Ist. Mat. I - CIA
24/11/22

$f: [a,b] \to \mathbb{R}$, derivabile $n+1$ volte, $x_0 \in [a,b]$

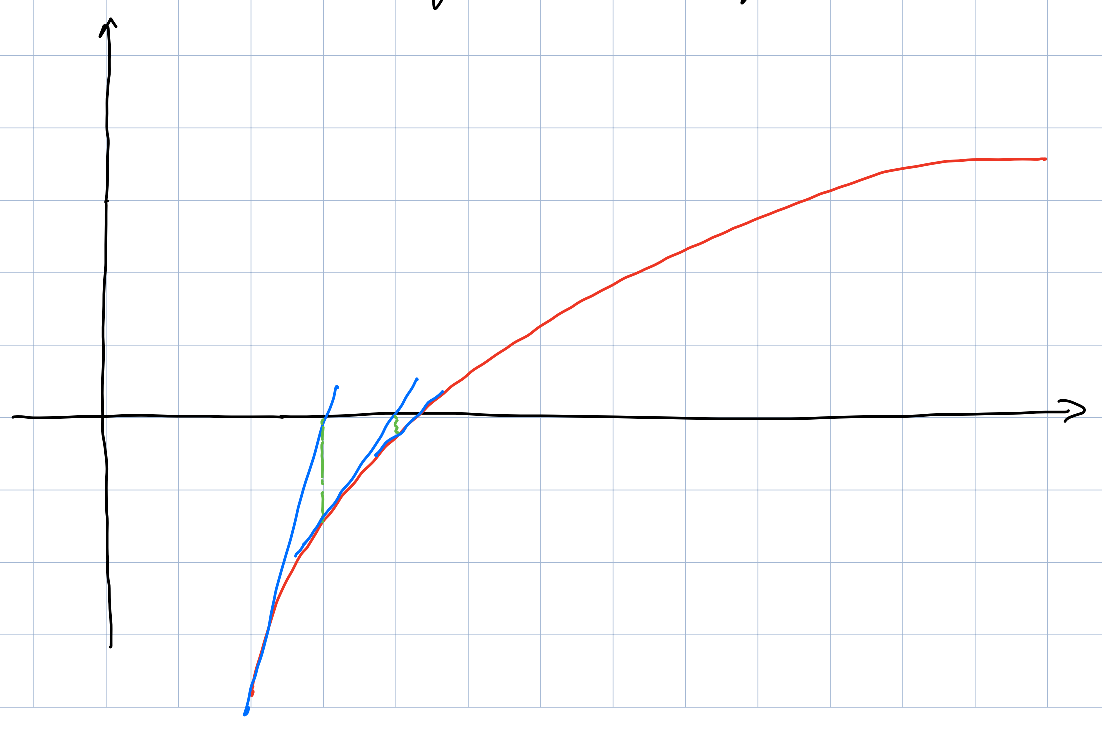
Taylor/Peano $\quad f(x) = P_n(x) + o((x-x_0)^n)$ $\quad$ dà info solo per $x \to x_0$

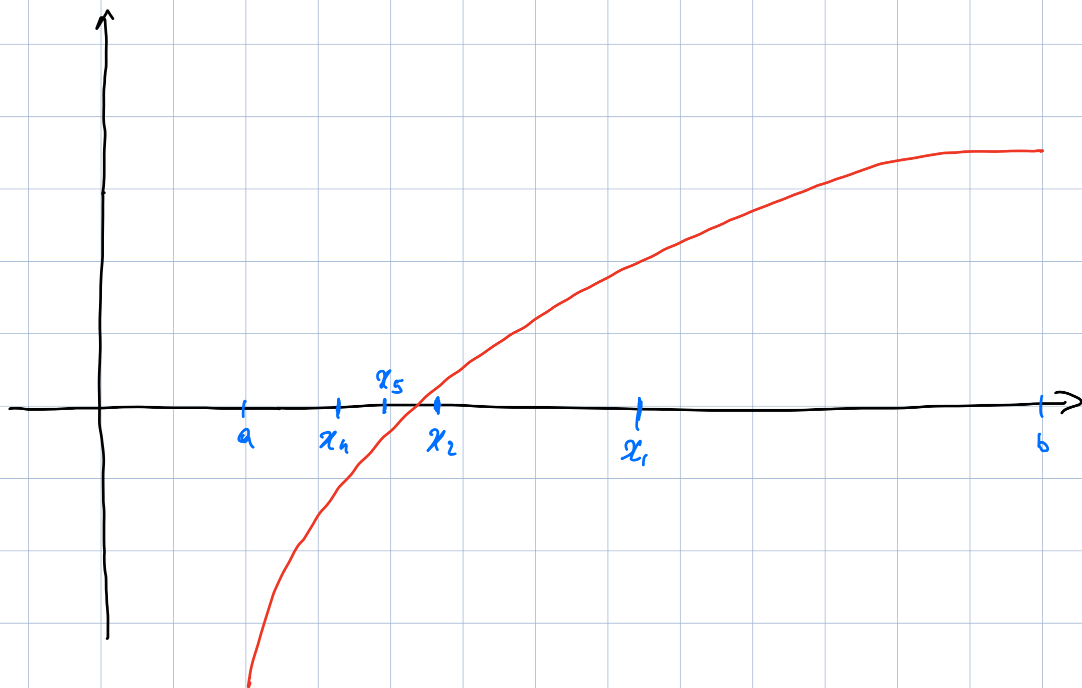
Taylor/Lagrange $\quad \forall x \in [a,b] \quad \exists c$ tra x_0 e x t.c.

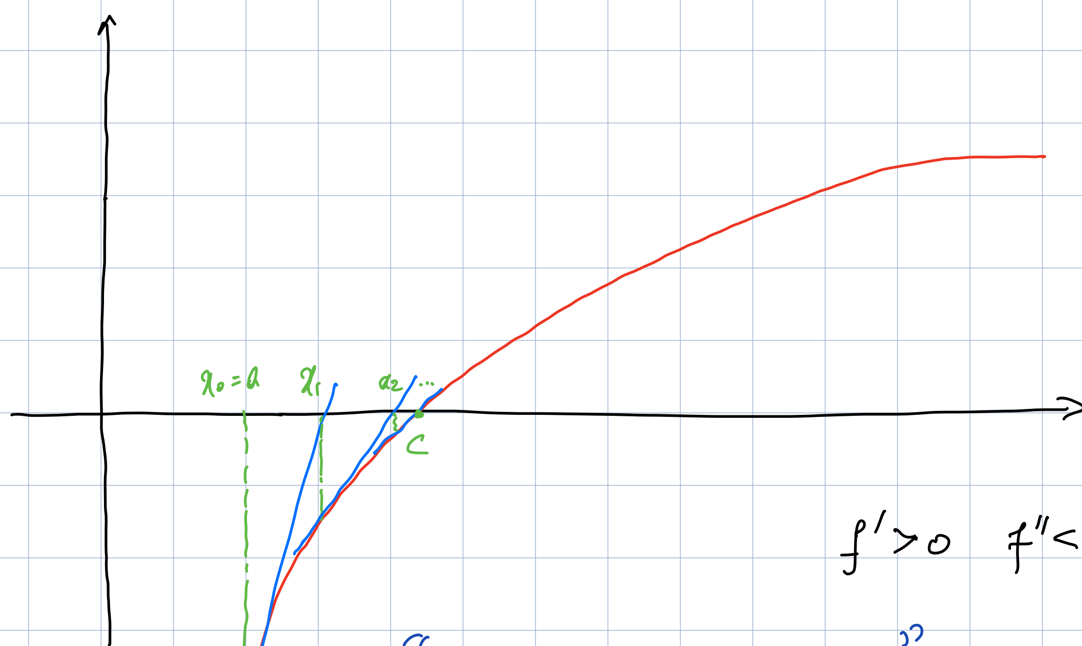
$$f(x) = P_n(x) + \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}$$

dà info globale $\forall x \in [a,b]$.

Metodo di ricerca zeri con tangenti è molto più veloce di bisezione.







$f' > 0 \quad f'' < 0$

$$\begin{cases} x_0 = a \\ x_{k+1} = x_k - \dfrac{f(x_k)}{f'(x_k)} \end{cases}$$

(funziona solo sapendo che gli x_k costruiti sono in $[a,b]$)

Formalizzo così: poiché $f(a) < 0 < f(b)$, $f' > 0$ so che ho un unico zero c. Pongo

$$g : [a,c] \to \mathbb{R} \qquad g(x) = x - \frac{f(x)}{f'(x)}$$

e provo che:

(1) $a < g(a)$

(2) $g(c) = c$

(3) g crescente su $[a,c]$.

Ammessa (1), (2), (3) abbiamo:

$$a < c \underset{(3)}{\Longrightarrow} g(a) < g(c) \underset{(2)}{\Longrightarrow} g(a) < c$$

$$\underset{(1)}{\Longrightarrow} a < g(a) < c \underset{(3)\dots}{\Longrightarrow} \underset{x_0}{a} < \underset{x_1}{g(a)} < \underset{x_2}{g^2(a)} < \underset{x_3}{g^3(a)} \underset{\dots}{\cdots} < c$$

$$\Longrightarrow x_k \nearrow \text{ e } < c \Longrightarrow \text{ ha limite } d.$$

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \xrightarrow[k \to \infty]{} d = d - \frac{f(d)}{f'(d)}$$

$$\Longrightarrow f(d) = 0 \Longrightarrow d = c.$$

Dimostro: (1) $a < g(a)$

(2) $g(c) = c$

(3) g crescente su $[a,c]$.

(1) $$g(a) = a - \frac{f(a)}{f'(a)} = a - \frac{<0}{>0} = a - (<0) > a$$

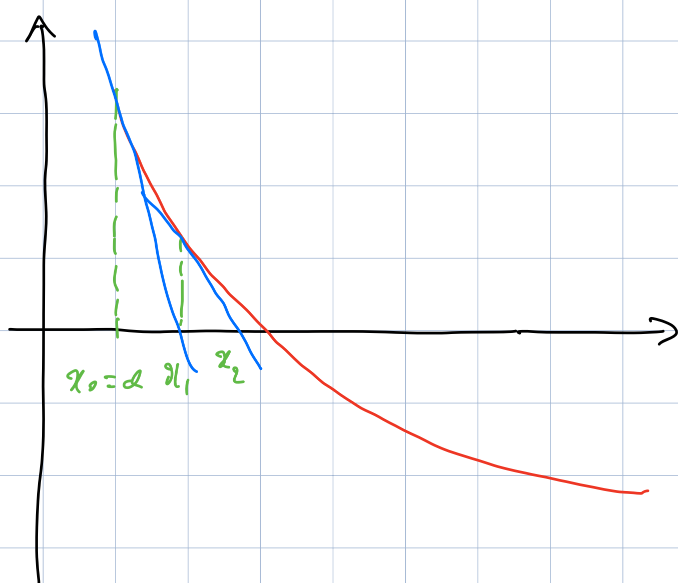
(2) $g(c) = c - \dfrac{f(c)}{f'(c)} = c - \dfrac{0}{\cdots} = c$

(3) $g(x) = x - \dfrac{f(x)}{f'(x)}$

$$\Rightarrow g'(x) = 1 - \frac{f'(x)\cdot f'(x) - f(x)\cdot f''(x)}{f'(x)^2}$$

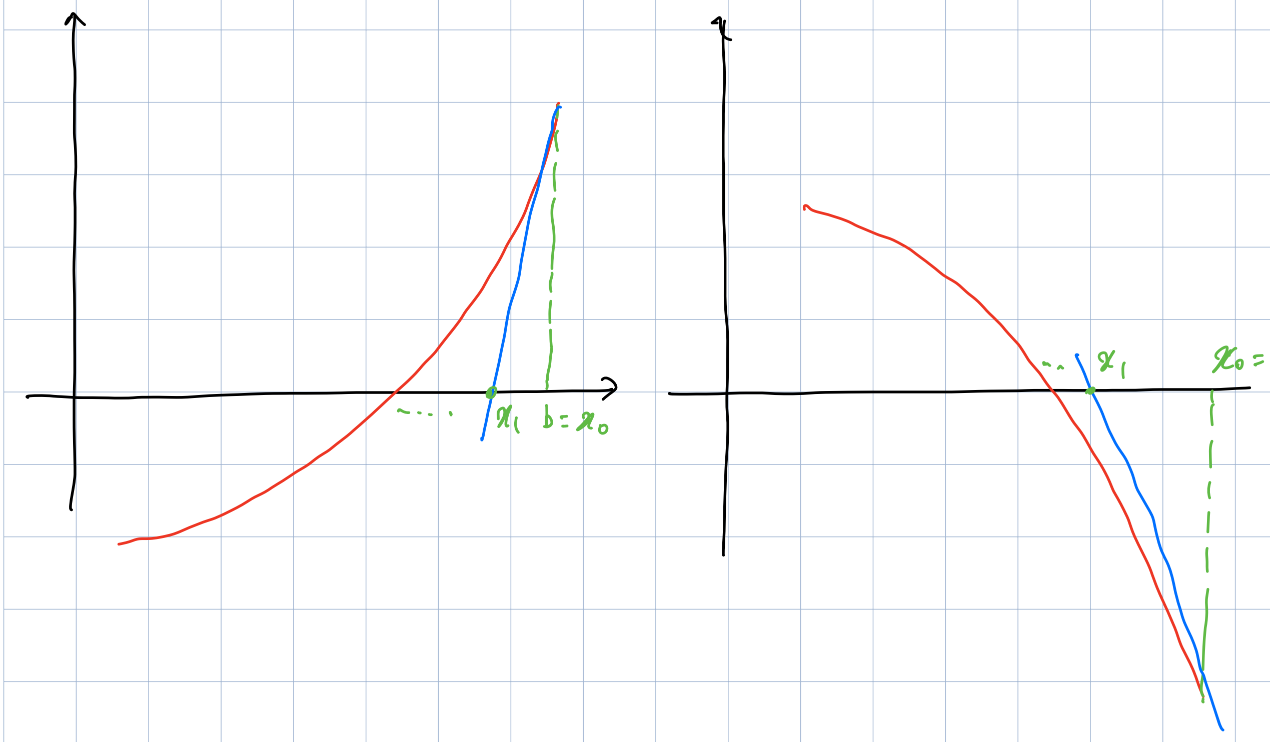
$$= \frac{f(x)\cdot f''(x)}{f'(x)^2} = \frac{(<0)\cdot(<0)}{(>0)^2} > 0$$

$\forall x \in (a, c)$. $\blacksquare$



$f' < 0 \quad f'' > 0$

$\begin{cases} x_0 = a \\ \cdots \end{cases}$



Serie numeriche.

$$\sum_{n=0}^{+\infty} a_n \qquad \left(\sum_{n=n_0}^{+\infty} a_n \right)$$

Es: $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n} \qquad \sum_{n=1}^{\infty} \frac{1}{n}$

$\displaystyle\sum_{n=0}^{\infty} a_n$ significa $\displaystyle\lim_{n\to\infty} s_n$ con $\displaystyle s_n = \sum_{k=0}^{n} a_k$

somma parziale n-esima

Se $\lim_{n\to\infty} s_n$ esiste finito diciamo $\sum_{n=0}^{\infty} a_n$ <u>convergente</u>

Se esiste $\pm\infty$ <u>divergente</u>.

<u>Es</u>: $\alpha \neq 1$ $\quad \sum_{n=0}^{\infty} \alpha^n = \lim_{n\to\infty} \left(\sum_{k=0}^{n} \alpha^k \right)$

$$= \lim_{n\to\infty} \frac{1-\alpha^{n+1}}{1-\alpha}$$

$\nearrow \frac{1}{1-\alpha}$ se $|\alpha|<1$

$\to +\infty$ se $\alpha>1$

$\searrow$ non esiste se $\alpha \leq -1$

<u>Oss</u>: data $(b_n)_{n=0}^{+\infty}$ con $\lim_{n\to\infty} b_n = 0$

posto $a_n = b_n - b_{n+1}$ si ha $\sum_{n=0}^{+\infty} a_n = b_0$

Infatti $\quad s_n = (b_0 - \cancel{b_1}) + (\cancel{b_1} - \cancel{b_2}) + (\cancel{b_2} - b_3) + \dots + (b_n - b_{n+1})$

$$= b_0 - b_{n+1} \longrightarrow b_0$$

<u>Es</u>: $b_n = \frac{1}{n} \qquad a_n = \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}$

$$\Longrightarrow \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

$$\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \dots = 1$$

Oss: se $\sum_{n=0}^{\infty} a_n$ converge allora $\lim_{n\to\infty} a_n = 0$.

Infatti: $a_n = S_n - S_{n-1} \longrightarrow S - S = 0$.

Oss: se $\sum_{n=0}^{\infty} a_n$ converge allora $\boxed{R_k = \sum_{n=k}^{\infty} a_n}$ converge e tende a 0 per $k \to \infty$.

$$\sum_{n=k}^{m} a_n = \sum_{n=0}^{m} a_n - \sum_{n=0}^{k-1} a_n = S_m - S_{k-1} \xrightarrow{m\to\infty} \underbrace{S}_{} \xrightarrow{k\to\infty} S - S = 0$$

Oss: se $\sum_{n=k}^{\infty} a_n$ converge $\Rightarrow \sum_{n=0}^{\infty} a_n$.

Oss: se $a_n \geq 0$ $\quad S_m = \sum_{k=0}^{m} a_n$ è crescente

$\Rightarrow$ o $\sum a_n$ converge o diverge a $+\infty$.

Anzi: converge $\Longleftrightarrow (S_m)_{m=0}^{\infty}$ è limitata.

Foglio 5 - Esc 3

(c) $f : [-\pi, \pi] \to \mathbb{R} \qquad f(x) = x^3 \cdot \log(3 + \cos(x))$

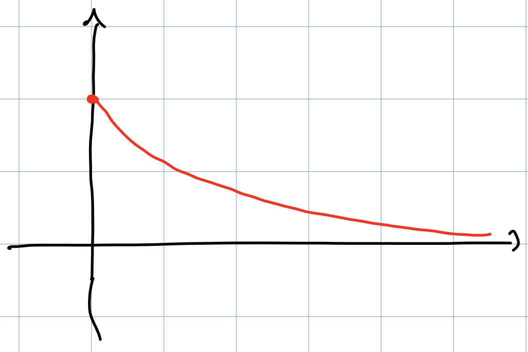
(W) OK

(Z) $f(-\pi) = -\pi^3 \cdot \log(2) < 0$

$f(\pi) = \pi^3 \cdot \log(2) > 0$ OK

(d) $f : [0, +\infty) \to \mathbb{R} \qquad f(x) = \frac{1}{1+x}$

(W) No (Z) No



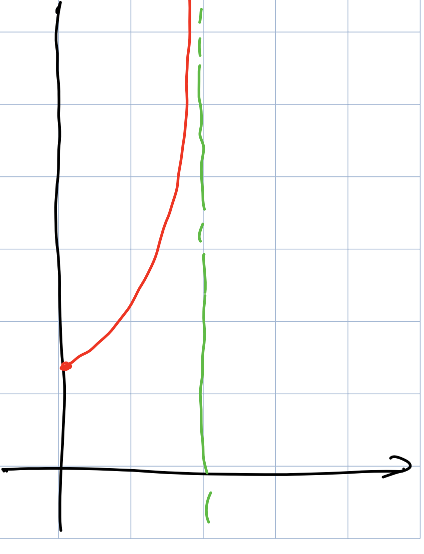
ha $\max = 1$

non ha min

$(\inf = 0)$

non ha zeri.

(e) $f : [0,1) \to \mathbb{R} \qquad f(x) = \frac{1}{\log(2-x)}$



(W), (Z) No

ha $\min = \log(2)$

$\sup = +\infty$

non ha zeri.

Ese 4 — Trovare $\mathrm{Im}(f)$.

(a) $f : [0,4] \to \mathbb{R}$ $\quad f(x) = x + \sqrt{x} - 1$.

$$f = \nearrow + \nearrow + \text{cost} = \nearrow$$

$$\mathrm{Im}(f) = [f(0), f(4)] = [-1, 5]$$

(b) $f : [-1,1] \to \mathbb{R}$ $\quad f(x) = \arctan(x) - \sqrt{1-x}$

$$f = \nearrow - (\searrow) = \nearrow + \nearrow = \nearrow$$

$$\mathrm{Im}(f) = \left[-\frac{\pi}{4} - \sqrt{2}, \frac{\pi}{4}\right]$$

(c) $f : [0, \pi/2] \to \mathbb{R}$ $\quad f(x) = \cos(x) - x^3$

$$f = \searrow - (\nearrow) = \searrow + \searrow = \searrow$$

$$\mathrm{Im}(f) = [f(\pi/2), f(0)] = \left[-\frac{\pi^3}{8}, 1\right]$$

(d) $f : [0,2] \to \mathbb{R}$ $\quad f(x) = \sin(x^3)$

$$\mathrm{Im}\left([0,2] \xrightarrow{x \mapsto x^3} \mathbb{R}\right) = [0,8] \supset [0, 2\pi]$$

$$\mathrm{Im}(f) = \mathrm{Im}(\sin) = [-1,1].$$

<u>Esp</u>: $f: I \to \mathbb{R}$, $J = \mathrm{Im}(f)$

invertibile $\underbrace{\text{con inversa continua}}_{\text{sì se } I \text{ è intervallo e } f \text{ è continua e invertibile}}$

(a) $f: [2,4] \to \mathbb{R}$, $f(x) = x - 2\sqrt{x}$

$f'(x) = 1 - 2 \cdot \frac{1}{2} x^{1/2-1} = 1 - \frac{1}{\sqrt{x}} > 0$ su $[2,4]$

$J = [2 - 2\sqrt{2}, 0]$ <u>sì</u>

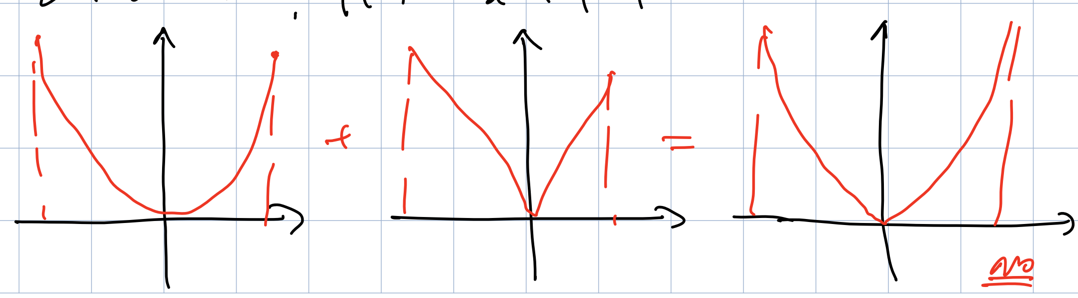
(b) $f: (0, +\infty) \to \mathbb{R}$ $f(x) = e^x + \log(x)$

f continua, $f = \nearrow + \nearrow = \nearrow$

$\lim_{x \to 0^+} f(x) = -\infty$ $\lim_{x \to +\infty} f(x) = +\infty \Rightarrow J = \mathbb{R}$.

Invertibile. Applicando il teorema sulla inversa continua su $[\varepsilon, M]$, $M > \varepsilon > 0$ trovo che f^{-1} è continua su $\mathbb{R}$.

(c) $f: [-1,1] \to \mathbb{R}$, $f(x) = x^2 + |x|$

f : 

(d) $f : [1,3] \to \mathbb{R}$ $\quad f(x) = \sin(x) + \cos(x)$

Con derivate: $\quad f'(x) = \cos(x) - \sin(x)$

$$\cos(x) = \sin(x) \iff \frac{\pi}{4} + k\pi \quad , \quad k \in \mathbb{Z}$$

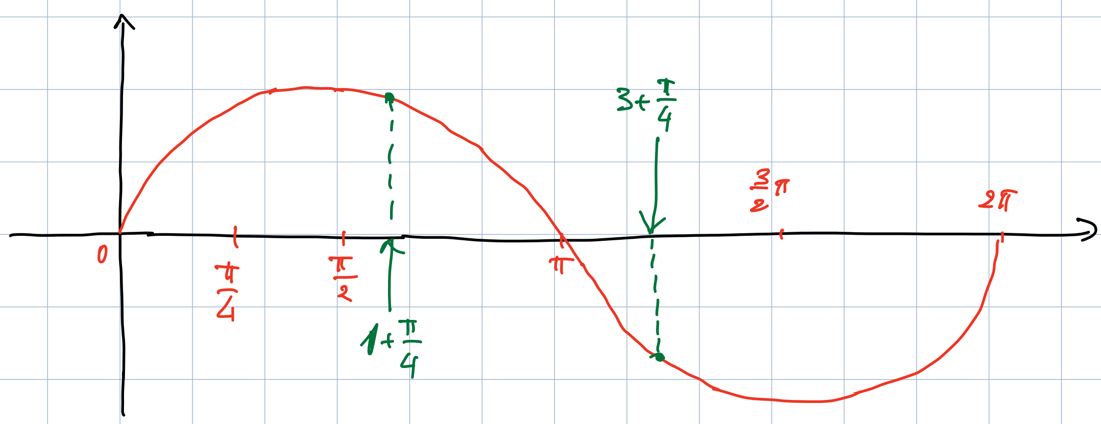
$$\frac{\pi}{4} + k\pi \notin [1,3]$$

Sì

Senza derivate: $\quad f(x) = \sqrt{2} \cdot \left(\frac{1}{\sqrt{2}} \sin(x) + \frac{1}{\sqrt{2}} \cos(x) \right)$

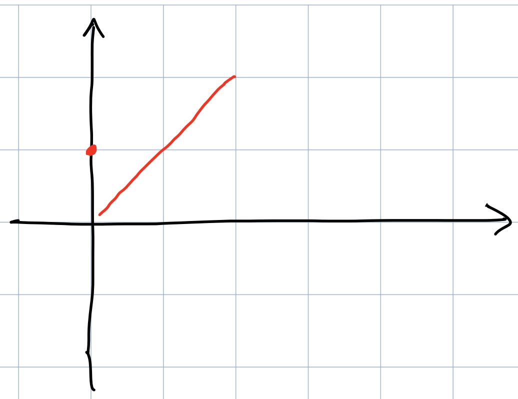
$$= \sqrt{2} \cdot \left(\cos(\pi/4) \cdot \sin(x) + \sin(\pi/4) \cdot \cos(x) \right)$$

$$= \sqrt{2} \cdot \sin\left(x + \frac{\pi}{4}\right).$$



decrescente.

(e) $f : [0,2) \to \mathbb{R}$ $\quad f(x) = x + 1 - \operatorname{sgn}(x)$



No invertibile
No continua.

(f) $f: [-2,0] \to \mathbb{R}$ $\quad f(x) = x^3 - 2x$

$f'(x) = 3x^2 - 2$ $\quad 3x^2 - 2 = 0 \iff x = \pm\sqrt{\frac{2}{3}}$

$-\sqrt{\frac{2}{3}} \in [-2,0]$ $\Longrightarrow$ non è invertibile

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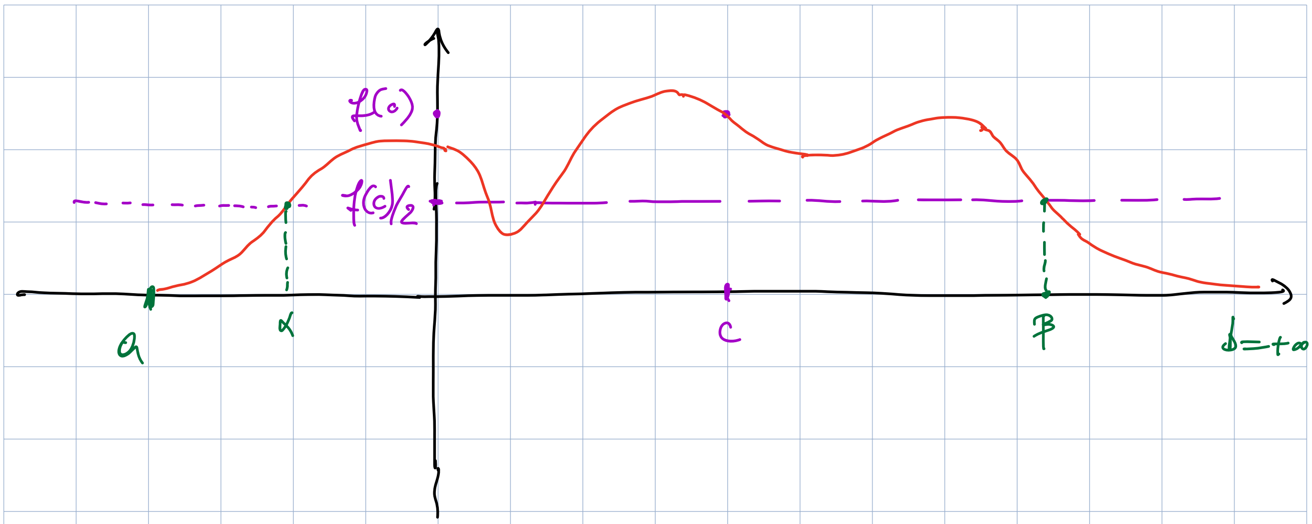
Pagine 144-145 Zanichelli complementi 8-11.

(8+) Provare che se $f: (a,b) \to \mathbb{R}$ è continua $\left(\begin{array}{l} a \in \mathbb{R} / a = -\infty \\ b \in \mathbb{R} / b = +\infty \end{array}\right)$
$f \geq 0$, $\lim\limits_{x \to a^+} f(x) = \lim\limits_{x \to b^-} f(x) = 0$

$\Rightarrow$ f ha max.

Se $f \equiv 0$ ok. Altrimenti scelgo c t.c. $f(c) > 0$;
esistono α, β t.c. $f(x) < f(c)/2$ per $x \in (a, \alpha)$
e per $x \in (\beta, b)$.



Se applico Weierstrass alla restrizione di f su $[\alpha,\beta]$ trovo max $M \geq f(c)$. Su $(a,b) \setminus [\alpha,\beta]$ ho $f(x) < f(c)/2$ $\Longrightarrow \max\limits_{(a,b)} f = M$.

Verificare che servono tutte le ipotesi

$\lim\limits_{a^+,b^-} = 0$, $f \geq 0$