Ist. Mat. I - CIA

30/11/22

$$\sum_{n=0}^{\infty} a_n = \lim_{n\to\infty} S_n$$

$$S_n = \sum_{n=0}^{n} a_n$$

<u>Oss</u>: $\sum_{n=0}^{\infty} a_n$ converge o diverge

$\Leftrightarrow$ $\sum_{n=N}^{\infty} a_n$ lo fa.

<u>Fatto</u>: $0 \le a_n \le b_n$

Se $\sum b_n < +\infty$ allora anche $\sum a_n < +\infty$.

Se $\sum a_n = +\infty$ allora anche $\sum b_n = +\infty$.

Applicazione $\boxed{\sum_{n=1}^{\infty} \frac{1}{n^2} < +\infty}$. Infatti:

$$\frac{1}{n^2} < \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1} = b_n - b_{n+1}$$

$$\sum b_n < +\infty$$

Conseguenza: $\sum_{n=1}^{\infty} \frac{1}{n^\alpha} < +\infty$ per $\alpha \geq 2$

Problema: Per quali α la $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$ converge?
(armonica generalizzata).

Risposta: sì per $\alpha > 1$; no per $0 < \alpha \leq 1$.
[Lo vedremo in parte]

Es: $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$

$\sum_{n=1}^{1000} \frac{1}{n} = 7.48\dots \qquad \sum_{n=1}^{10000} \frac{1}{n} = 9.78\dots \longrightarrow +\infty$

$\sum_{n=1}^{1000} \frac{1}{\sqrt{n}} = 61,80\dots \qquad \sum_{n=1}^{10000} \frac{1}{\sqrt{n}} = 198,54\dots \longrightarrow +\infty$

Fatto: $a_n, b_n > 0$. Se $\lim_{n\to\infty} \frac{a_n}{b_n} = L \neq 0$
allora $\sum a_n$ e $\sum b_n$ hanno stesso comportamento

Infatti, preso $\varepsilon = L/2$ si ha che

$$\frac{L}{2} \leq \frac{a_n}{b_n} \leq \frac{3}{2}L \qquad \text{per } n \geq N$$

$$\frac{L}{2} \cdot b_n \leq a_n \leq \frac{3}{2}L \cdot b_n \qquad \text{per } n \geq N$$

$\Longrightarrow$ ciascuna maggiore un multiplo dell'altra (non importa se vale per $n \geq N$ per primo fatto).

Criterio della radice (per $a_n \geq 0$):

se $\sqrt[n]{a_n} \longrightarrow L$ allora
- se $L < 1$, $\sum a_n < +\infty$
- se $L > 1$, $\sum a_n = +\infty$

Infatti:

- se $L < 1$ scelgo $\varepsilon > 0$ t.c. $L + \varepsilon < 1$; allora

$$\sqrt[n]{a_n} < L + \varepsilon \quad \text{per } n \geq N$$

$$\Rightarrow a_n < (L+\varepsilon)^n \quad \text{per } n \geq N$$

confronto per $n \geq N$

successione geometrica di ragione $L + \varepsilon < 1$

so che $\sum (L+\varepsilon)^n < +\infty$

$\Longrightarrow$ OK

- se $L > 1$ scelgo $\varepsilon > 0$ t.c. $L - \varepsilon > 1$; allora

$$\sqrt[n]{a_n} > L - \varepsilon \quad \text{per } n \geq N$$

$$\Rightarrow a_n > \underbrace{(L-\varepsilon)^n}_{\to +\infty} \quad \text{per } n \geq N$$

$(L - \varepsilon > 1)$

Es: $\sum_{n=1}^{\infty} \underbrace{\left(\frac{1}{2}+\frac{1}{n}\right)^n}_{a_n}$

$$\sqrt[n]{a_n} = \frac{1}{2}+\frac{1}{n} \longrightarrow \frac{1}{2} < 1 \Longrightarrow \text{converge}$$

Criterio del rapporto $(a_n > 0)$:

Se $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$

- se $L<1$, $\sum a_n < +\infty$
- se $L>1$, $\sum a_n = +\infty$

Infatti: • Se $L<1$ prendo $\varepsilon>0$ t.c. $L+\varepsilon<1$;

$\frac{a_{n+1}}{a_n} < L+\varepsilon$ per $n \geq N$; dunque

$$a_{N+1} < (L+\varepsilon)\, a_N$$
$$a_{N+2} < (L+\varepsilon)\, a_{N+1} < (L+\varepsilon)^2 \cdot a_N$$
$$a_{N+3} < (L+\varepsilon)\, a_{N+2} < (L+\varepsilon)^3 \cdot a_N$$
$$\dots$$
$$a_{N+k} < \underbrace{(L+\varepsilon)^k}_{\sum (L+\varepsilon)^k < +\infty \text{ poiché } L+\varepsilon<1} \cdot \underbrace{a_N}_{\text{costante} > 0}$$

confronto che vale da un certo punto ... $\Longrightarrow$ OK

• Se $L > 1$ prendo $\varepsilon > 0$ t.c. $L - \varepsilon > 1$; dunque

$\frac{a_{n+1}}{a_n} > L - \varepsilon$ per $n \geq N$; come sopra

$$a_{N+k} > \underbrace{(L-\varepsilon)^k}_{\to +\infty \ (L-\varepsilon>1)} \cdot \underbrace{a_N}_{cost}$$

Def: $\sum_{n=0}^{\infty} a_n$ si dice assolutamente convergente se

$$\sum_{n=0}^{\infty} |a_n| < +\infty.$$

Teorema: se $\sum a_n$ è assolutamente convergente allora è convergente.

Es: $\sum_{n=1}^{\infty} \frac{\cos(n)}{n^2}$

$$\underbrace{\sum_{n=1}^{\infty} \left| \frac{\cos(n)}{n^2} \right|}$$

$$\left| \frac{\cos(n)}{n^2} \right| \leq \frac{1}{n^2}$$

↳ termini pos. dominata da convergente $\Longrightarrow$ convergente

$$\Longrightarrow \sum_{n=1}^{\infty} \frac{\cos(n)}{n^2} \text{ converge.}$$

__Dim:__ chiamo:

$$S_m = \sum_{n=0}^{m} a_n \; ; \; S_m^+ = \sum_{\substack{n=0 \\ a_n \geq 0}}^{m} a_n \; ; \; S_m^- = \sum_{\substack{n=0 \\ a_n < 0}}^{m} (-a_n)$$

$$\overline{S}_m = \sum_{n=0}^{m} |a_n|$$

Allora: $S_m = S_m^+ - S_m^-$

$$\overline{S}_m = S_m^+ + S_m^-$$

ipotesi ass. conv.

$$\downarrow$$

$$\overline{S}$$

$\Rightarrow S_m^+, S_m^-$ limitate crescenti

$\Rightarrow$ hanno limiti S^+, S^-

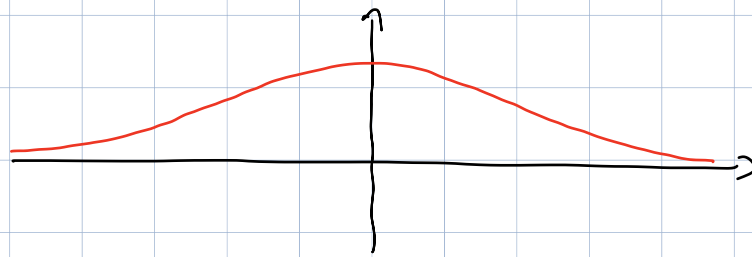
$\Rightarrow S_m \longrightarrow S^+ - S^-$. ▨

$f : (a,b) \to \mathbb{R}$ continua

$f(x) \geq 0 \;\; \forall x \quad \lim_{a^+, b^-} f = 0 \Rightarrow$ ha max.

• serve $\lim = 0$ $\qquad f(x) = x^2$ su $\mathbb{R}$

• può non avere min $\qquad f(x) = \dfrac{1}{1+x^2}$ su $\mathbb{R}$



• sempre $f \geq 0$ $\qquad f(x) = -\dfrac{1}{1+x^2}$ su $\mathbb{R}$

Ese ⑨ (Z pag 144-5):

$f : (a,b) \to \mathbb{R}$ continua

$\lim\limits_{x \to a^+} f(x) \cdot \lim\limits_{x \to b^-} f(x) < 0$

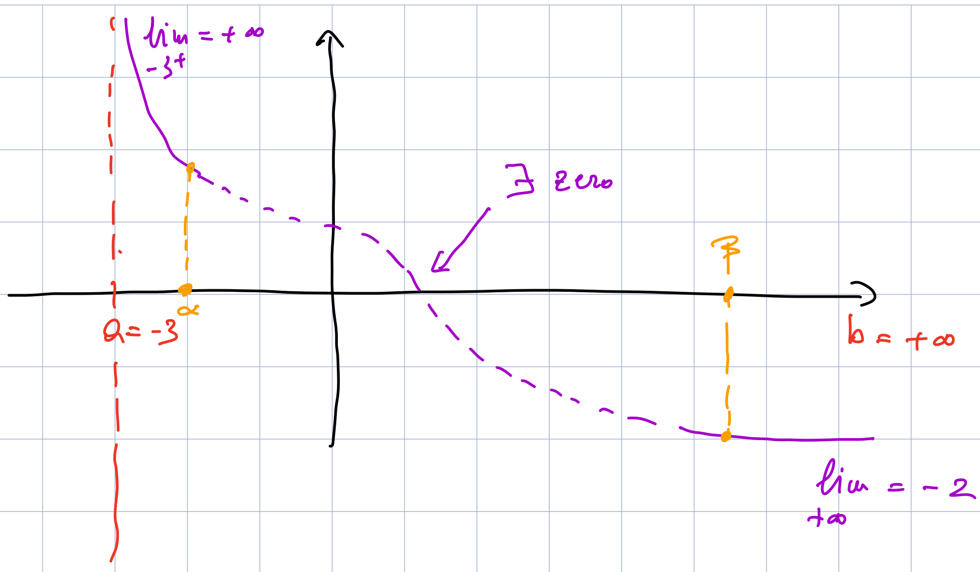
(compreso $a = -\infty$, $b = +\infty$, $\lim = \pm\infty$; escluso $\lim = 0$)

$\Rightarrow$ f ha almeno uno zero.

Infatti: se f ha un limite >0 assume valori >0

" " " " " <0 " " <0

Preso $\alpha < \beta$ con $f(\alpha) \cdot f(\beta) < 0$ si applica il teorema di esistenza zeri alla restrizione di f a $[\alpha, \beta]$



(10) Provare che $\underbrace{\tan(x) + e^x - 5}_{f(x)} = 0$ ha soluzione in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

f continua su $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$; $\lim_{x \to \pm\frac{\pi}{2}^{\mp}} f(x) = \pm\infty$

$\Rightarrow$ applico (9)

(11) Provare che $f(x) = x^3 e^{-x^2}$ su $\mathbb{R}$ ha max/min.

Applico ⑧ a f ristretta a $[0,+\infty)$ e trovo max
che è anche max per f perché è >0
e $f(x) \leq 0$ per $x \leq 0$

Analogamente minimo su $(-\infty, 0]$.

Derivate arcsin, arccos, arctan, arccot.

$f(x) = \sin(x)$ $\qquad f: \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1,1]$

$g(y) = \arcsin(y)$ $\qquad g = f^{-1}$

$y = \sin(x)$ $\qquad x = \arcsin(y)$

Teoria: $g'(y) = \dfrac{1}{f'(g(y))} = \dfrac{1}{f'(x)} = \dfrac{1}{\cos(x)}$

$\cos^2(x) + \sin^2(x) = 1$

$\Rightarrow \cos^2(x) + y^2 = 1$

$\Rightarrow \cos^2(x) = 1 - y^2$ $\qquad \Longrightarrow \cos(x) = \sqrt{1-y^2}$

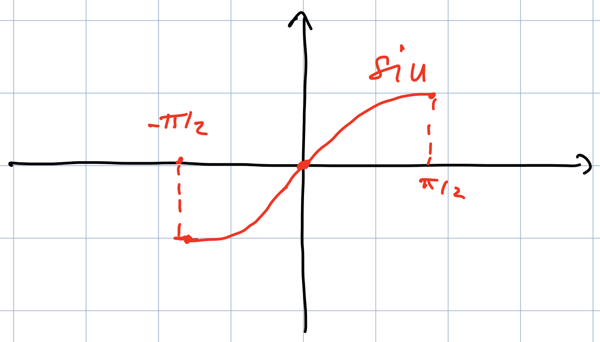
poiché $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

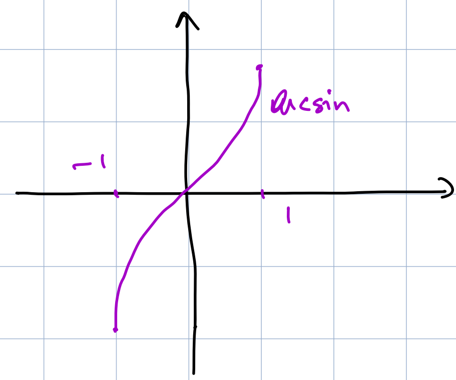
$\Longrightarrow \cos(x) \geq 0$

$$\Longrightarrow \frac{1}{\cos(x)} = \frac{1}{\sqrt{1-y^2}} \qquad \text{per } y \in (-1,1)$$

$$\boxed{D(\arcsin(y)) = \frac{1}{\sqrt{1-y^2}} \quad \text{per } y \in (-1,1)}$$

Non esiste per $y = \pm 1$ (tende a $\mp \infty$)





$\boxed{\text{arccos}}$ $\quad y = \cos(x) \quad x \in [0, \pi]$

$\quad x = \arccos(y) \quad y \in [-1, 1]$

$$D(\arccos(y)) = \frac{1}{-\sin(x)}$$

$\cos^2(x) + \sin^2(x) = 1$

$y^2 + \sin^2(x) = 1$

$\sin^2(x) = 1 - y^2 \implies \sin(x) = \sqrt{1-y^2}$

vero poiché $\sin(x) \geq 0$ su $[0, \pi]$

$$\boxed{D(\arccos(y)) = -\frac{1}{\sqrt{1-y^2}} \quad y \in (-1,1)}$$

tende a $\pm\infty$ in ± 1

$\boxed{\text{arctan}}$ $\quad y = \tan(x) \quad x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\quad x = \arctan(y) \quad y \in \mathbb{R}$

$$D(\arctan(y)) = \frac{1}{D(\tan(x))} = \frac{1}{1+\tan^2(x)} = \frac{1}{1+y^2}$$

$\boxed{\text{arccot}}$ $\quad D(\text{arccot}(y)) = -\frac{1}{1+y^2}$

$\boxed{\text{Z, pag. 160}}$ Determinare che tipo di punto sia 0 per la f assegnata.

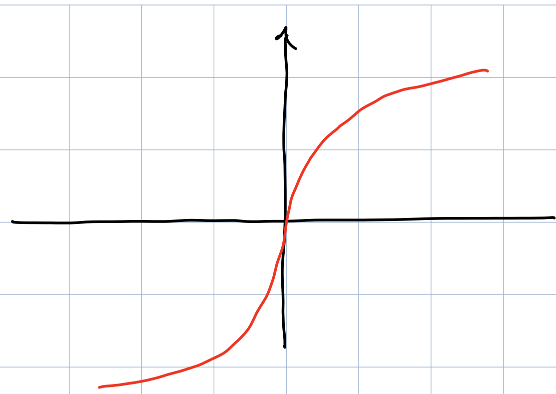
(Sempre: se scrivo solo $f(x) = \dots$ intendo $f: D \to \mathbb{R}$, $D = \{x : \text{l'espressione ha senso}\}$.)

$x^{1/3}$ $\quad x^{1/3} = \sqrt[3]{x}$ è $\mathbb{R} \to \mathbb{R}$ dispari

$$\left(x^{1/3}\right)' = \frac{1}{3} x^{-2/3} \longrightarrow +\infty \quad \text{in } 0^\pm$$

$$\frac{1}{3} x^{-2/3} = \frac{1}{3}\sqrt[3]{1/x^2}$$

Derivata in 0: $\displaystyle \lim_{h \to 0} \frac{h^{1/3} - 0}{h} = +\infty$

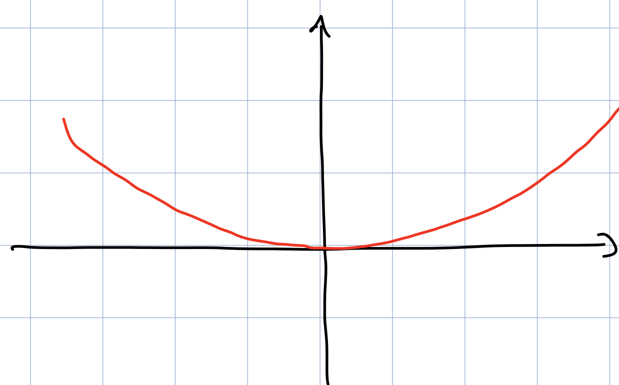


punto a tangente verticale

$x^{4/3}$

$x^{4/3} = \sqrt[3]{x^4}$ $\quad \mathbb{R} \to \mathbb{R}$ $\quad$ pari

$(x^{4/3})' = \frac{4}{3} x^{1/3} \to 0$ per $x \to 0^\pm$



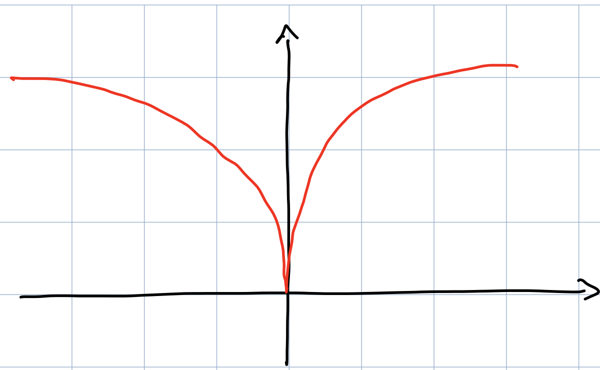
tangente orizzontale

$x^{2/3}$

$x^{2/3} = \sqrt[3]{x^2}$ $\quad \mathbb{R} \to \mathbb{R}$ $\quad$ pari

$(x^{2/3})' = \frac{2}{3} \cdot x^{-1/3} \to \pm\infty$ in $0^\pm$

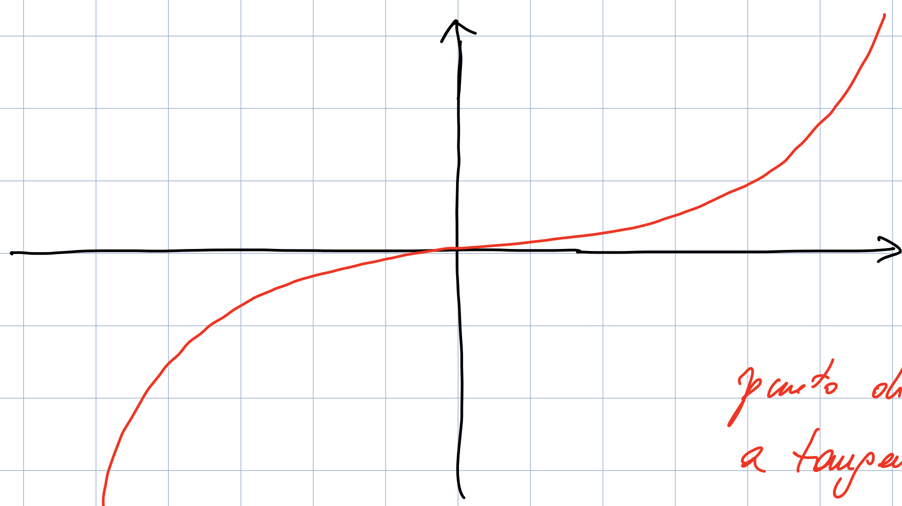
$$\lim_{h \to 0} \frac{h^{2/3} - 0}{h} = \pm\infty$$



cuspide a
tangente vert.

$x^{5/3}$ $\quad$ $x^{5/3} = \sqrt[3]{x^5}$ $\quad$ $\mathbb{R} \to \mathbb{R}$ $\quad$ dispari

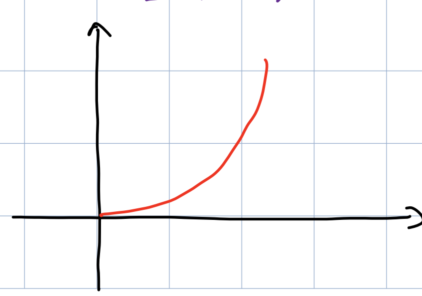
$$(x^{5/3})' = \frac{5}{3}x^{2/3} \qquad (x^{5/3})'' = \underbrace{\frac{5}{3}\cdot\frac{2}{3}\cdot x^{-1/3}}_{\text{concorde con } x}$$



punto di flesso
a tangente orizz.

$x^{3/2}$ $\quad$ $x^{3/2} = \sqrt[2]{x^3}$ $\quad$ $[0, +\infty) \to \mathbb{R}$

$$(x^{3/2})' = \frac{3}{2}x^{1/2}$$



tangente
orizzontale