Ist. Mat. I – CIA
31/3/23

Foglio 8

$\boxed{1}$ $v_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ $v_2 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$

- lin. indip.
- estenderli a base $\mathcal{B}$
- coord $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ risp. a $\mathcal{B}$

• $\alpha_1 v_1 + \alpha_2 v_2 = 0 \overset{?}{\Rightarrow} \alpha_1 = \alpha_2 = 0$

$$\begin{cases} 1 \cdot \alpha_1 + 0 \cdot \alpha_2 = 0 \\ 2\alpha_1 + 2 \cdot \alpha_2 = 0 \\ 0\alpha_1 + 1 \cdot \alpha_2 = 0 \end{cases} \Rightarrow \alpha_1 = \alpha_2 = 0 \quad \checkmark$$

• affermo che $v_1, v_2, e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ sono base $\mathcal{B}$

$\dim(\mathbb{R}^3) = 3$; 3 vettori lin. indip. sono base; infatti

$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 e_1 = 0 \overset{?}{\Rightarrow} \alpha_1 = \alpha_2 = \alpha_3 = 0$

$$\begin{cases} \alpha_1 + \alpha_3 = 0 \\ 2\alpha_1 + 2\alpha_2 = 0 \\ \alpha_1 = 0 \end{cases} \quad \text{sì} \quad \checkmark$$

• $$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + \gamma \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} \alpha + \gamma = 1 \\ 2\alpha + 2\beta = 2 \\ \beta = 3 \end{cases} \qquad \begin{cases} \beta = 3 \\ \alpha = -1 \\ \gamma = 2 \end{cases} \qquad \text{coordinate: } \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$$

Oppure: $B = \left(v_1, v_2, \begin{pmatrix}1\\2\\3\end{pmatrix}\right) \Rightarrow$ Coord: $\begin{pmatrix}0\\0\\1\end{pmatrix}$

$\boxed{2}$ $W: \begin{cases} x + y + z = 0 \\ x + hy + (2-h)z = 0 \\ x + h^2 y + (4-3h)z = 0 \end{cases}$. dim/base.

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & h & 2-h \\ 1 & h^2 & 4-3h \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & h-1 & 1-h \\ 0 & h^2-1 & 3-3h \end{vmatrix} = (h-1)^2 \begin{vmatrix} 1 & -1 \\ h+1 & -3 \end{vmatrix}$$

$$= (h-1)^2 \cdot (h-2)$$

$\dim W = 0$ per $h \neq 1, 2$

$h = 1$: $W: x + y + z = 0$ $\dim(W) = 3 - 1 = 2$

$z = -x - y$

$$\begin{pmatrix}1\\0\\-1\end{pmatrix}\begin{pmatrix}0\\1\\-1\end{pmatrix}$$ base

$h = 2$ $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 4 & -2 \end{pmatrix}$ $\det = 0$; righe lin. dip.

$$\begin{cases} x + y + z = 0 \\ x + 2y = 0 \end{cases}$$

$\begin{cases} x = -2y \\ z = y \end{cases}$ $\begin{pmatrix}-2\\1\\1\end{pmatrix}$ base $\dim(W) = 3 - 2 = 1$

$\boxed{3}$

$$v_1 = \begin{pmatrix}1\\1\\1\\0\end{pmatrix} \quad v_2 = \begin{pmatrix}0\\1\\1\\1\end{pmatrix} \quad v_3 = \begin{pmatrix}1\\1\\0\\0\end{pmatrix}$$

- lin. indip.
- aggiungere v_4 t.c. siano lin indip ($\Rightarrow$ sono base)
- $W: \begin{cases} y=0 \\ z+u=0 \end{cases}$; trovare $T = \langle v_1, v_2, v_3 \rangle$

  $T \cap W$, $T+W$

$$\begin{cases} a+c=0 \\ a+b+c=0 \\ a+b=0 \\ b=0 \end{cases} \Rightarrow a=b=c=0 \quad \text{lin. indip.}$$

$$v_4 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} : \quad \begin{cases} a+c+d=0 \\ a+b+c=0 \\ a+b=0 \\ b=0 \end{cases} \Rightarrow a=b=c=d=0.$$

Determino $T \cap W$ prendendo un generico el. di T, cioè

$$a\begin{pmatrix}1\\1\\1\\0\end{pmatrix} + b\cdot\begin{pmatrix}0\\1\\1\\1\end{pmatrix} + c\begin{pmatrix}1\\1\\0\\0\end{pmatrix}$$

e imponendo che esso soddisf. le equaz. di W $\begin{cases} y=0 \\ z+u=0 \end{cases}$

$$\begin{cases} a+b+c=0 \\ (a+b)+b=0 \end{cases} \qquad \begin{cases} a=-2b \\ c=b \end{cases}$$

$$T \cap W = \left\langle -2\begin{pmatrix}1\\1\\1\\0\end{pmatrix} + \begin{pmatrix}0\\1\\1\\1\end{pmatrix} + \begin{pmatrix}1\\1\\0\\0\end{pmatrix} \right\rangle = \left\langle \begin{pmatrix}-1\\0\\-1\\1\end{pmatrix} \right\rangle$$

$T+W$ è il più piccolo sottospazio che contiene $T \cup W$ cioè $\mathrm{Span}(T \cup W)$.

Osservo che $v_4 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ sta in W

Già visto sopra $\langle \underbrace{v_1, v_2, v_3}_{\subset T}, \underbrace{v_4}_{\subset W} \rangle = \mathbb{R}^4$

$\Longrightarrow$ $T + W = \mathbb{R}^4$.

Formula di Grassmann : V sp. vett. di dim $< +\infty$
$T, W \subset V$ sottosp. vett.

$$\Rightarrow \underset{3}{\underset{\|}{\dim(T)}} + \underset{(4-2=2)}{\underset{\|}{\dim(W)}} = \underset{1}{\underset{\|}{\dim(T \cap W)}} + \underset{4}{\underset{\|}{\dim(T+W)}}$$

$\boxed{4}$ $W_1 : x + 2y + 3z + u = 0$

$W_2 : \begin{cases} x + y = 0 \\ x + z = 0 \\ x - y + z = 0 \end{cases}$

dim / basi $W_1, W_2, W_1 \cap W_2, W_1 + W_2$

W_1 : $\dim = 4 - 1 = 3$

$x = -2y - 3z - u$

$$\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{base}$$

W_2 : $\dim \overset{?}{=} 4 - 3$ (potrebbe essere anche $4 - 2 = 2$)

$\begin{cases} y = -x \\ z = -x \\ x + x - x = 0 \end{cases}$ $\begin{cases} x = 0 \\ y = 0 \\ z = 0 \end{cases}$ base : $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ $\dim = 1$

$W_1 \cap W_2$: impongo che un vettore di W_1, cioè $a\begin{pmatrix}-2\\1\\0\\0\end{pmatrix}+b\begin{pmatrix}-3\\0\\1\\0\end{pmatrix}+c\begin{pmatrix}-1\\0\\0\\1\end{pmatrix}$ sia uguale a uno di W_2, cioè $d\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$

$$\begin{cases} -2a-3b-c = 0 \\ a=0 \\ b=0 \\ c=d \end{cases} \qquad \begin{cases} a=0 \\ b=0 \\ c=0 \\ d=0 \end{cases}$$

$W_1 \cap W_2 = \{0\}$ $\qquad$ $\dim = 0.$

$$W_1+W_2 = \Big\langle \underbrace{\begin{pmatrix}-2\\1\\0\\0\end{pmatrix}, \begin{pmatrix}-3\\0\\1\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}}_{\text{lin. indip}} \Big\rangle$$

$\Rightarrow W_1+W_2 = \mathbb{R}^4$

$$\underset{\substack{\parallel\\3}}{\dim(W_1)} + \underset{\substack{\parallel\\1}}{\dim(W_2)} = \underset{\substack{\parallel\\0}}{\dim(W_1\cap W_2)} + \underset{\substack{\parallel\\4}}{\dim(W_1+W_2)}$$

$\boxed{5}$ $F:\mathbb{R}^3 \to \mathbb{R}^3$ lin.

$F(e_1) = 2e_1 - e_2$

$F(e_2) = e_1 + e_3$

$F(e_3) = -e_1 + e_2 - e_3$

- calcolare $F(3e_1 - 5e_2 - e_3)$
- base di $\mathrm{Ker}(F)$

iniettiva? suriettiva?

$$\begin{aligned} F(3e_1-5e_2-e_3) &= 3F(e_1) - 5F(e_2) - F(e_3) \\ &= 6e_1 - 3e_2 \\ &\quad -5e_1 \qquad\quad -5e_3 \\ &\quad +e_1 - e_2 \quad + e_3 \end{aligned}$$

$= 2e_1 - 4e_2 - 4e_3$

Oss: $A \in M_{m\times n}(\mathbb{R})$; $g: \mathbb{R}^n \to \mathbb{R}^m$
$g(x) = A \cdot x$

$$g(e_1) = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} = \text{I colonna di } A.$$

$g(e_j) =$ j-esima colonna di A.

Nell'esercizio ho $F(x) = A \cdot x$ dove:

$$A = \begin{pmatrix} 2 & 1 & -1 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

cioè A è la matrice associata a F rispetto alla base canonica sia in partenza sia in arrivo.

$$F(3e_1 - 5e_2 - e_3) = \begin{pmatrix} 2 & 1 & -1 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -5 \\ -1 \end{pmatrix} = \begin{pmatrix} 6-5+1 \\ -3-1 \\ -5+1 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ -4 \end{pmatrix}$$

$\mathrm{Ker}(F)$: Risolvo $F(xe_1 + ye_2 + ze_3) = 0$ ovvero
$A \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$

$$\begin{pmatrix} 2 & 1 & -1 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 2x + y - z = 0 \\ -x + z = 0 \\ y - z = 0 \end{cases} \quad \begin{cases} x = z \\ y = z \\ 2x + z - z = 0 \end{cases} \quad \begin{cases} x = 0 \\ y = 0 \\ z = 0 \end{cases}$$

$\mathrm{Ker}(F) = \{0\}$ $\quad \dim = 0.$

In generale $g: U \to W$ lineare
iniettiva $\Longleftrightarrow \ker(g) = 0$.

$\Rightarrow F$ iniettiva.

<u>Oss</u>: se $g: U \to W$ è lineare e $v_1, \dots, v_n$ base di U
allora $g(v_1), \dots, g(v_n)$ sono generatori di $Im(f)$.

Per F: $Im(F) = \langle \underbrace{\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}} \rangle$

lin. indip:

$$\begin{cases} 2a + b - c = 0 \\ -a + c = 0 \\ b - c = 0 \end{cases} \Rightarrow a = b = c = 0$$

$\Rightarrow \dim(Im(F)) = 3 \quad \Rightarrow Im(f) = \mathbb{R}^3$

In generale: $g: V \to W$ lineare

$$\underset{\substack{\| \\ 0}}{\dim(\ker(g))} + \underset{\substack{\| \\ 3}}{\dim(Im(g))} = \underset{\substack{\| \\ 3}}{\dim(V)}$$

$\boxed{6}$ $F: \mathbb{R}^4 \to \mathbb{R}^3$ con matrice rispetto alle basi canoniche

$$A = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 2 & 1 & 1 & 3 \\ 1 & 1 & 0 & 2 \end{pmatrix}$$

- $F(\underbrace{3e_1 - 5e_2 - e_3})$
- basi $\ker(f)$, $Im(f)$
  F iniettiva / surgettiva ?

$$F(3e_1 - 5e_2 - e_3 + 0\cdot e_4) = \begin{pmatrix}1&0&1&1\\2&1&1&3\\1&1&0&2\end{pmatrix}\begin{pmatrix}3\\-5\\-1\\0\end{pmatrix} = \begin{pmatrix}2\\0\\-2\end{pmatrix}$$

$$Ker(f): \begin{cases} x + z + u = 0 \\ 2x + y + z + 3u = 0 \\ x + y + 2u = 0 \end{cases} \qquad \begin{cases} u = -x - z \\ y = -2x - z - 3u \\ \quad = -2x - z + 3x + 3z = x + 2z \\ x + x + 2z - 2x - 2z = 0 \end{cases}$$

$$\begin{cases} u = -x - z \\ y = x + 2z \end{cases}$$

$$\dim = 2 \qquad \text{base} \qquad \begin{pmatrix}1\\1\\0\\-1\end{pmatrix} \begin{pmatrix}0\\2\\1\\-1\end{pmatrix}$$

$$Im(f) = \left\langle \begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\3\\2\end{pmatrix} \right\rangle$$

Da un insieme di generatori estraggo base:

- li esamino nell'ordine in cui sono (o in un altro)
- tengo il primo non nullo
- passo dopo passo esamino un vettore

Lui appartiene al generato di quelli già tenuti? → sì: lo butto
→ no: lo tengo

$$\begin{pmatrix}0\\1\\1\end{pmatrix} \begin{pmatrix}1\\1\\0\end{pmatrix} \cancel{\begin{pmatrix}1\\2\\1\end{pmatrix}} \cancel{\begin{pmatrix}1\\3\\2\end{pmatrix}}$$

✓ ✓ $I + II$ $2 \cdot I + II$

$\Longrightarrow \dim(\mathrm{Im}(f)) = 2.$

$$f: \mathbb{R}^4 \to \mathbb{R}^3 \qquad \underset{\overset{\|}{2}}{\dim(\mathrm{Ker}(f))} + \underset{\overset{\|}{2}}{\dim(\mathrm{Im}(f))} = \underset{\overset{\|}{4}}{\dim(\mathbb{R}^4)}$$

$\boxed{7}$ $\quad v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} \quad v_3 = \begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix}$

- $\mathcal{B} = (v_1, v_2, v_3)$ base $\qquad \cancel{\mathcal{B} = \{v_1, v_2, v_3\}}$
- matrice di $id_{\mathbb{R}^3}$
  rispetto a base canonica in partenza e $\mathcal{B}$ in arrivo.
- coordinate di $e_1 + e_2$.

$$\begin{vmatrix} 1 & 1 & 0 \\ 0 & -2 & 3 \\ 1 & 0 & -1 \end{vmatrix} = 2 + 3 = 5 \neq 0 \Longrightarrow \text{base.}$$

Matrice associata a $f: V \to W$ rispetto a
$(v_1, \dots, v_n)$ di V
$(w_1, \dots, w_m)$ di W
è quella che ha come j-esima colonna le coord. di $f(v_j)$ rispetto a $(w_1, \dots, w_m)$.

$$id_{\mathbb{R}^3}(e_1) = e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{2}{5}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \frac{3}{5}\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + \frac{2}{5}\begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix}$$

$$id_{\mathbb{R}^3}(e_2) = e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{5}\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + \frac{1}{5}\begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix}$$

$$id_{\mathbb{R}^3}(e_3) = e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \frac{3}{5}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \frac{3}{5}\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} - \frac{2}{5}\begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix}$$

$$\begin{cases} a+b=1 \\ -2b+3c=0 \\ a-c=0 \end{cases} \quad \begin{cases} c=a \\ b=\frac{3}{2}a \\ a=\frac{2}{5} \end{cases} \quad \Bigg| \quad \begin{cases} a+b=0 \\ -2b+3c=1 \\ a-c=0 \end{cases} \quad \begin{cases} c=a \\ b=-a \\ 2a+3a=1 \end{cases} \quad \begin{cases} a=1/5 \\ b=-1/5 \\ c=1/5 \end{cases}$$

$$\begin{cases} a+b=0 \\ -2b+3c=0 \\ a-c=1 \end{cases} \quad \begin{cases} b=-a \\ c=a-1 \\ 2a+3a-3=0 \end{cases}$$

$$A = \begin{pmatrix} 2/5 & 1/5 & 3/5 \\ 3/5 & -1/5 & -3/5 \\ 2/5 & 1/5 & -2/5 \end{pmatrix}$$

coord di e_1+e_2 = coord e_1 + coord e_2

$$= \begin{pmatrix} 3/5 \\ 2/5 \\ 3/5 \end{pmatrix}$$