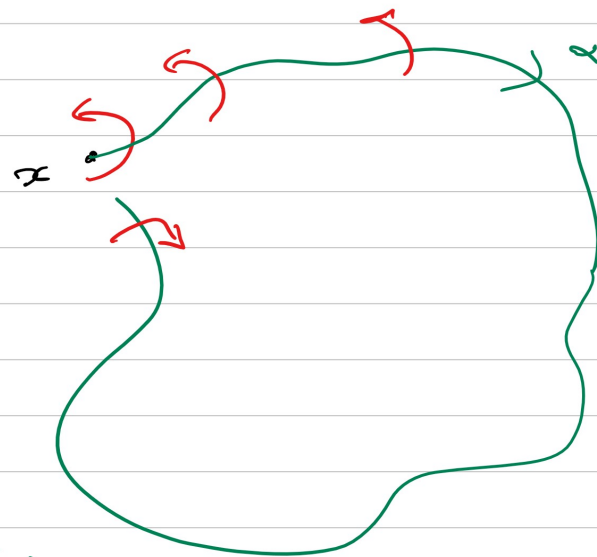
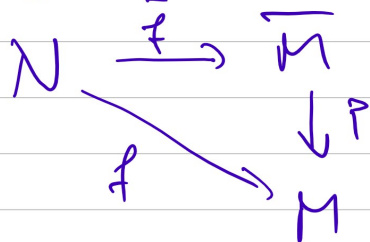


Oss: l'orientazione di una varietà ha senso per varietà TOP

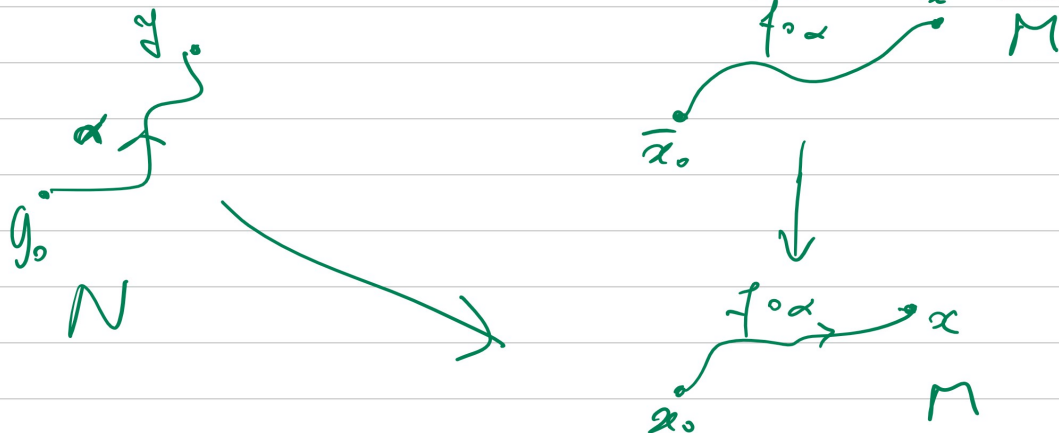
M varietà non orientabile $\Rightarrow \exists \bar{M} \xrightarrow{p} M$ $\bar{M} = \{(x, o) : x \in M, o \text{ orient. vicino a } x\}$

Prop: se $f: N \rightarrow M$ riv. con N orient, M no $\exists$

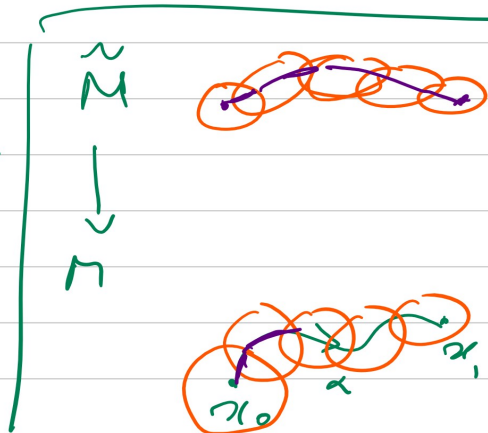


Dim: scegli $x_0 \in M$, $y_0 \in f^{-1}(x_0)$, $\bar{x}_0 \in \bar{f}^{-1}(x_0)$.

Definisco $\bar{f}(y_0) = \bar{x}_0$ e $\forall y \in N$ definisco $\bar{f}(y)$ così:



Per ogni $\bar{f}(y) = \bar{x}$.

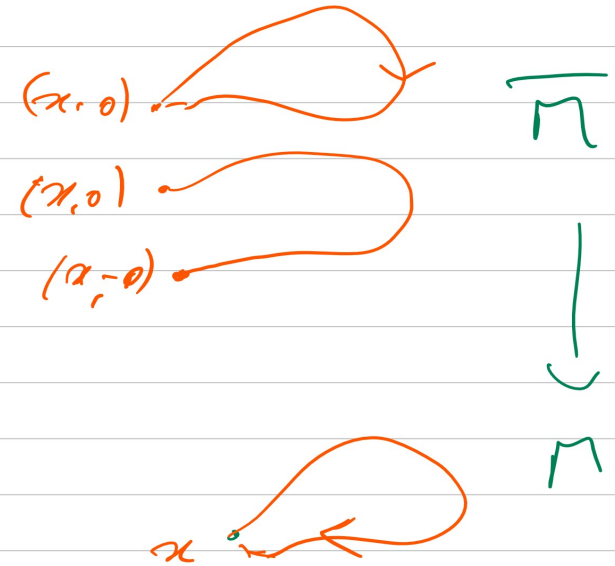
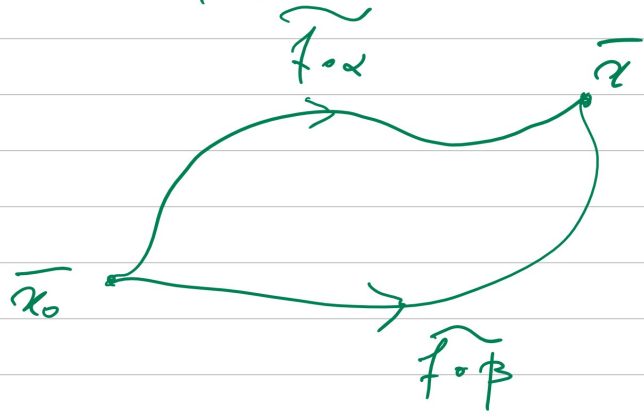


Se uno un altro β da $y_0 = y$ ho α, β^{-1} laccio in N

$\Rightarrow$ preserve orientazione $\Rightarrow f(\alpha \cdot \beta^{-1}) = f(\alpha) \cdot f(\beta)^{-1}$ preserve orientazione in M

$\Rightarrow$ si solleva a laccio in $\bar{M}$

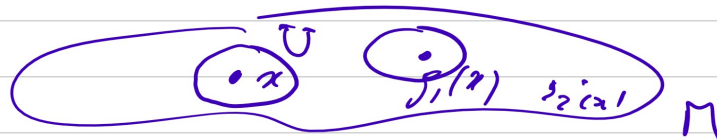
$\Rightarrow \tilde{f}(\alpha) \in \tilde{f}(\beta)$ hanno stesso estremo



G gruppo agisce su M e ho omom $G \rightarrow \text{Aut}(M)$ (TOP DIFF PL)

$(g \circ h)(\alpha) = g(h(\alpha))$

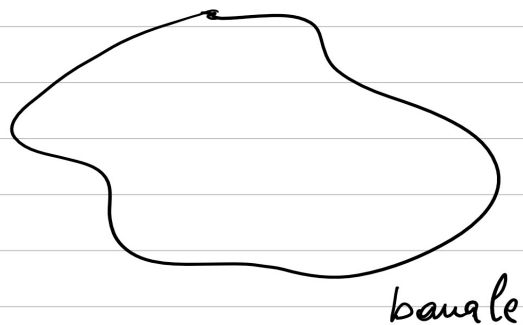
Ese: G finito che agisce su M senza pt fissi $\Rightarrow \pi: M \rightarrow M/G$ riv. tre a due



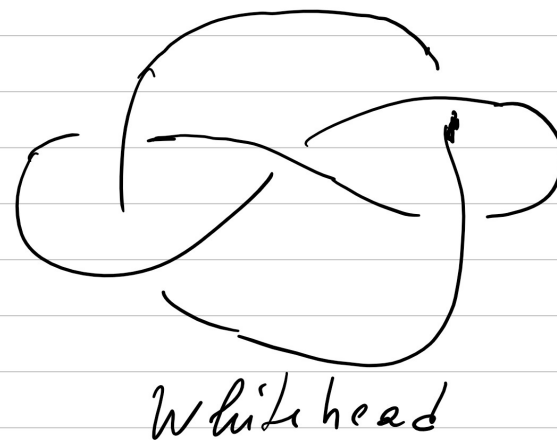
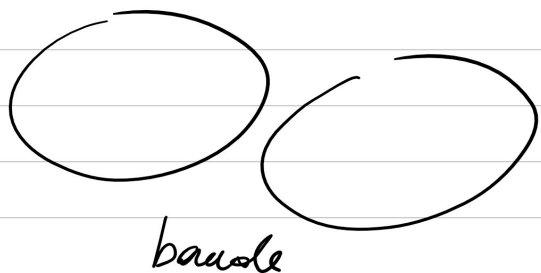
Nodi e link

④

nodi



link



"uguoli" se deformabili l'uno nell'altro.

Fatto: i tre nodi e i tre link sopra sono distinti.

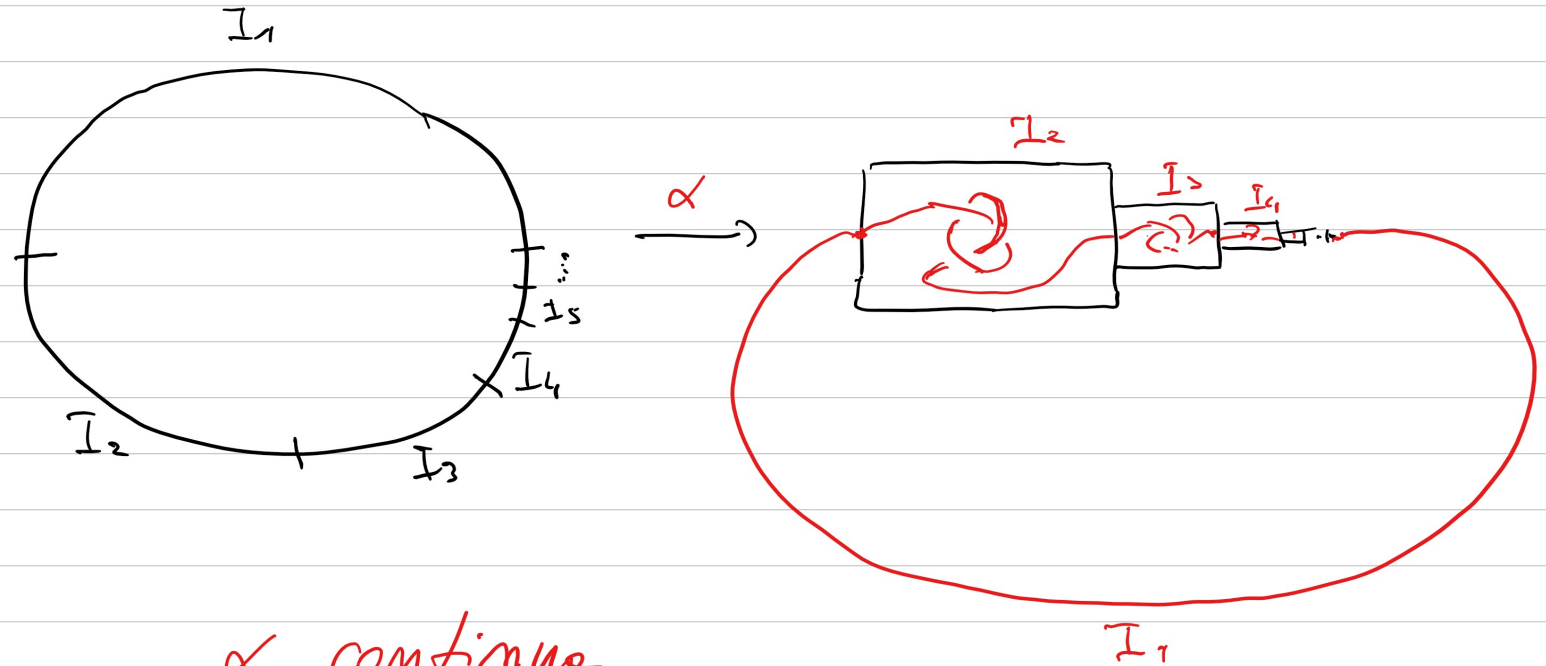
"Def": $\alpha: S^1 \rightarrow \mathbb{R}^3$ continuous injective

$\alpha_0 \sim \alpha_1 \iff \exists \alpha_t \quad \alpha: S^1 \times [0,1] \rightarrow \mathbb{R}^3$ continuous injective

ISOTOPIA

Non ve bene

Wild embedding:



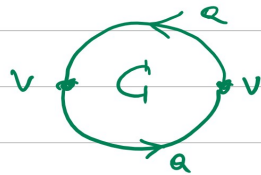
α continuous
immagine è "infinitamente annodata"

(3)

Ese: se M è orientabile, M/G orient $\iff$ tutti gli α di G preservano orientazione

Es: $P^2(\mathbb{R})$ non orientabile; $P^2(\mathbb{R}) = S^2 / \langle I_3, -I_3 \rangle$

$\det(-I_3) = -1 \implies -I_3$ non preserva orientazione.



$$\begin{aligned}\partial V &= 0 \\ \partial a &= 0 \\ \partial C_1 &= 2a\end{aligned}$$

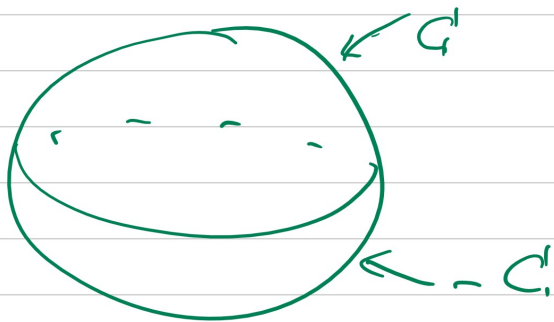
$$\begin{aligned}H_0 &= \mathbb{Z} \\ H_1 &= \mathbb{Z}/2 \\ H_2 &= 0\end{aligned}$$

Es: $P^3(\mathbb{R}) = S^3 / \langle I_4, -I_4 \rangle$ $\det(-I_4) = +1 \implies P^3(\mathbb{R})$ orient.

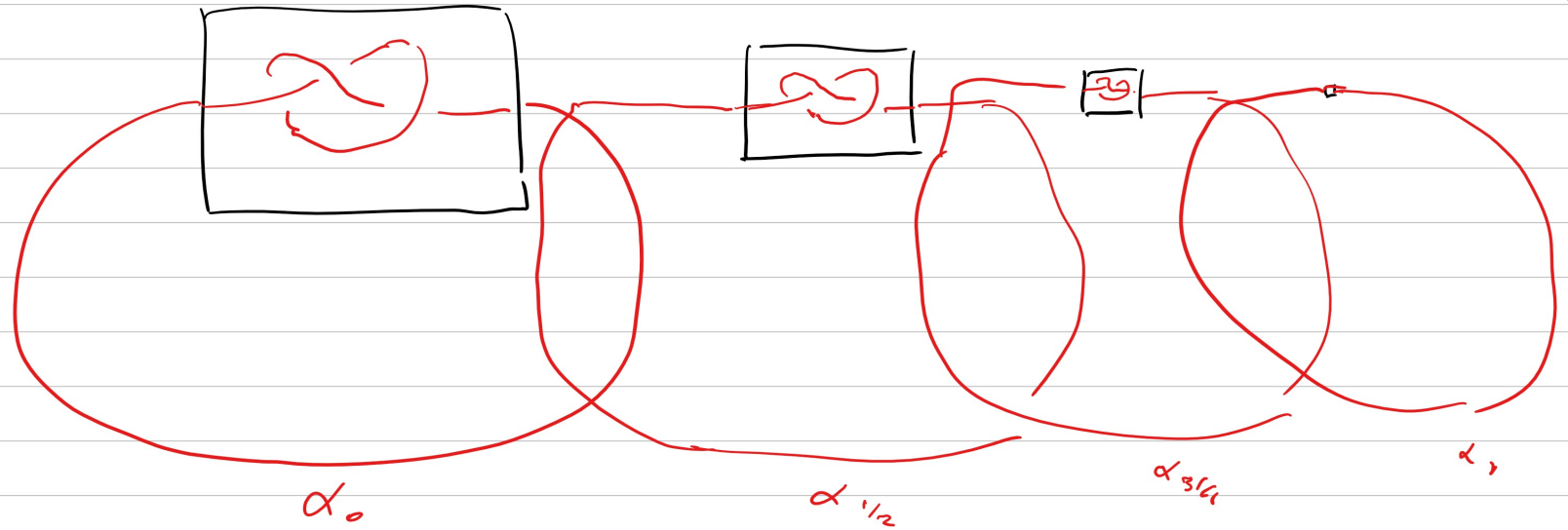
$$P^3(\mathbb{R}) = P^2(\mathbb{R}) \cup D$$

$$\begin{aligned}\partial V &= 0 \\ \partial a &= 0 \\ \partial C_1 &= 2a \\ \partial D &= 0\end{aligned}$$

$$\implies H_3 = \mathbb{Z}$$



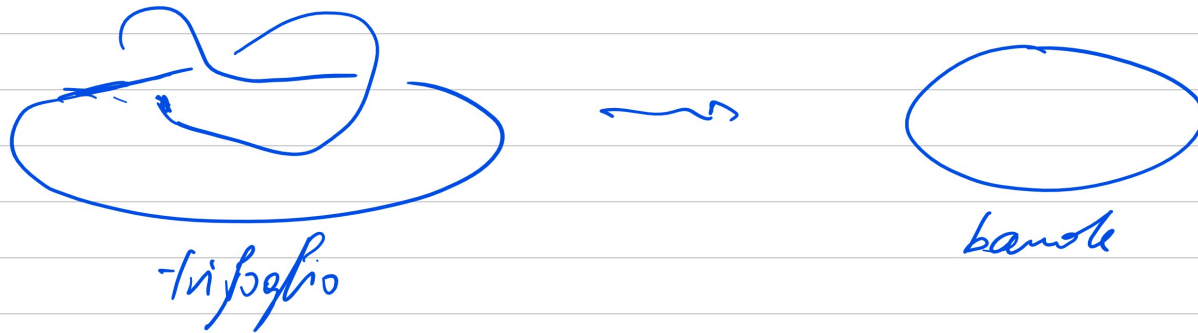
(6)

Wild isotopy:

$$\alpha_t : S^1 \rightarrow \mathbb{R}^3$$

$$\alpha : S^1 \times [0, 1] \rightarrow \mathbb{R}^3$$

continuous



Due modi per aggiungere: DIFF PL

Spesso è naturale pensare i nodi in $S^3 = \mathbb{R}^3 \cup \{\infty\}$.

DIFF: modo: $\alpha: S^1 \rightarrow \mathbb{R}^3$ diff con $\alpha'(z) \neq 0 \quad \forall z$

$\widehat{\text{modo}}$: $\alpha: S^1 \rightarrow S^3$ diff. con $\alpha'(z) \neq 0 \quad \forall z$ $\square \square \square$

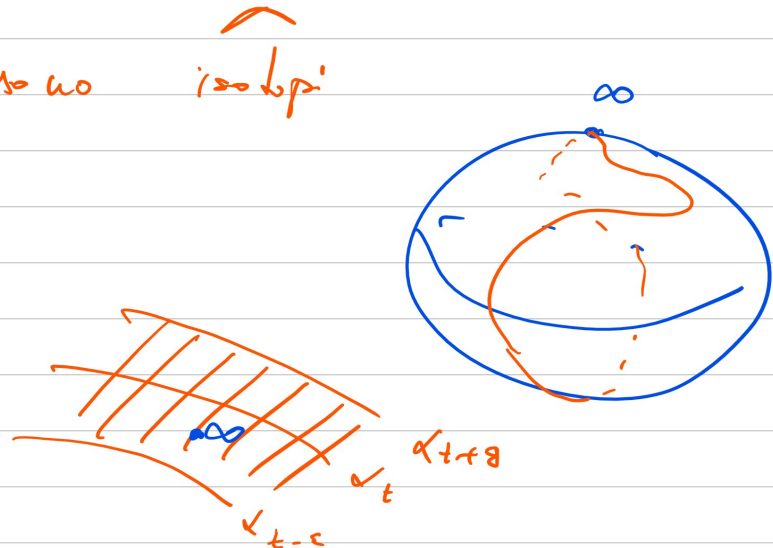
isotopia tra α_0, α_1 $\alpha: S^1 \times [0,1] \rightarrow \mathbb{R}^3$ diff. $\alpha'_t(z) \neq 0 \quad \forall t, z$

$\widehat{\text{isotopia}}$ $\dots \dots \dots \rightarrow S^3$

Fatti:

- $\widehat{\text{modo}}$ è $\widehat{\text{modo}}$; isotopi sono isotopi
- $\widehat{\text{modo}}$ è isotopo a modo
modi isotopi sono isotopi

$\Rightarrow \mathbb{R}^3, S^3$ sono buone.

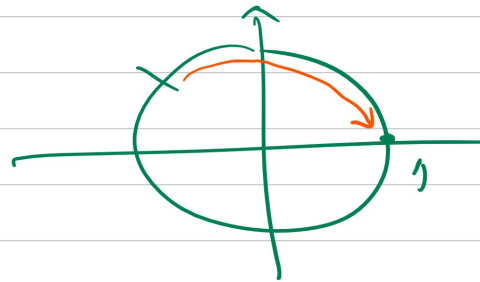


Prop: dato $K \subset S^3$ esistono al meno di isotopie al più
due $\alpha: S^1 \rightarrow S^3$ di $\mathcal{H}$ con $\alpha'(z) \neq 0 \forall z$ t.c. $\alpha(S^1) = K$.
iniettive

Dim: $\alpha_0, \alpha_1: S^1 \rightarrow S^3$ iniettive con $\alpha_0(S^1) = \alpha_1(S^1) = K$

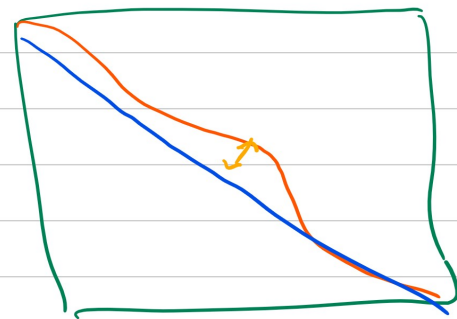
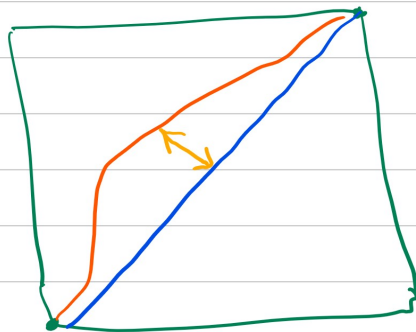
considero $\gamma: S^1 \rightarrow S^1$ $\gamma_0 = \alpha_1^{-1} \circ \alpha_0$; basta vedere che $\gamma_0: S^1 \rightarrow S^1$
è isotopia e $\gamma_1 = \begin{cases} \text{id} \\ \text{congiungo} \end{cases}$ ($\alpha_t = \alpha_1 \circ \gamma_t$) -

• wlog/isotopia $\gamma_0(1) = 1$



(se no ruoto)

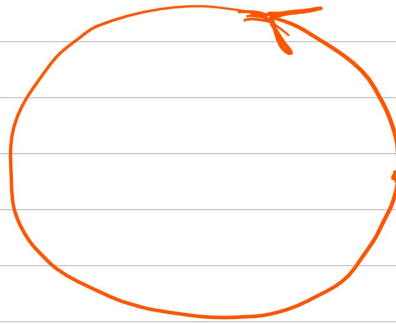
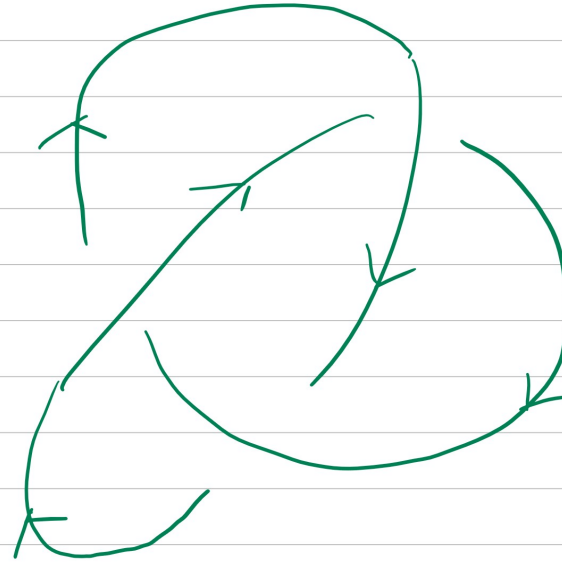
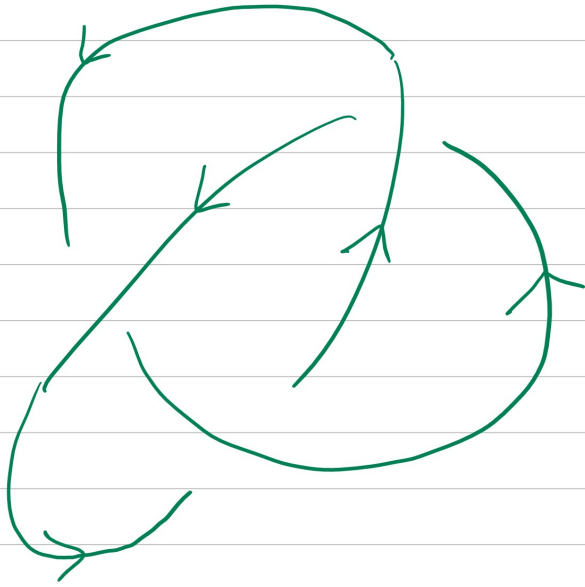
• $S^1 = [0, 2\pi] / 0 \sim 2\pi$



$$\gamma_t(z) = (1-t)\gamma_0(z) + t \cdot z$$

$$\gamma_t(z) = (1-t)\gamma_0(z) + t \cdot \bar{z}$$

VA



PROBLEMA :

dato K orientato
 è esso isotopo
 a $-K$ (lui con
 orientaz. opposta?)

INVERTIBILE

Posso fare teoria per "nodi orientati" = nodo con verso -

Loro hanno una sola parametrizzazione / isotopie -

Fatto: ce
 ne sono
 che invert.
 che no -

TRIFOGLIO, BANACE
di

Prop: dati $\alpha_0, \alpha_1 : S^n \rightarrow S^3$ sono fatti equivalenti:

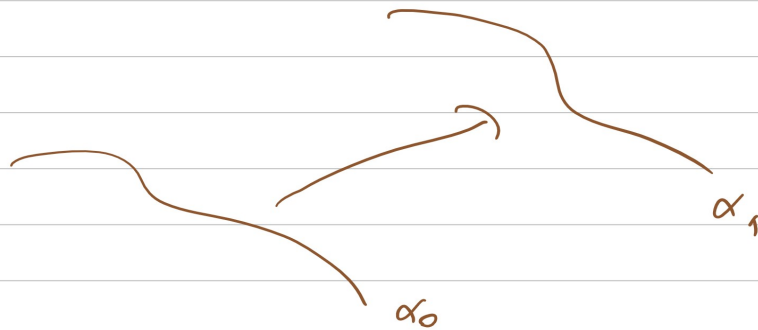
(1) α_0, α_1 isotopi

(2) $\exists f_t : S^3 \rightarrow S^3$ t.c. $f_0 = \text{id}$ $f_1 \circ \alpha_0 = \alpha_1$

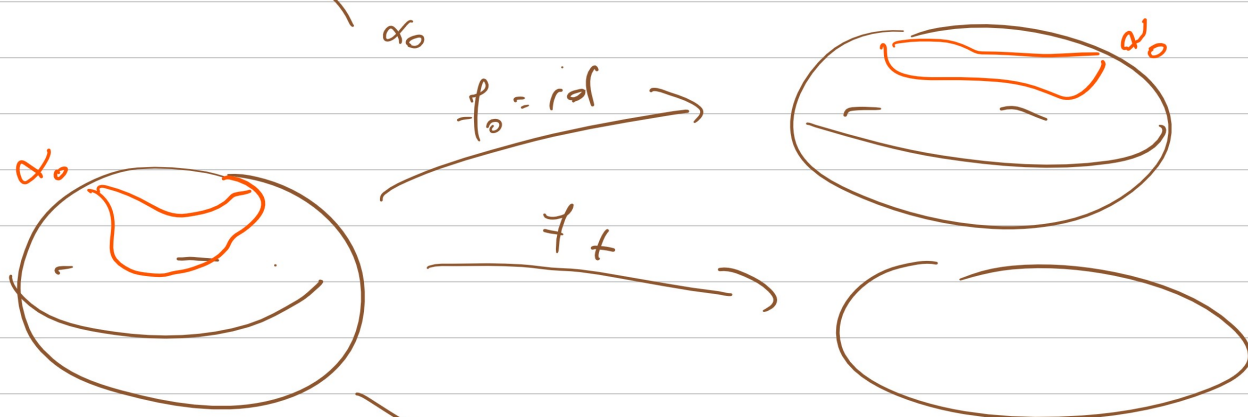
(3) $\exists f : S^3 \rightarrow S^3$ che preserve l'orientazione t.c. $f \circ \alpha_0 = \alpha_1$.

Dimo: (2) $\Rightarrow$ (1)
(2) $\Rightarrow$ (3)

(1)

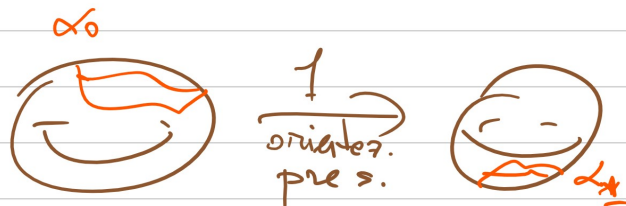


(2)



isotopia ambiente

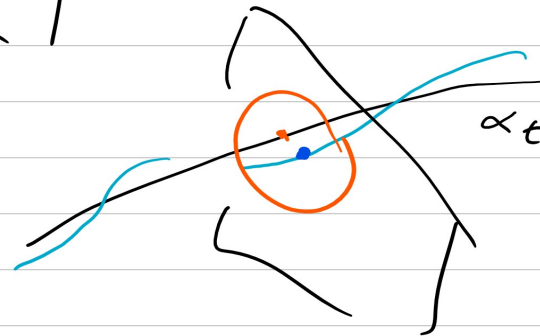
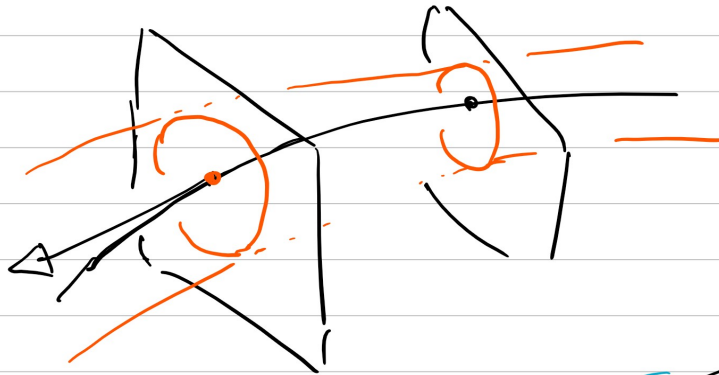
(3)



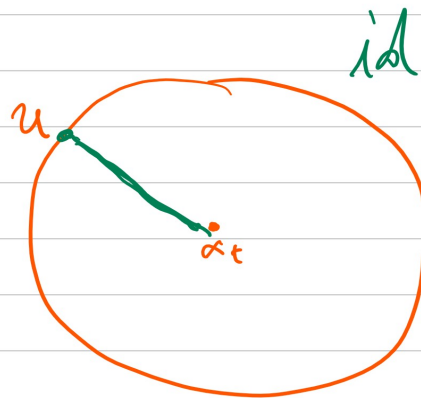
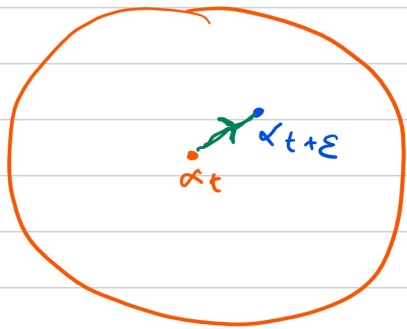
(1) $\Rightarrow$ (2) Devo vedere che una isotopia del nodo si estende all'ambiente.

$(\alpha_t)_{t \in [0,1]}$

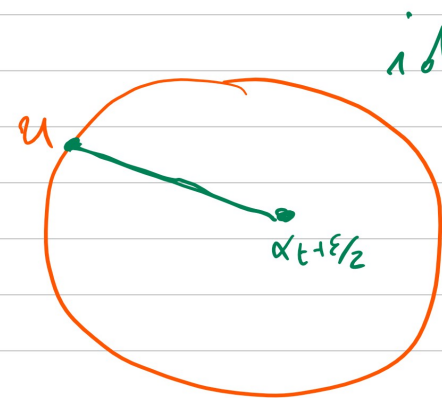
usando compattezza di $[0,1]$ basta vedere che per ε molto piccolo l'isotopia tra α_t e $\alpha_{t+\varepsilon}$ si estende all'ambiente.



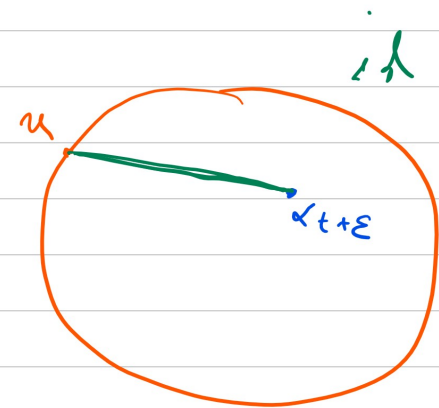
$\exists U$ intorno di α_t
 $U = D^2 \times S^1$ $\alpha_t = \{0\} \times S^1$



$f_\varepsilon = id$



$f_{t+\varepsilon/2}$



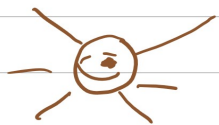
$f_{t+\varepsilon}$

(12)

(3) $\Rightarrow$ (2) Basta vedere che $f: \mathbb{S}^3 \rightarrow \mathbb{S}^3$ che
preserva orientazione è isotopo a id.

Idea: wlog $f(0) = 0$ $f(\infty) = \infty$ $df_0 = A \in GL_+(n, \mathbb{R})$

- isotopo f a A facendole diventare A su pelle di saggio $\nearrow$



- $GL_+(n, \mathbb{R})$ è connesso per archi $\Rightarrow \exists A_t$ $A_0 = A$ $A_1 = I_n$

(• rendere continuo $G-S'$
• $SO(m)$ connesso per archi per induzione su m) - $\square$

_____ o _____

Esempio: $(\tilde{\Sigma}, S, 7, 3; \underline{[4, 2, 1]}, [5, 2], \pi_3)$

(16)

Quali Γ possiamo realizzare in tale $\mathcal{L}^{\pi_3}$? ($\Gamma \subset \tilde{\Sigma}$)

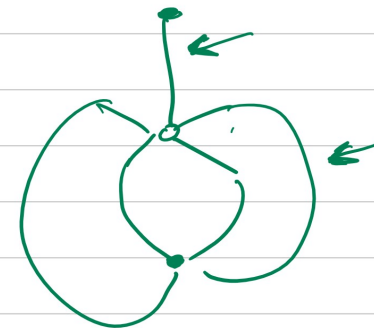
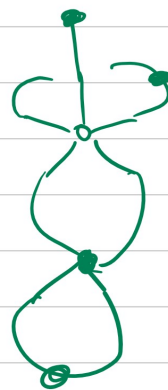
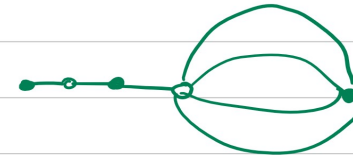
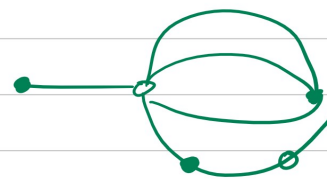
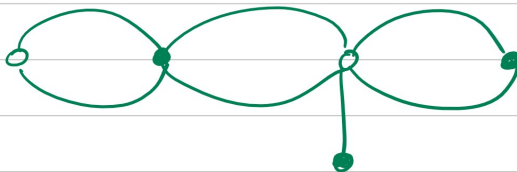
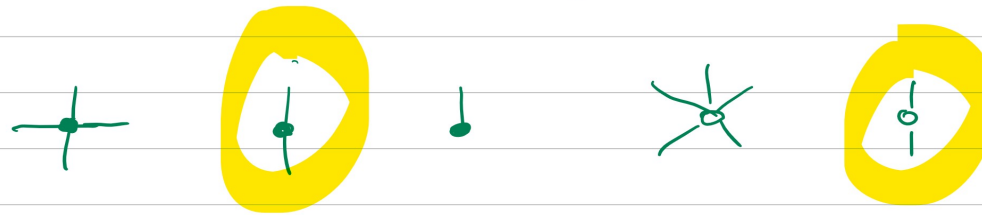
RH: $2(1-g) - (3+2+l_3) = 7(2-3)$

$$2 - 2g - 5 - l_3 = -7$$

$$2g = 4 - l_3$$

$$\begin{array}{ll} g=0 & l_3=4 \\ g=1 & l_3=2 \end{array}$$

Assattamente:

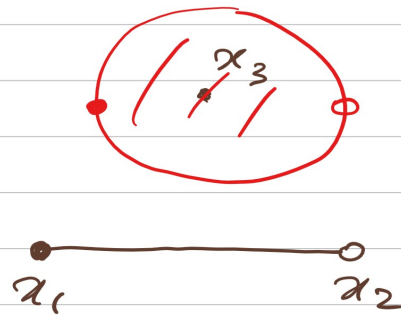


gli Γ_i hanno valenze $\pi_1 = (d_{ij})_{j=1}^{l_1}$

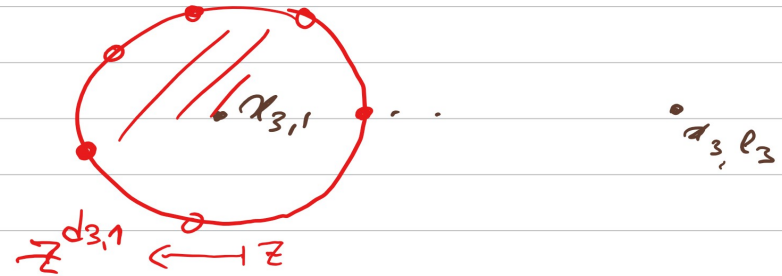
(15)

$i=0$ hanno valenze π_2

ogni segmento unisce 



f
←



$\tilde{\Sigma} \setminus \Gamma$ consiste di l_3 dischi chiusi di cui il j -esimo ha sul bordo, alternati, $d_{3,j}$ e $d_{3,j}$.

Γ denso d'enfant

Risultato: $f \longrightarrow \Gamma$



$$\text{val}(\bullet) = \pi_1$$

$$\text{val}(o) = \pi_2$$

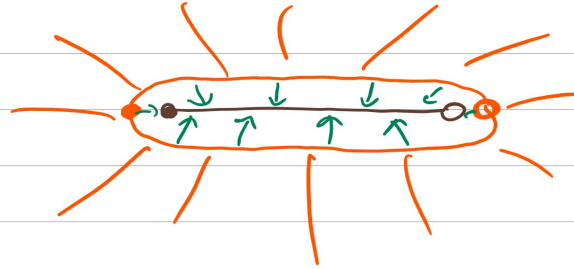
$$\text{lumpR}(\tilde{\Sigma} \setminus \Gamma) = \pi_3$$

Fatto: dato Γ con esiste f : da definire prima su Γ e poi su ogni disco complementare

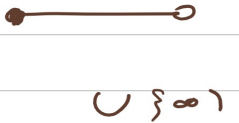
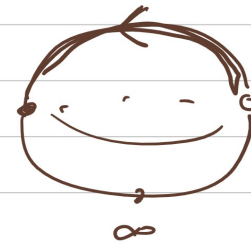
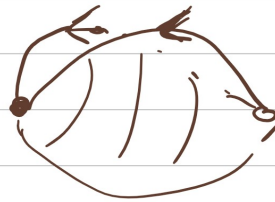
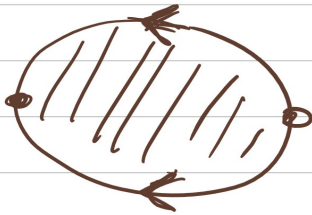
$S \setminus t$ è aperto $\overline{S \setminus t} = S$

(14)

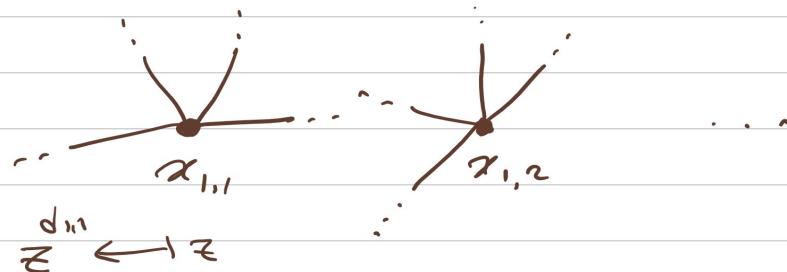
Considero invece che $S \setminus t$, $S \setminus \{t\}$



Considero il complementare di t come un disco chiuso con bordo ∂



Considero $f^{-1}(t) = \Gamma$ grafo con vertici colorati $\bullet$ $\circ$ $\times$:



Problema di Hurwitz:

Doti $(\tilde{\Sigma}, \Sigma, d, m; \pi_1, \dots, \pi_m)$
*candidato dato da
 ramificazione*

$\pi_i = \text{partiz. d. } d$
 $l_i = \text{lungh. d. } \pi_i$
 $\tilde{m} = l_1 + \dots + l_m$

$$\chi(\tilde{\Sigma}) - \tilde{m} = d(\chi(\Sigma) - m)$$

RH

- ...
- $m \cdot d \equiv \tilde{m} \pmod{2}$ - ovvio escluso se $\Sigma, \tilde{\Sigma}$ sono non orient.

esiste un riv. ramificato $f: \tilde{\Sigma} \rightarrow \Sigma$ di grado d

con m pt. di ramificazione e rispettivi prod. loc. $\pi_1, \dots, \pi_m$?
*realizzabile
 o eccezionale ?*

Dessin d'enfant.

$(\tilde{\Sigma}, S, d, 3; \pi_1, \pi_2, \pi_3)$

Suppongo che esista $f: \tilde{\Sigma} \rightarrow S$ che lo realizzo con pt. ram. x_1, x_2, x_3

Prendo un segmento

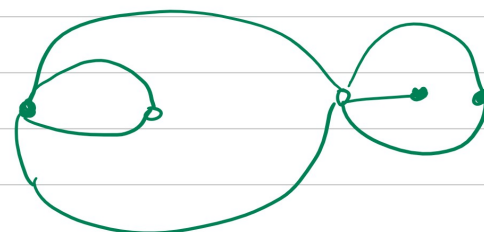
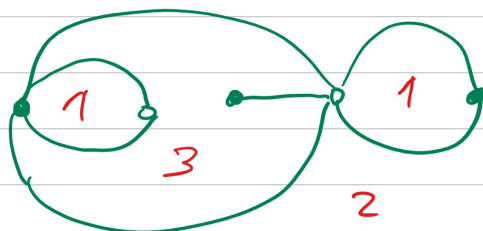
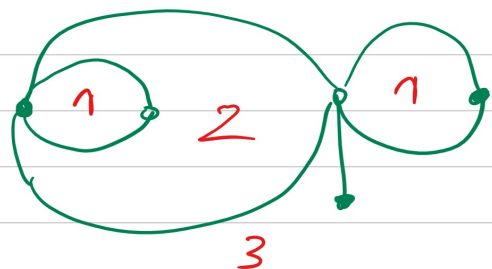
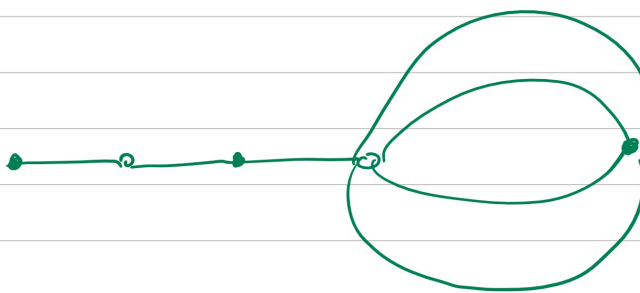
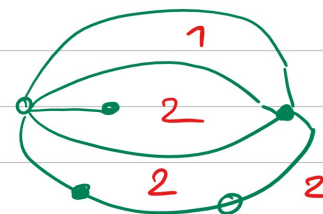
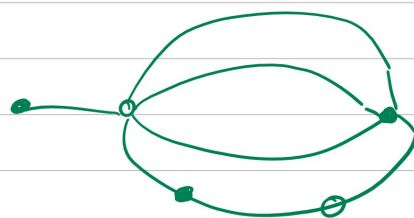
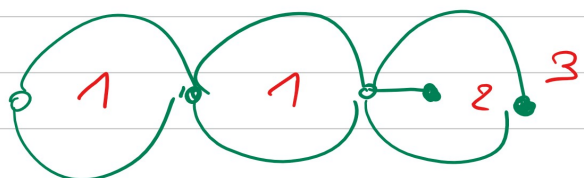
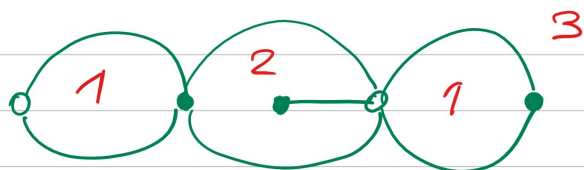
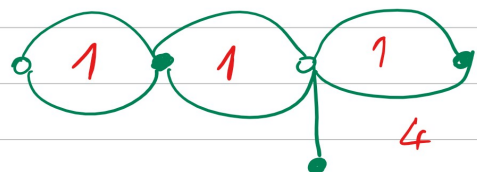


in S .

x_3

caso $l_3 = 4 \quad \tilde{\Sigma} = \mathbb{R}$

(17)



Realizzo $[4, 1, 1, 1]$ $[3, 2, 1, 1]$ $[2, 2, 2, 1]$

to be

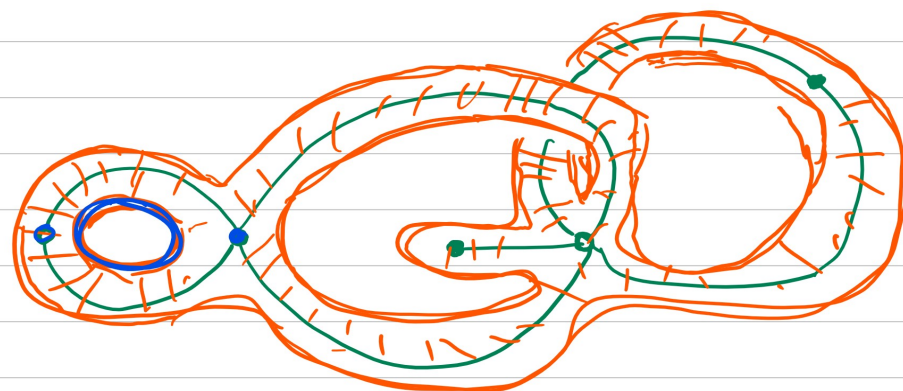
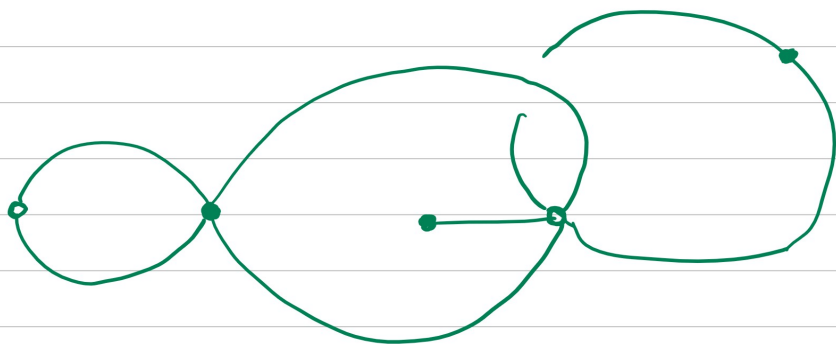
caso $l_3 = 2$ $\tilde{\Sigma} = T$

(18)

per disegnare $\Gamma \subset \tilde{\Sigma} \neq S$

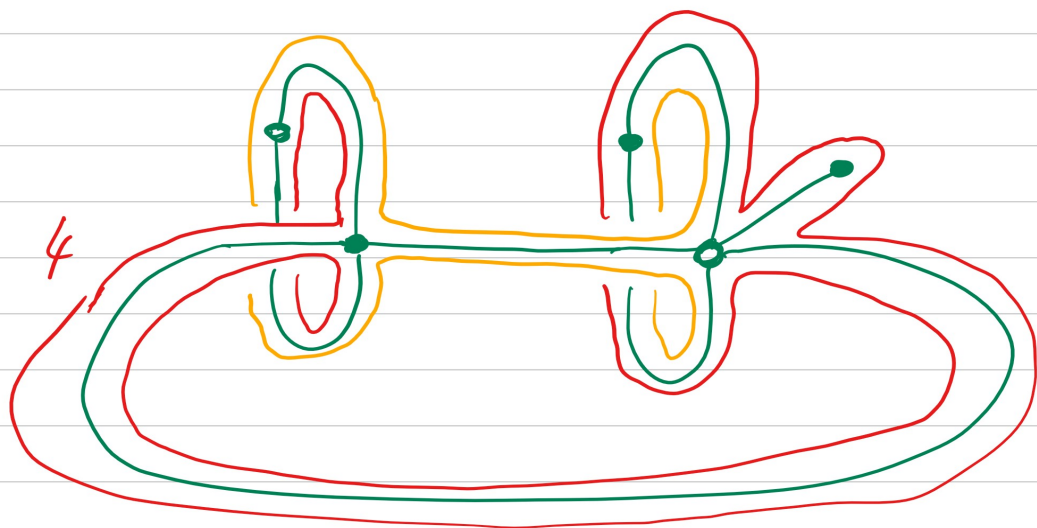
disegno $\Gamma \subset \mathbb{R}^2$ con punti doppi
(intersezioni che appaiono in $\mathbb{R}^2$ ma
non esistono in $\tilde{\Sigma}$).

$\tilde{\Sigma}$ = intorno di Γ in $\mathbb{R}^2$
ignorando pti doppi + dischi attaccati alle circonf. $\perp$ bordo

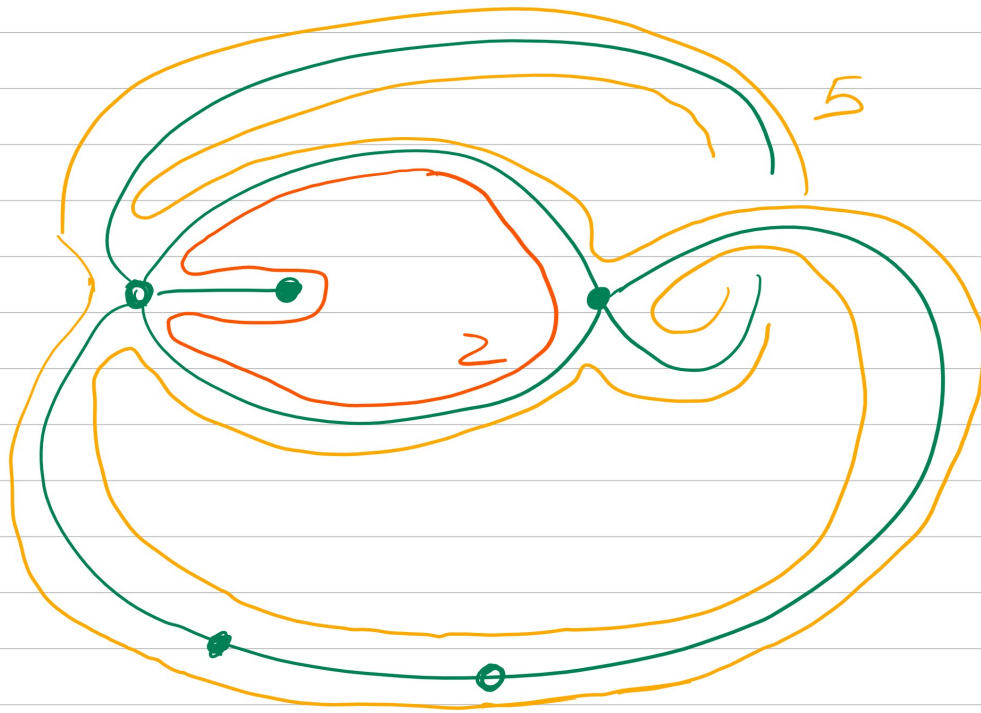


$$\chi = 5 - 7 + 2 = 0 \Rightarrow \tilde{\Sigma} = T$$

$$\pi_3 = [6, 1]$$



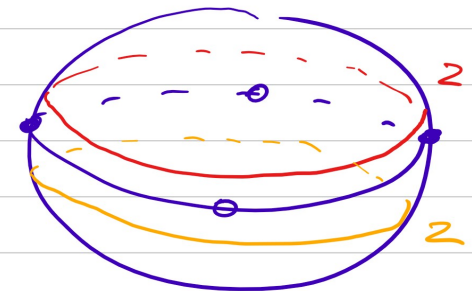
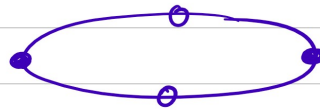
$$\pi_3 = [4, 3]$$



$$\pi_3 = [5, 2]$$

Oss: abbiamo realizzato tutte le possibilità -

Es: $(S, S, 4, 3; [2, 2], [2, 2], [3, 1])$ è realizzabile.



Approccio al Pb. di Hurwitz via monodromie.

(20)

Thm: $(\tilde{\Sigma}, \Sigma, d, M; \pi_1, \dots, \pi_m)$ con Σ di genere g è realizzabile se e solo se esistono

$$\alpha_1, \beta_1, \dots, \alpha_g, \beta_g, \theta_1, \dots, \theta_m \in \mathcal{S}_d \quad \text{t.c.}$$

$$\cdot \prod_{p=1}^g [\alpha_p, \beta_p] \cdot \prod_{i=1}^m \theta_i = \text{id}$$

• i cicli di θ_i hanno lunghezza π_i

• $\langle \alpha_p, \beta_p, \theta_i \rangle$ agisce transitivamente su $\{1, \dots, d\}$.

$$\mathcal{S}_d = \{ f: \{1, \dots, d\} \hookrightarrow \text{bij} \}$$

$$\sigma \in \mathcal{S}_d$$

$$\sigma = (a_1^{(1)} \dots a_{m_1}^{(1)}) \cdot (a_1^{(2)} \dots a_{m_2}^{(2)}) \dots$$

$$\{a_j^{(i)}\}_{\substack{j=1 \dots m_i \\ i=1 \dots m}} = \{1, \dots, d\}$$

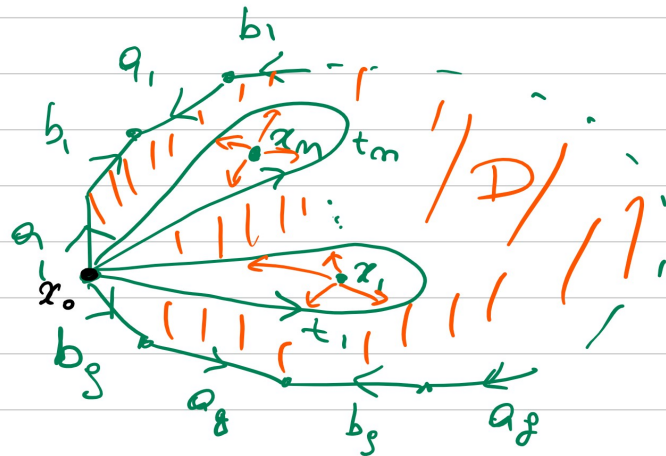
$$(a_1, \dots, a_m) =$$

ciclo

Dimo: dato $f: \tilde{\Sigma} \rightarrow \Sigma$ che realizza

prendo $f^*: \tilde{\Sigma}^* \rightarrow \Sigma^*$

$$\Sigma^* = g \cdot T \setminus \{mpt_i\}$$



$$\pi_1(\Sigma^*) =$$

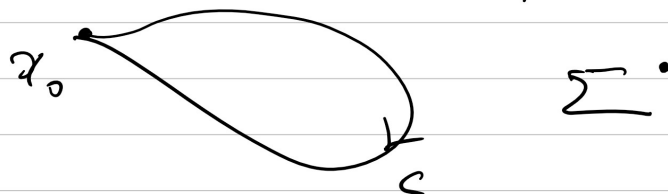
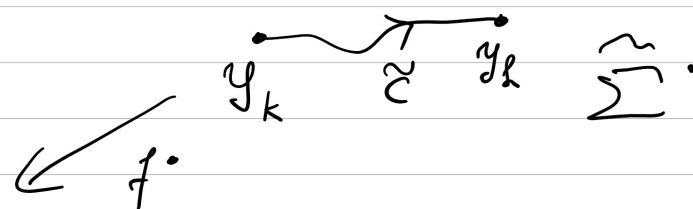
$$\langle a_p, b_p, t_i \mid \partial D \rangle$$

$$\prod_{p=1}^g [a_p, b_p] \cdot \prod_{i=1}^m t_i$$

$$f^*(x_0) = \{y_1, \dots, y_d\}$$

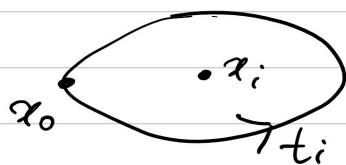
$$p : \pi_1(\Sigma^*) \rightarrow \mathcal{S}_d$$

$$p(c)(k) = h \text{ se}$$

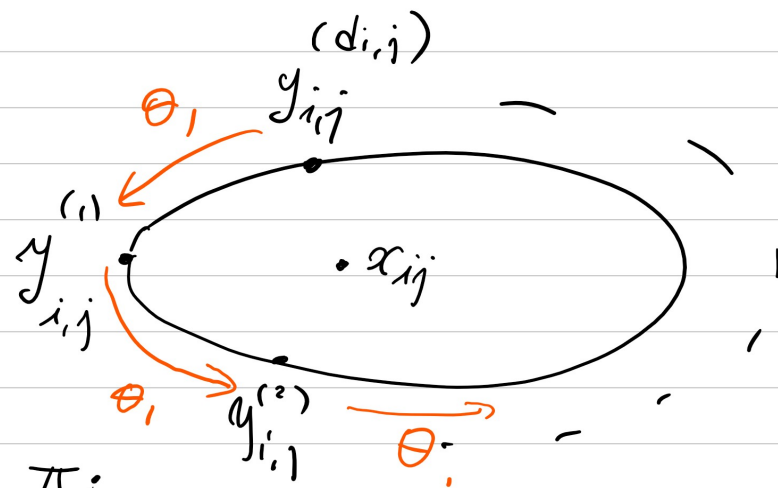


$$\alpha_p = p(a_r) \quad \beta_p = p(b_r) \quad \theta_i = p(t_i)$$

$$\cdot \quad \prod [\alpha_p \beta_p] \cdot \prod \theta_i = id \quad \checkmark$$



$$\xleftarrow{f} \mathbb{Z}^{d_{ij}} \leftarrow \mathbb{Z}$$



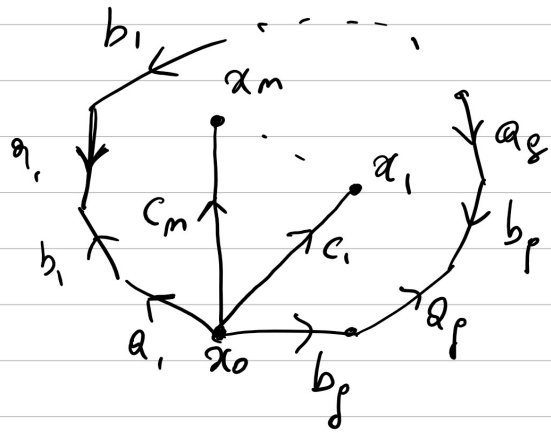
$\Rightarrow$ cicli di θ_i hanno lunghezza π_i

$$\cdot \quad y_k \xrightarrow{\tilde{c}} y_h \quad \Rightarrow \quad h = p((f \circ \tilde{c}))(k) \quad \underline{\underline{OK}}$$

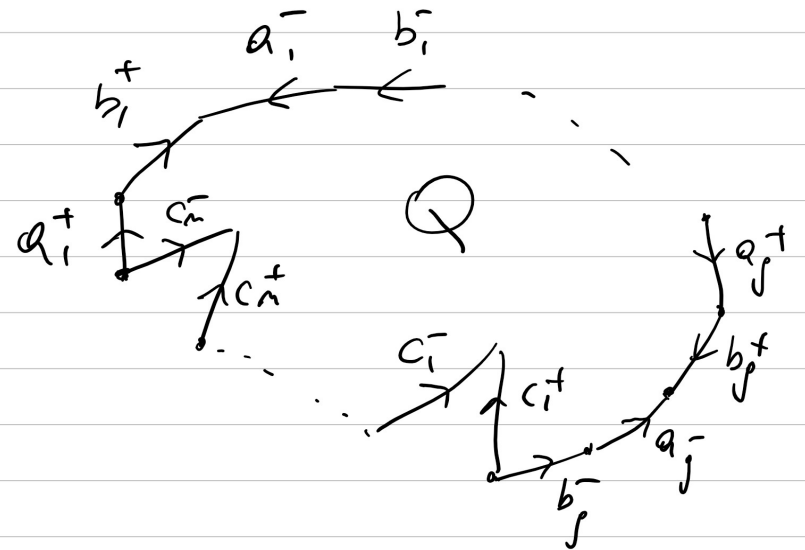
$\tilde{\Sigma}$ connessa

Vicinanze appoggio di avere $\alpha_p, \beta_p, \theta_i$

(22)



Taglio Σ lungo
tali a_p, b_p, c_i



$$\Sigma_1 = \mathbb{Q} / \begin{array}{l} a_p^- = a_p^+ \\ b_p^- = b_p^+ \\ c_i^- = c_i^+ \end{array}$$

Prendo d copie $Q_1, \dots, Q_d$ di Q con tali di Q_m $a_{m,p}^\pm, b_{m,p}^\pm, c_{m,i}^\pm$

$$\widehat{\Sigma} = (Q_1 \sqcup \dots \sqcup Q_d) / \begin{array}{l} a_{m,p}^- = a_{\alpha_p(m),p}^+ \\ b_{m,p}^- = b_{\beta_p(m),p}^+ \\ c_{m,i}^- = c_{\theta_i(m),i}^+ \end{array}$$

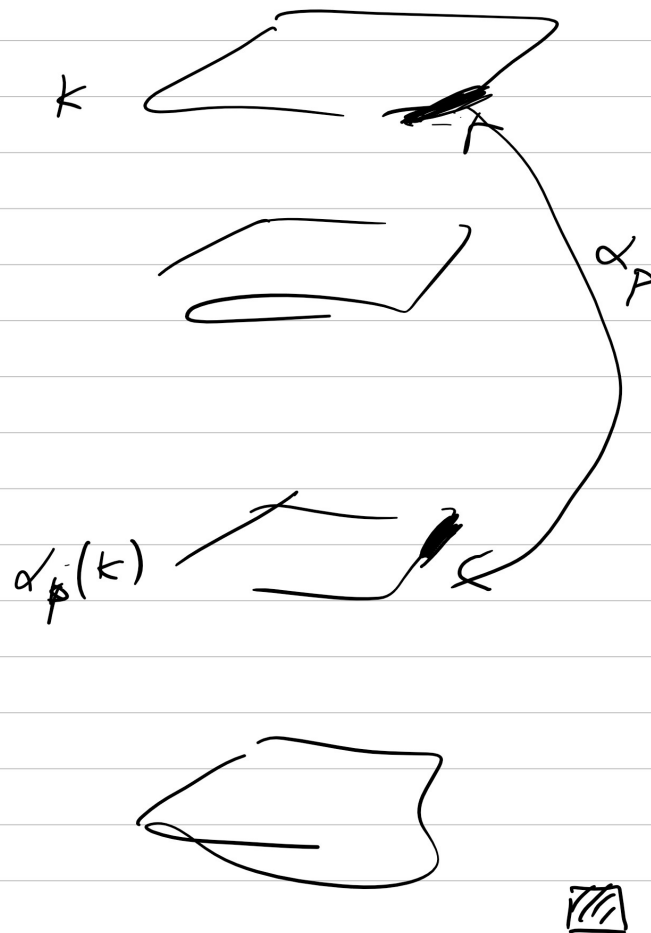
Fatti: • $\tilde{\Sigma}$ è superficie connessa (transitiva?)

• è ben def. un riv. transf. $\tilde{\Sigma} \rightarrow \Sigma$
 ottenuto da $\mathbb{P}_m \rightarrow \mathbb{Q}$ con id

• $\tilde{\Sigma}$ orientabile e $\chi(\tilde{\Sigma})$ è
 determinata da Σ, d, n, n tramite RH

$$\Rightarrow \chi(\tilde{\Sigma}) = \chi(\tilde{\Sigma})$$

$$\Rightarrow \tilde{\Sigma} = \tilde{\Sigma}.$$



Thm: $(\tilde{\Sigma}, \Sigma, d, m; \pi_1, \dots, \pi_m)$

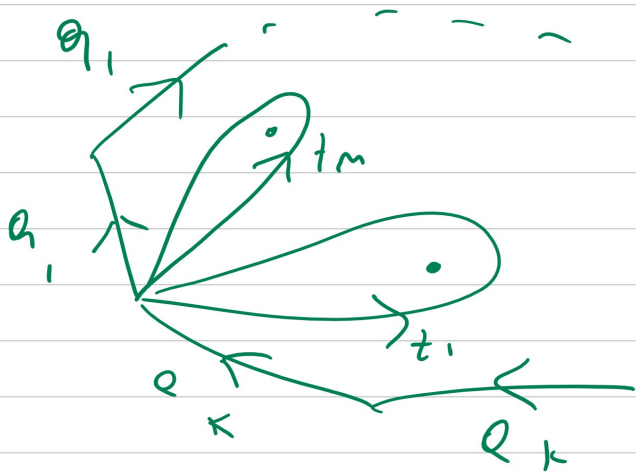
(24)

con $\Sigma = k \cdot \mathbb{P}$ $\bar{e}$ realizzabile $\Leftrightarrow$

$\exists \alpha_1, \dots, \alpha_k, \theta_1, \dots, \theta_m \in \mathcal{S}_d$ t.c.

- $\prod \alpha_p^2 \cdot \prod \theta_i = \text{id}$
- cicli di θ_i lunghezza π_i
- $\langle \alpha_p, \theta_i \rangle$ transitivo
- ogni α_p contiene solo cicli di lunghezza pari $\Leftrightarrow \tilde{\Sigma}$ orientabile

Dimo: $\exists \Leftrightarrow \alpha_p, \theta_i$



$$\pi(\tilde{\Sigma}) = \langle \alpha_p, \theta_i \mid \prod \alpha_p^2 \cdot \prod \theta_i \rangle$$

$$p: \pi(\tilde{\Sigma}) \rightarrow \mathcal{S}_d$$

$$\alpha_p = p(\alpha_p) \quad \theta_i = p(\theta_i)$$

$$\exists \alpha_i, b_i \rightarrow \exists f$$

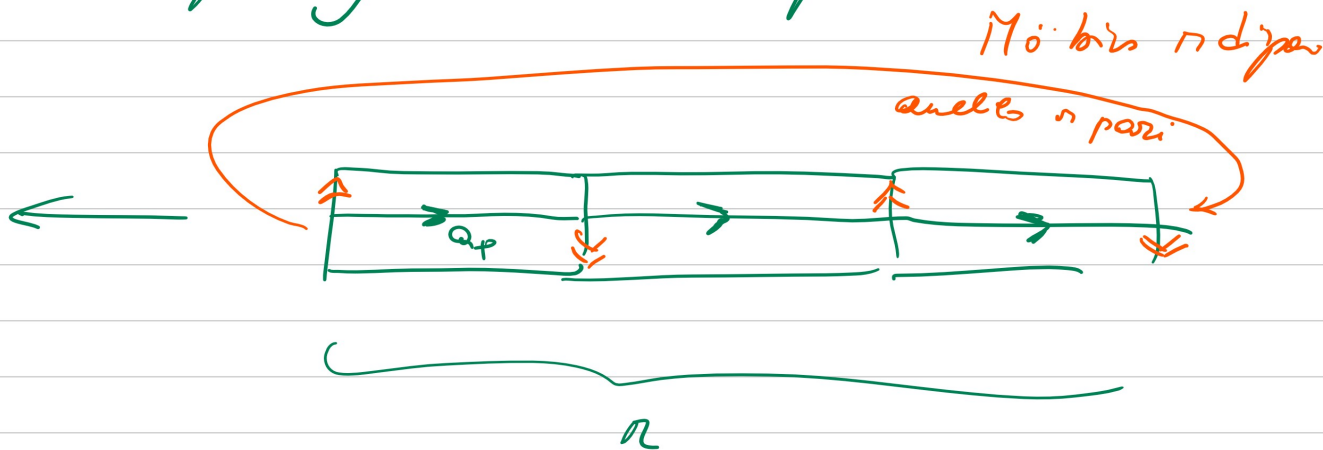
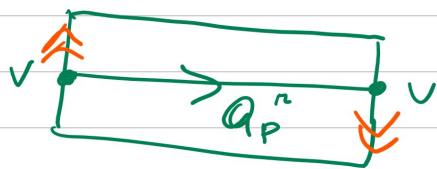
(25)

$$\text{taglio} \leadsto Q \leadsto Q_1 \sqcup \dots \sqcup Q_s / \sim = \tilde{\Sigma} \longrightarrow \Sigma$$

che realizza il $\mathcal{D}_0$

$$\chi(\tilde{\Sigma}) = \chi(\Sigma) \quad \text{per RH}$$

Fatto: un riv. di $k.P^2$ è orientabile se e solo se
 (potenza di un
 qui generatore a_p del p -esimo $\mathbb{P}^2$ si solleva a un
 laccio orientabile - preserv. Inoltre un laccio che
 solleva a_p^n è orientabile preserv. $\Leftrightarrow n$ è pari:



Usando cio' risulta che
 $\tilde{\Sigma}$ orientabile $\iff$ gen. α_p ha cicli di lunghezza pari

$$\Rightarrow \chi(\tilde{\Sigma}) = \chi(\Sigma)$$

e sono entrambe orientabili o entrambe non

$$\Rightarrow \tilde{\Sigma} = \tilde{\Sigma}.$$