Ist. Mat. I - CIA

13/10/22

$\boxed{7b}$

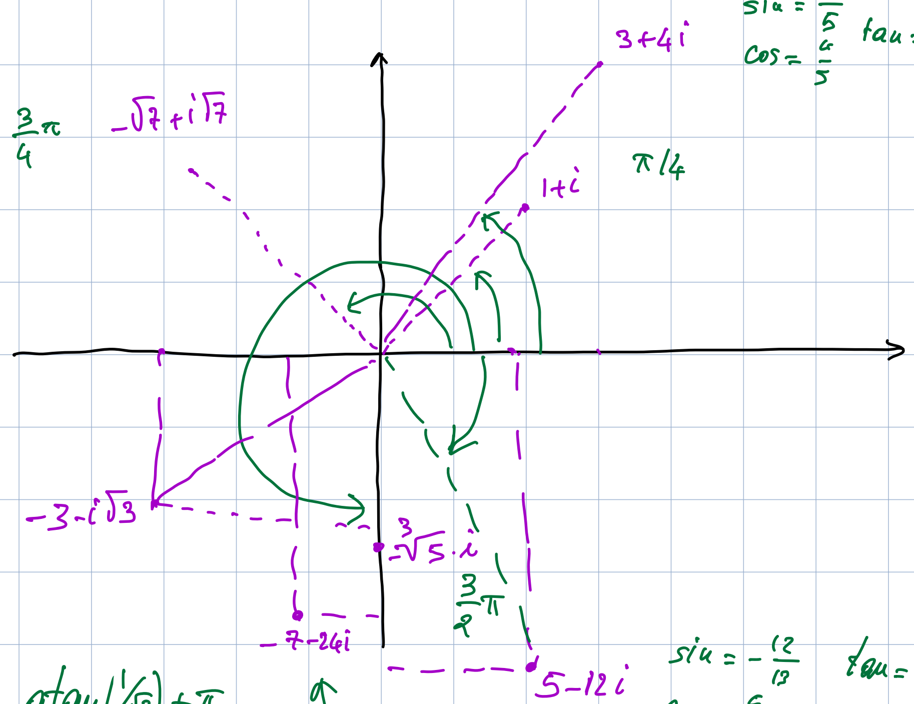
$$z = \frac{\overline{2-3i} + 4|2-i|^2 \cdot i}{7i + (1+5i)\overline{(4+i)}}$$

$$z = \frac{2+3i + 4\cdot 5 \cdot i}{7i + 9 + 19i} = \frac{2+23i}{9+26i} = \frac{(2+23i)(9-26i)}{9^2+26^2} = \dots$$

8

$\sin = \frac{3}{5}$ $\tan = \frac{4}{3}$ $\operatorname{atan}\left(\frac{4}{3}\right)$

$\cos = \frac{4}{5}$



$\cos = -\frac{\sqrt{3}}{2}$

$\sin = -\frac{1}{2}$

$\tan = \frac{1}{\sqrt{3}}$

$\operatorname{atan}(1/\sqrt{3}) + \pi = \frac{7}{6}\pi$

$\sin = -\frac{12}{13}$ $\tan = -\frac{12}{5}$

$\cos = \frac{5}{13}$

$\operatorname{atan}\left(-\frac{12}{5}\right) = -\operatorname{atan}\left(\frac{12}{5}\right)$

$\cos = -\frac{24}{25}$

$\sin = -\frac{7}{25}$

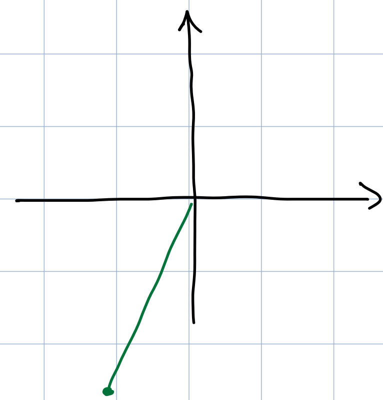
$\tan = \frac{7}{24}$

$\operatorname{atan}\left(\frac{7}{24}\right) + \pi$

$\boxed{9a}$ $\exp\left(\ln(4) - i\frac{4}{3}\pi\right)$

$$e^{\ln(4)} \cdot e^{-i\frac{4}{3}\pi} = 4\cdot\left(-\frac{1}{2} - i\cdot\frac{\sqrt{3}}{2}\right)$$

$$= -2 - 2i\sqrt{3}$$



$\boxed{10}$ (a) $z^2 + iz + 2 = 0$

$$(z-z_1)(z-z_2) = z^2 - (z_1+z_2)\cdot z + z_1 z_2$$

$$\begin{cases} z_1 + z_2 = -i \\ z_1 z_2 = 2 \end{cases} \qquad \begin{matrix} z_1 = -2i \\ z_2 = i \end{matrix}$$

oppure:

$$\Delta = -1 - 8 = -9$$

$$\sqrt{\Delta} = \pm 3i$$

$$z_{1,2} = \frac{-i \pm 3i}{2} = \begin{cases} i \\ -2i \end{cases}$$

(b) $2(2+i)z^2 + 3(1-5i)z - 13 - i = 0$

$$\Delta = 9\left(1 - 10i - 25\right) + 8(2+i)(13+i)$$

$$= 9(-24 - 10i) + 8(25 + 15i)$$

$$= -216 - 90i + 200 + 120i = -16 + 30i$$

Cerco le radici □ di Δ come $\boxed{\alpha + i\beta}$, cioè

voglio $\alpha^2 - \beta^2 + 2i\alpha\beta = -16 + 30i$

$\sqrt{\Delta} = \pm(3+5i)$

cioè $\begin{cases} \alpha^2 - \beta^2 = -16 \\ \alpha\beta = 15 \end{cases}$

$\boxed{\alpha = 3 \quad \beta = 5}$

$$z_{1,2} = \frac{-3(1-5i) \pm (3+5i)}{4(2+i)} = \begin{cases} 20i \cdot \frac{2-i}{20} = 1+2i \\ (-6+10i)\frac{2-i}{20} = \frac{1}{10}(-1+13i) \end{cases}$$

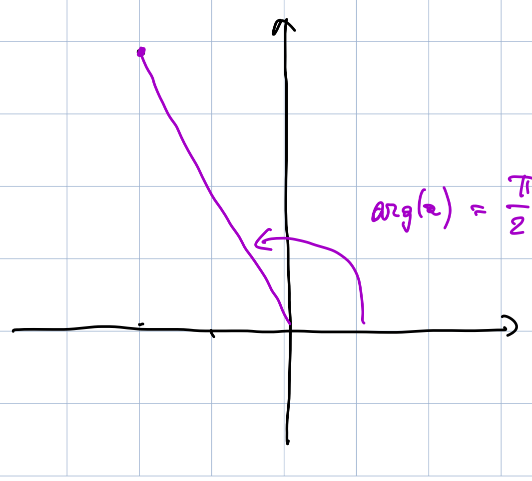
$$\frac{-b \pm \sqrt{\Delta}}{2a}$$

$$\frac{1}{10}(-3+5i)(2-i)$$

$\boxed{e}$ $z^4 = \underbrace{-8 + 8\sqrt{3}\,i}_{a}$

$$\sqrt[4]{|a|} \cdot e^{i\left(\frac{\arg(a)}{4} + \frac{2k\pi}{4}\right)} \qquad k = 0,1,2,3$$

$$|a| = 8 \cdot \sqrt{1+3} = 16 \qquad \sqrt[4]{|a|} = 2$$



$\arg(a) = \frac{\pi}{2} + \frac{\pi}{6} = \frac{2}{3}\pi$

$$z_{1,2,3,4} = 2 \cdot e^{i\left(\frac{\pi}{6} + k\frac{\pi}{2}\right)} \qquad k = 0,1,2,3$$

$$z_1 = 2 \cdot \left(\frac{\sqrt{3}}{2} + \frac{i}{2} \right) = \sqrt{3} + i$$

$$z_2 = -1 + i\sqrt{3}$$

$$z_3 = -\sqrt{3} - i$$

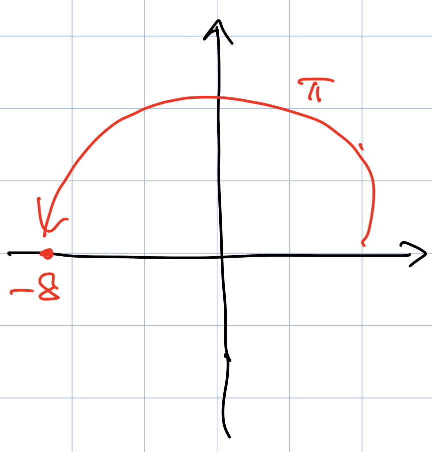
$$z_4 = 1 - i\sqrt{3}$$

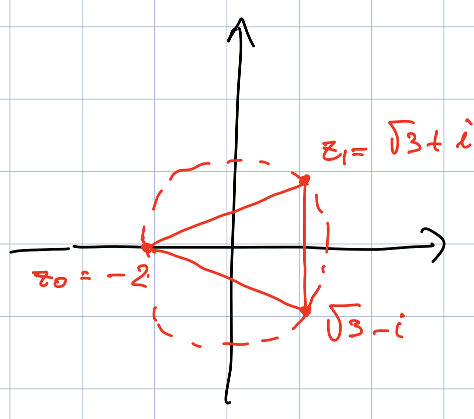
$\boxed{d}$ $\quad z^3 = \underbrace{-8}_{a}$ $\qquad |a| = 8$

$\arg(a) = \pi$

$$z_{0,1,2} = 2 \cdot e^{i\left(\frac{\pi}{3} + \frac{2k}{3}\pi\right)} \qquad k = 0,1,2$$

$z_0 =$





$\boxed{f}$ $\quad 3(1+i)z^3 - 4(3+i)z^2 + (13-i)z + 2(i-2) = 0$

$z = 1$

$$\begin{array}{rr} 3 & +3i \\ -12 & -4i \\ +13 & -i \\ -4 & +2i \end{array} = 0$$

$$\begin{array}{c|ccc|c} & 3+3i & -12-4i & 13-i & -4+2i \\ & & & & \\ 1 & & 3+3i & -9-i & 4-2i \\ \hline & 3+3i & -9-i & 4-2i & \diagup \end{array}$$

$z_1 = 1$ $\qquad z_{2,3}$ radici di $(3+3i)z^2 - (9+i)z + 4-2i$

(trovarle)

molt. $= 1$

$\boxed{g}$ $\quad 3(1+i)z^3 - 2(3+2i)z^2 + (3+7i)z - 2(1+2i) = 0$

i

$$\begin{array}{rr} 3 & -3i \\ 6 & +4i \\ -7 & +3i \\ -2 & -4i \end{array} = 0$$

$$
\begin{array}{c|ccc|c}
 & 3+3i & -6-4i & 3+7i & -2-4i \\
i & & -3+3i & 1-9i & 2+4i \\
\hline
 & 3+3i & -9-i & 4-2i & \checkmark
\end{array}
$$

$z_1 = i$ $\qquad$ $z_{2,3}$ le radici di $(3+3i)z^2-(9+i)z+(4-2i)$

$-3-3i-9i+1+4-2i \neq 0$

molt. $= 1$

$\boxed{h}$ $\qquad \boxed{z^3 \cdot \bar{z} + 8i = 2z(z+2i\bar{z})}$

equazione non polinomiale sui complessi

$$z^2 \cdot |z|^2 + 8i - 2z^2 - 4i|z|^2 = 0$$

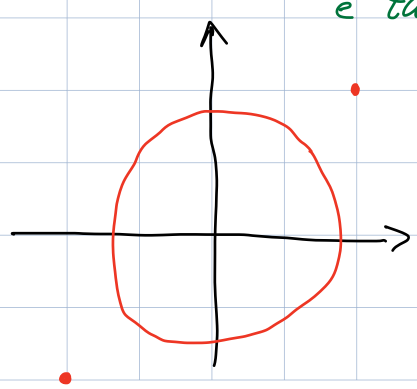
$$z^2(|z|^2-2) + 4i(2-|z|^2) = 0$$

$$(z^2-4i)(|z|^2-2) = 0$$

OSS: scrivendo $z = x+iy$ e sostituendo trovo sistema di 2 equaz. polinomiali in x, y reali

soluz: $\qquad z = \pm\sqrt{2}(1+i)$

e tutti gli z con $|z| = \sqrt{2}$



infinite soluzioni

$$\lim \left(a_n^{b_n} \right) = \left(\lim a_n \right)^{\lim b_n}$$

se $a_n > 0$, $\lim a_n > 0$ se finiti

Data $a_n^{b_n}$ posso prendere $\ln$

$$\ln\left(a_n^{b_n}\right) = b_n \cdot \ln(a_n)$$

Fatto: se $b_n \cdot \ln(a_n) \longrightarrow L$ allora $a_n^{b_n} \longrightarrow e^L$ dove

$$e^{+\infty} = +\infty \qquad e^{-\infty} = 0$$

Forme indeterminate esponenziali:

$$1^{\pm\infty} \rightsquigarrow \pm\infty \cdot 0$$
$$0^0 \rightsquigarrow 0 \cdot (-\infty)$$
$$(+\infty)^0 \rightsquigarrow 0 \cdot (+\infty)$$

Già visto $\left(1 + \frac{1}{n}\right)^n$ di tipo 1^∞ ha limite e

Fatto: $a_n \to \pm\infty$ allora $\left(1 + \frac{1}{a_n}\right)^{a_n} \longrightarrow e$

Forma determinata $\frac{L}{\infty} = 0$ se L è finito

Invece $\frac{L}{0}$ con $L \neq 0$; $\left|\frac{L}{0}\right| = +\infty$

ma se al denominatore ho segni che cambiano il lim. non esiste.

Es: $\lim_{n\to\infty} \frac{\frac{n^2-1}{n^2+1}}{(-1)^n \cdot \frac{1}{n^4+7}}$ $\frac{1}{0}$ ma il lim non esiste

Risultati di permanenza del segno:

- $\lim_{n\to\infty}(a_n) > 0$ allora $a_n > 0$ per n grande
  $\exists N$ t.c. $a_n > 0$ per $n \geq N$

- $a_n \geq 0$ ed esiste $\lim(a_n)$ allora $\lim(a_n) \geq 0$

(non è vero che $a_n > 0$ allora $\lim(a_n) > 0$)

Teoremi del confronto:

- $a_n \leq b_n \leq c_n$ (per $n \geq N$)

se $\lim(a_n) = \lim(c_n) = L$
allora $\lim(b_n) = L$

- $|a_n| \leq b_n$ (per $n \geq N$)
  se $\lim(b_n) = 0$ allora $\lim(a_n) = 0$

- se $(a_n)_{n=0}^{+\infty}$ è limitata e $\lim(b_n) = 0$
  allora $\lim(a_n \cdot b_n) = 0$.

Alcuni esempi di forme indeterminate calcolabili:

- $\infty - \infty$: $a_n = p(n) - q(n)$
  $p(x), q(x)$ polinomi con coeff. direttore (quello di grado più alto) positivo

Fatto: ciascuno dei due tende a $+\infty$
Se $\deg(p(x)) > \deg(q(x))$ $\lim = +\infty$
Se $\deg(p(x)) < \deg(q(x))$ $\lim = -\infty$

Esempio: $\lim\limits_{n \to \infty} (n^2 - n) = +\infty$

Verifica formale: dato $K > 0$ cerco N t.c.
$n^2 - n > K$ per $n \geq N$.

$x^2 - x > K$ valida per $x >$ radice di $x^2 - x = K$

$$\frac{1+\sqrt{1+4K}}{2}$$

Se prendo $N = \left[\frac{1+\sqrt{1+4K}}{2}\right] + 1$ ho che $n^2 - n > K$ per $n \geq N$.

$\infty - \infty$: $\lim_{n\to\infty} \left(\sqrt{n+1} - \sqrt{n}\right)$

$$\sqrt{n+1} - \sqrt{n} = \frac{\left(\sqrt{n+1} - \sqrt{n}\right)\cdot\left(\sqrt{n+1} + \sqrt{n}\right)}{\sqrt{n+1} + \sqrt{n}}$$

$$= \frac{n+1-n}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}} \longrightarrow \frac{1}{\infty + \infty} = 0$$

- $\frac{\infty}{\infty}$ $\lim_{n\to\infty} \frac{p(n)}{q(n)}$ $p(x), q(x) \in \mathbb{R}[x]$

Fatto: se $\deg(p(x)) > \deg(q(x))$

$$\lim \frac{p(n)}{q(n)} = \pm\infty$$

$\pm$ = prodotto segni dei coeff. direttori

$$\lim \frac{7x^5 + (\text{grado} \leq 4)}{4\cdot x^4\ (\text{grado} \leq 3)} = +\infty$$

$$\lim \frac{7x^5 + (\text{grado} \leq 4)}{-4x^4 + (\text{grado} \leq 3)} = -\infty$$

se $\deg(p(x)) < \deg(q(x))$

$$\lim \frac{p(n)}{q(n)} = 0$$

se $\deg(p(x)) = \deg(q(x))$

$$\lim \frac{p(n)}{q(n)} = \text{rapporto coeff direttori}$$

$$\lim \frac{7x^4 + (\text{grado} \leq 3)}{-5x^4 + (\text{grado} \leq 3)} = -\frac{7}{5}$$

$$\frac{a_k \cdot n^k + a_{k-1} n^{k-1} + \dots}{b_h \cdot n^h + b_{h-1} \cdot n^{h-1} + \dots} =$$

$k > h$

$$\frac{\overbrace{a_k \cdot n^{k-h} + a_{k-1} \cdot n^{k-h-1} + \dots + a_0 \cdot \frac{1}{n^h}}^{\operatorname{sgn}(a_k) \cdot \infty}}{\underbrace{b_h + \underbrace{b_{h-1} \cdot \frac{1}{n} + \dots + b_0 \cdot \frac{1}{n^h}}_{\underbrace{0 + \dots + 0}_{0}}}_{b_h}}$$

$k = h$

$$\frac{\overbrace{a_k + \overbrace{a_{k-1} \cdot \frac{1}{n} + \dots + a_0 \cdot \frac{1}{n^k}}^{0 + \dots + 0}}^{a_k}}{b_k + \underbrace{b_{k-1} \cdot \frac{1}{n} + \dots + b_0 \frac{1}{n^k}}_{0 + \dots + 0}} \to \frac{a_k}{b_k}$$

$$\overbrace{\phantom{a_k + a_{k-1}\cdot\frac{1}{n} + \dots + a_0\cdot\frac{1}{n^k}}}$$

$$b_r$$

$$k<h \qquad \frac{\overbrace{a_k + \underbrace{a_{k-1}\cdot\frac{1}{n} + \dots + a_0\cdot\frac{1}{n^k}}_{}}^{a_k}}{\underbrace{b_h\cdot n^{h-k} + b_{h-1}\cdot n^{h-k-1} + \dots + b_h\cdot\frac{1}{n^k}}_{\operatorname{sgn}(b_h)\cdot\infty}} \longrightarrow 0$$

$$0 + \dots + 0$$