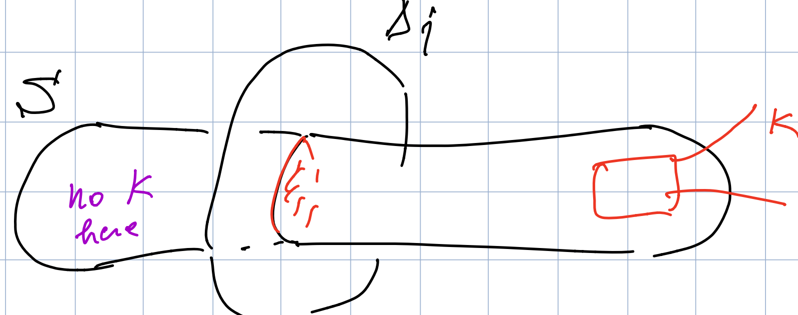


Suppose except at the ends of α_i

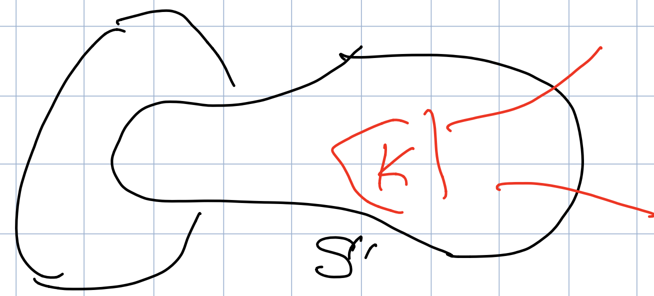
$S \cap \Sigma$, $S \pitchfork D_j$

Suppose $S \cap \Delta_j$ contains circle ; take innermost one on Δ_j





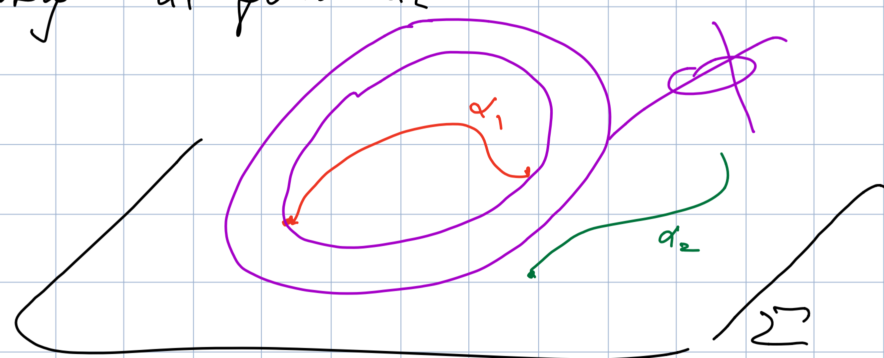
surger



Therefore can assume $S \cap \Delta_j$ contains no circles.

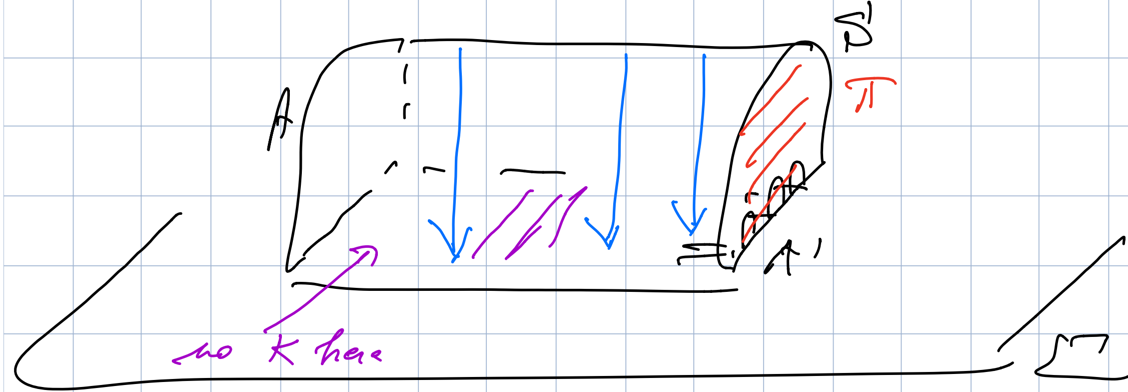
If there is a circle of $S \cap \Sigma$ that does not separate α_1 from α_2 then it bounds on Σ a disc disjoint from K $\Rightarrow$ again can surger to reduce intersection.

Eventually $S \cap \Sigma$ collection of parallel circles separating α_1 from α_2



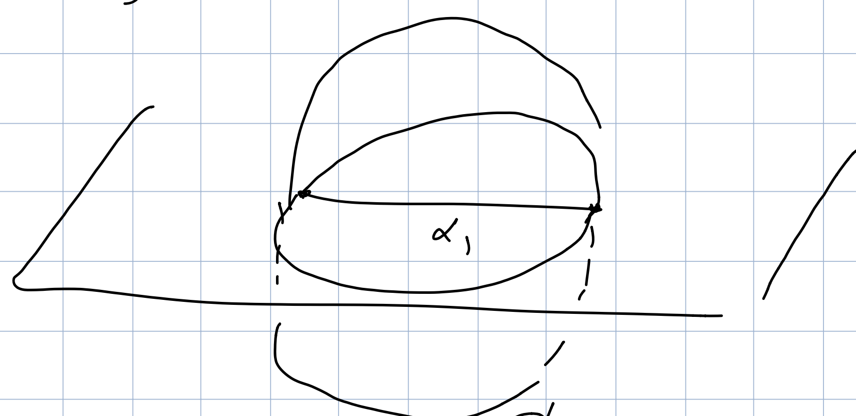
If there is more than one circle two of them are parallel on S (also on Σ). Take two consecutive ones on S: together bound annulus $A \subset S$; also bound annulus $A' \subset \Sigma$ (could happen that A' meets S not at its ∂ only but A meets Σ at ∂ only)

Now $A \cup A' = T = \partial \Pi$



can push Π below Σ thus removing intersection

Eventually $S \cap \Sigma = 1$ circle, $S \cap \Delta_i = \phi$



S trivial sphere separating α_1 from rest of K

$\Rightarrow$ gives trivial splitting. $\boxtimes$

Prop: if $K_{p,q}$ is non-trivial ((p,q) coprime, $p,q \geq 2$).
then $K_{p,q}$ is prime.

Proof: suppose $K_{p,q} = T = \partial \mathbb{T}_1 = \partial \mathbb{T}_2$.

Take $S \cong S^2$, $S \cap K_{p,q} = 2$ pts
suppose $S \pitchfork T$. Take $\gamma \subset S \cap T$ innermost γ circle on S.

$\Rightarrow \gamma = \partial D \quad D \subset \mathbb{T}_1 \quad D \cong D^2$.

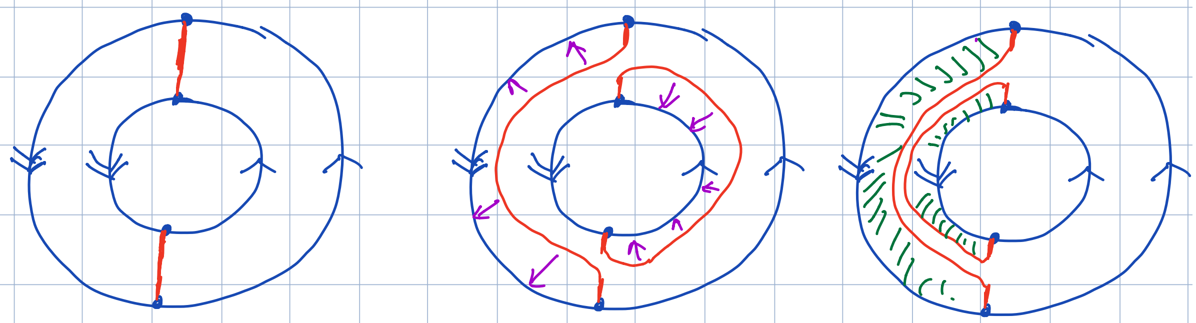
Possibilities:

1. $\gamma = \partial \Delta$, $\Delta \subset T \Rightarrow \Delta \cap K_{p,q} = \emptyset$
   $\Delta \cup D = S^2 = \partial B^3$ can isotope D across B^3
   part of Δ reducing intersection

2. $\gamma \,//\, K_{p,q}$, $\gamma \cap K_{p,q} = \emptyset$
   $\Rightarrow K_{p,q} = \gamma$ up to isotopy
   but bounds $D \Rightarrow$ trivial: absurd.

3. $\gamma \cap K_{p,q} = 2$ pts.

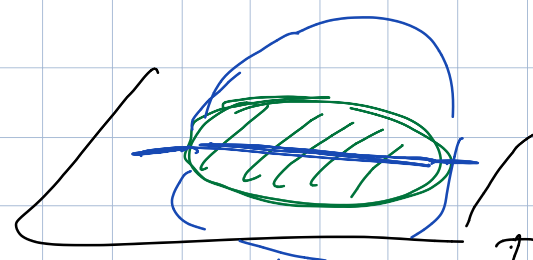
Cut T open along $K_{p,q}$ getting annulus; look at γ in this annulus:



3.0 — two circles: no

3.1 — $\sigma = K_{p,q}/\text{isotopy}$ absurd

3.2 — $\sigma = \sigma \# \Delta$, Δ CT



S^2 gives trivial splitting. ∎

By what said before: $K_1 \# \dots \# K_n$ well-defined.

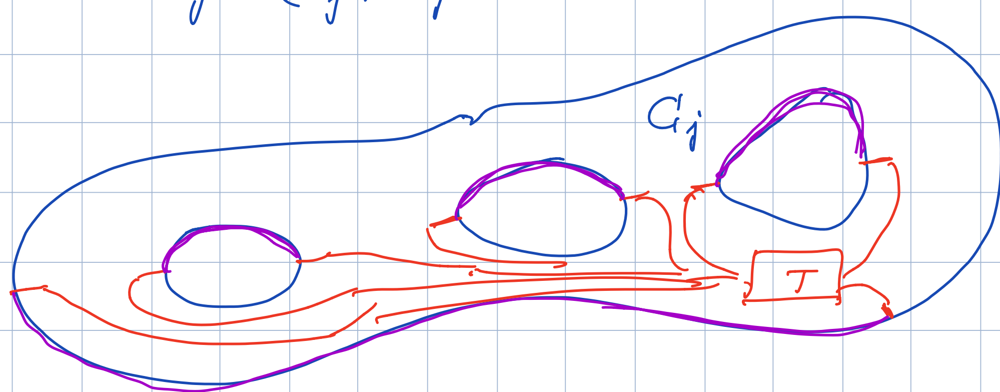
<u>Thm</u>: if K is non-trivial then K can be uniquely expressed as $K_0 \# \dots \# K_n$ with K_i prime up to reordering.

<u>Proof</u>: existence of decomposition from genus.

Uniqueness. Note first that a decomposition

$$K = K_0 \# \dots \# K_n$$

can be realized by a system $\mathcal{S} = S_1 \cup \dots \cup S_m$ where $S_i \cong S^2$, $S_i \cap S_j = \emptyset$ for $i \neq j$, $S_i \cap K = 2$ pts, the components of $\overline{S^3 \setminus (S_0 \cup \dots \cup S_m)} = C_1 \cup \dots \cup C_n$ and $K_j = (C_j, C_j \cap K)$:



namely $K_j = C_j \cap K \cup$ one arc on each component of ∂C_j.

Follows from fact that $\#$ well-defined and can be realized as explained above

trading 

Suppose $\mathcal{S} = S_1 \cup \ldots \cup S_m$
$\mathcal{S}' = S_1' \cup \ldots \cup S_{m'}'$
give two expressions of K as $\#$ of primes.

Can suppose that any two $S \subset \mathcal{S}$, $S' \subset \mathcal{S}'$ either coincide or are transverse (possibly disjoint).

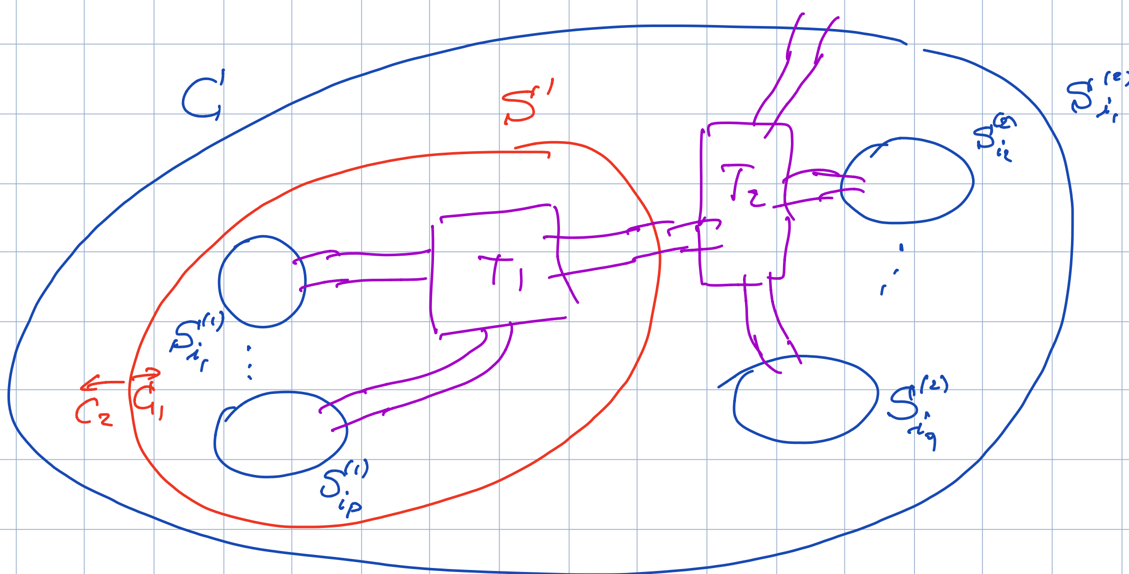
Claim: if there exists $S' \subset \mathcal{S}'$ s.t. $S' \cap \mathcal{S} = \emptyset$ then there exists $S \subset \mathcal{S}$ s.t. $S \not\subset \mathcal{S}'$ and $\mathcal{S}_{NEW} = (\mathcal{S} \setminus S) \cup S'$ new system of spheres providing the same expression of K as $\#$ of primes as $\mathcal{S}$.

Note: after this operation (on the symmetric one)
- the number of spheres in $\mathcal{S}$ and in $\mathcal{S}'$ is unchanged
- the decompositions given by $\mathcal{S}$ and by $\mathcal{S}'$ are unchanged
- the number of spheres shared by $\mathcal{S}$ and $\mathcal{S}'$ is increased.

$\Longrightarrow$ the process comes to an end at which all spheres in $\mathcal{S}'$ meet $\mathcal{S}$ and all spheres in $\mathcal{S}$ meet $\mathcal{S}'$; so either $\mathcal{S} = \mathcal{S}'$ or there is some transverse intersection.

Proof of Claim. Have $S' \subset \mathcal{S}'$ with $S' \cap \mathcal{S} = \emptyset$

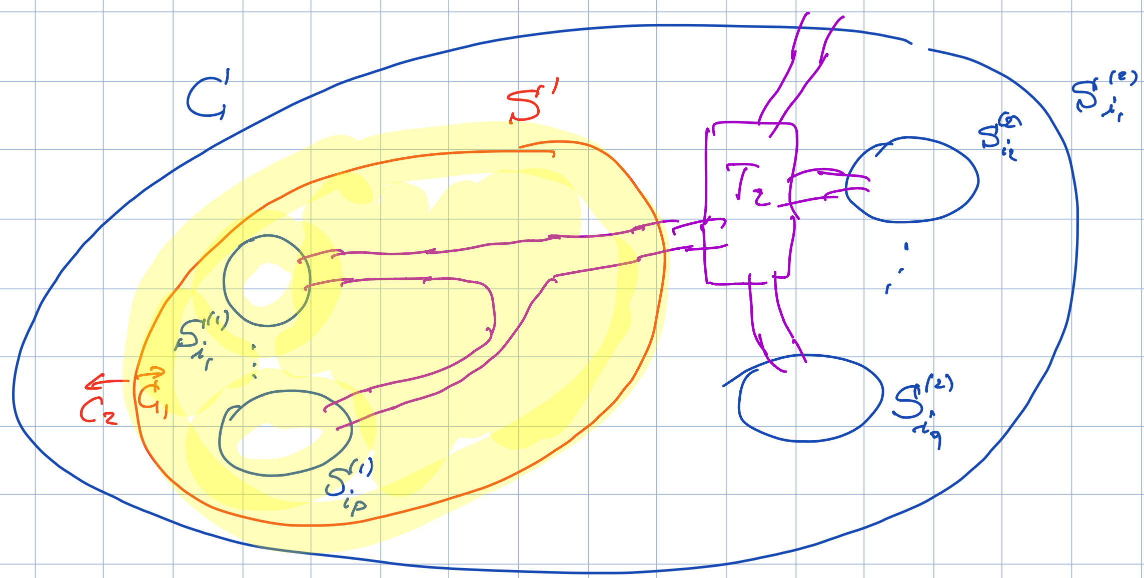
$\Rightarrow \exists$ component C of $S^3 \setminus \mathcal{S}$ that contains S'.



Since $\widehat{(C_1, C_1 \cap K)}$ is prime $\Rightarrow S'$ must give a trivial splitting of it

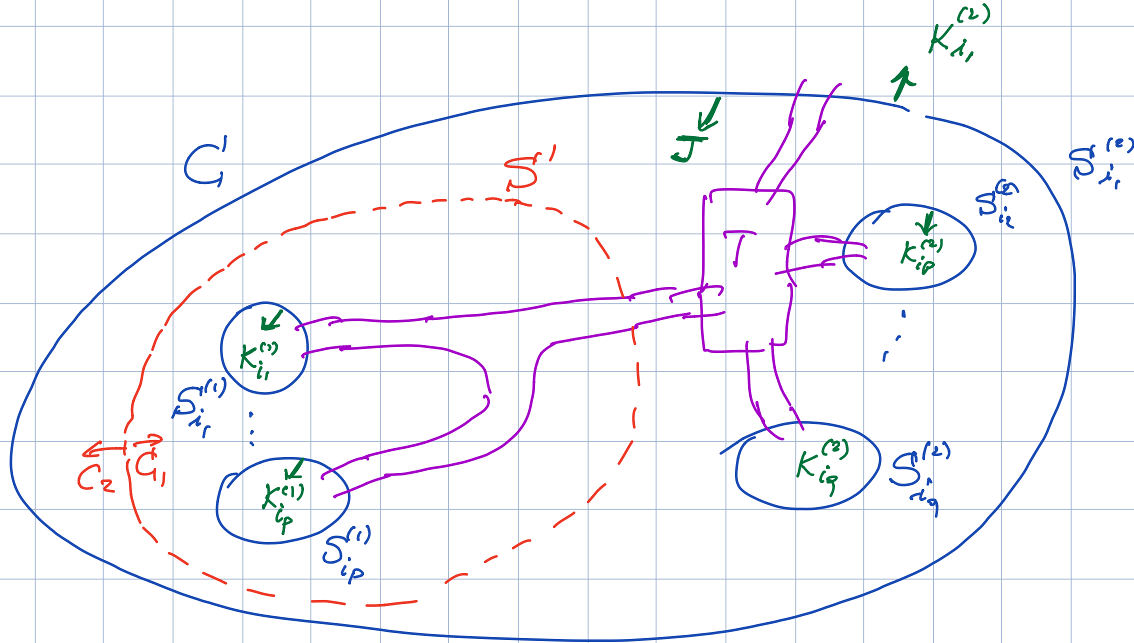
$\Rightarrow$ either T_1 or T_2 is trivial tangle.

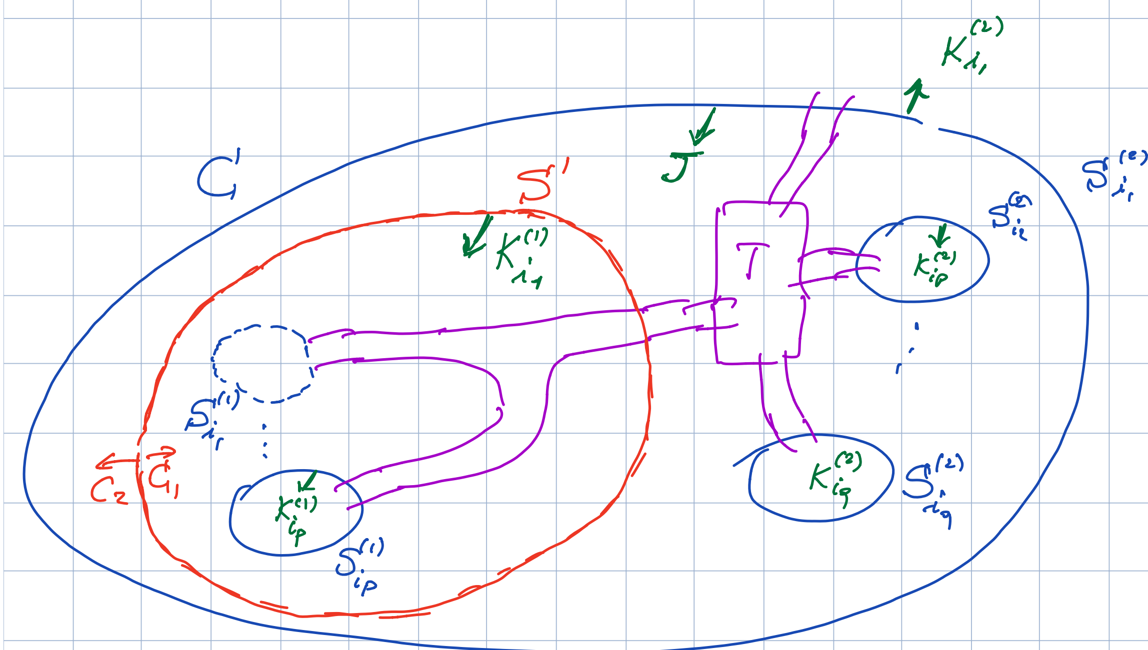
Suppose it's T_1 in picture.



I notice that not all of $S^{(1)}_{i_1}, \dots, S^{(1)}_{i_p}$ can belong to $\mathcal{J}'$ for otherwise one component of the splitting given by $\mathcal{J}'$ would be trivial.

We conclude by showing that any of the components of ∂C_1 not in $\mathcal{J}'$ can be traded with S'.





wlog $S_{i_1}^{(1)} \not\subset J'$ $\Rightarrow (J \setminus S_{i_1}^{(1)}) \cup S'$

gives same decomposition of K
on $\#$ as Σ did.

Claim proved.

Left to remove transverse intersection.