Ist. Mat. I - CIA
3/11/23

$f : I \to \mathbb{R}$ $\qquad f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$D(\sin(x)) = \cos(x)$

$$\frac{\sin(x+h) - \sin(x)}{h} = \frac{\sin(x)\cdot\cos(h) + \cos(x)\cdot\sin(h) - \sin(x)}{h}$$

$$= \underbrace{\sin(x)\cdot \underbrace{\frac{\cos(h)-1}{h^2}}_{\to -\frac{1}{2}} \cdot \underbrace{h}_{\to 0}}_{\to 0} + \cos(x)\cdot \underbrace{\frac{\sin(h)}{h}}_{\to 1}$$

$D(\cos(x)) = -\sin(x)$ $\qquad$ Ese

$D(e^x) = e^x$

$$\frac{e^{x+h} - e^x}{h} = e^x \cdot \underbrace{\frac{e^h - 1}{h}}_{\to 1}$$

$D(\log(x)) = \dfrac{1}{x}$

$$\frac{\log(x+h) - \log(x)}{h} = \frac{\log(1 + h/x)}{h/x} \cdot \frac{1}{x}$$

$$\to 1$$

Ese: $D(\log|x|) = \frac{1}{x}$

$x > 0$ ✓ $\qquad D(\log(-x)) = \frac{1}{-x} \cdot (-1) = \frac{1}{x}$

Teo: $(f \circ g)' = (f' \circ g) \cdot g'$

$$D(f \circ g) = ((Df) \circ g) \cdot g'$$

$$f(g(x))' = f'(g(x)) \cdot g'(x)$$

Dim: $\dfrac{f(g(x+h)) - f(g(x))}{h}$

$$= \frac{f(\overbrace{g(x+h)}^{g(x)+l}) - f(g(x))}{\underbrace{g(x+h) - g(x)}_{\frac{g(x)+l-g(x)}{l}}} \cdot \underbrace{\frac{g(x+h) - g(x)}{h}}_{\to g'(x)}$$

$$\frac{f(g(x)+l) - f(g(x))}{l} \to f'(g(x))$$

Oss: g continua in x (poiché $\exists g'(x)$)

$\Rightarrow g(x+h) \to g(x) \quad h \to 0$

$\Rightarrow l \to 0 \quad h \to 0$ ∎

Ese: $\quad D\left(\log\left(x^2+\cos(x)\right)\right)$

$$= \frac{2x - \sin(x)}{x^2+\cos(x)} .$$

Oss: $\quad f : [a,b] \to \mathbb{R}$ monotona $\quad c \in [a,b]$

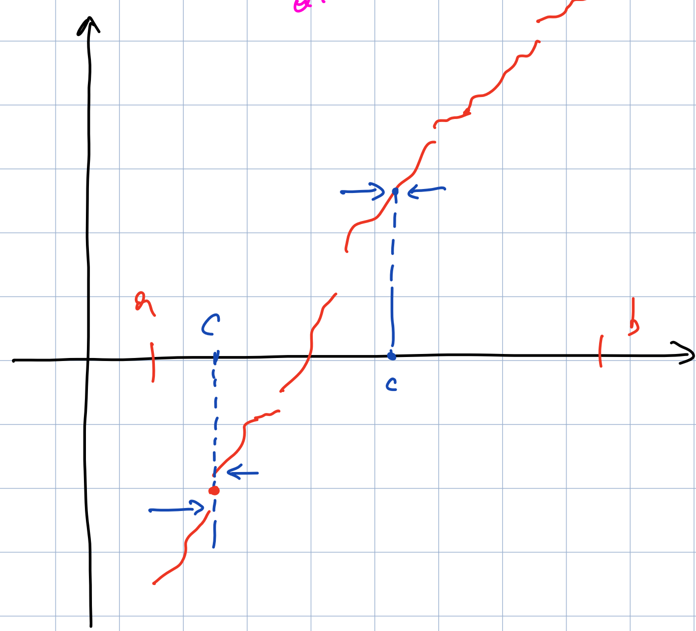
$\Rightarrow \exists \lim_{x \to c^{\pm}} f(x)$ dati da

$$\lim_{x\to c^-} f(x) = \sup \{ f(x) : x < c \}$$
$$\lim_{x\to c^+} f(x) = \inf \{ f(x) : x > c \}$$

se crescente

Ese

Oss: $c = a$, $\lim_{a^+}$; $c = b$ $\lim_{b^-}$



Teo: se $f:[a,b]\to[c,d]$ è continua e invertibile allora f^{-1} è continua.

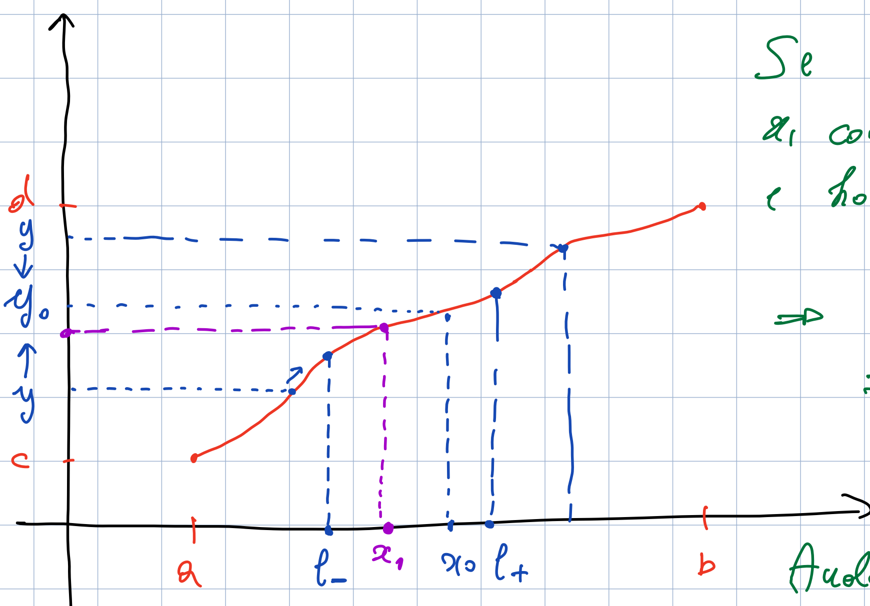
Dimo: f è str. monotona; suppongo crescente.
Anche f^{-1} è str. crescente.
Prendo $y_0=f(x_0)\in[c,d]$; devo vedere che

$$\lim_{y\to y_0} f^{-1}(y)=f^{-1}(y_0)=x_0.$$

$$\exists\, l_- = \lim_{y\to y_0^-} f^{-1}(y)=\sup\{f^{-1}(y) : y<y_0\}$$

$$l_+=\lim_{y\to y_0^+} f^{-1}(y)=\inf\{f^{-1}(y) : y>y_0\}.$$

Devo vedere che $l_-=l_+=x_0$.
Certamente $l_-\le x_0\le l_+$.
Supponiamo per assurdo $l_-<x_0$ oppure $l_+>x_0$



Se $l_-<x_0$ prendo x_1 con $l_-<x_1<x_0$ e ho $f(l_-)<\underbrace{f(x_1)}_{y_1}<f(x_0)$

$\Rightarrow$ $y_1<y_0$
$f^{-1}(y_1)=x_1>l_-$
assurdo
$l_-=\sup\{f^{-1}(y) : y<y_0\}$

Analogo se $l_+>x_0$. ∎

Teo: se $f:[a,b]\to[c,d]$ è invertibile e
$f'(x)\neq 0$, in $y=f(x)$

$$\exists\ (f^{-1})'(y)=\frac{1}{f'(x)}.$$

Cioè: $$(f^{-1})'=\frac{1}{f'\circ f^{-1}}$$

$$(f^{-1})'(y)=\frac{1}{f'(f^{-1}(y))}$$

Dim: pongo $g=f^{-1}$

$$\frac{g(y+h)-g(y)}{h}$$

$x=g(y)$
$k=g(y+h)-g(y)$

Fatto: g continua dunque per $h\to 0$ ho $k\to 0$

$$g(y+h)=g(y)+k=x+k$$
$$\Rightarrow f(x+k)=y+h=f(x)+h$$

$$\frac{g(y+h)-g(y)}{h}=\frac{k}{f(x+k)-f(x)}=\frac{1}{\underbrace{\frac{f(x+k)-f(x)}{k}}_{\to f'(x)}}$$

$\blacksquare$

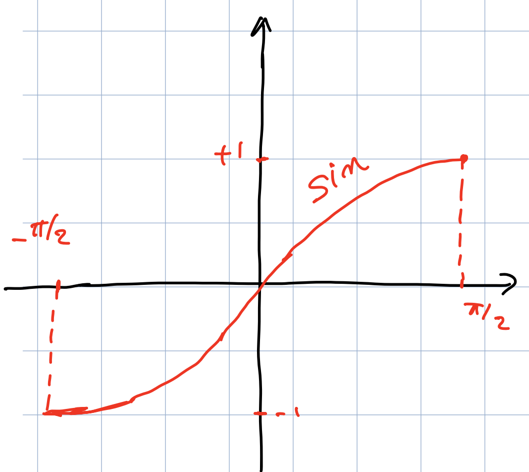
$$D(\arcsin(x)) = \frac{1}{\sqrt{1-x^2}} \quad \text{su} \quad (-1,1)$$

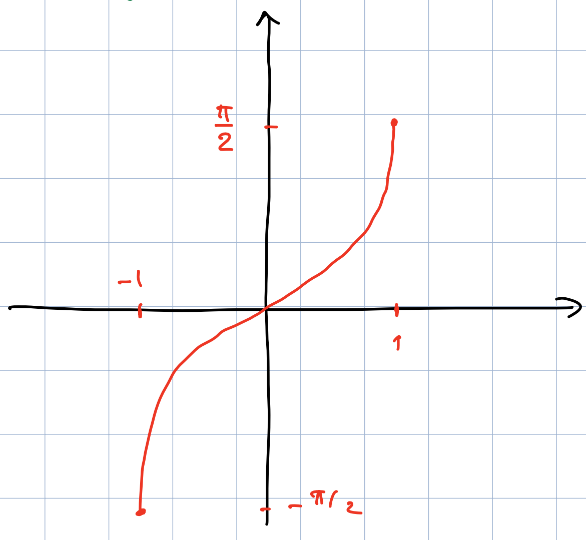
$$D(\arcsin(x)) = \frac{1}{D(\sin)(\arcsin(x))} = \frac{1}{\cos(\arcsin(x))}$$

$t = \arcsin(x) \qquad -1 \leq x \leq 1 \qquad -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$

$\sin(t) = x$

$$\cos(t) \geq 0 \Rightarrow \cos(t) = +\sqrt{1-\sin^2(t)}$$
$$= \sqrt{1-x^2}$$

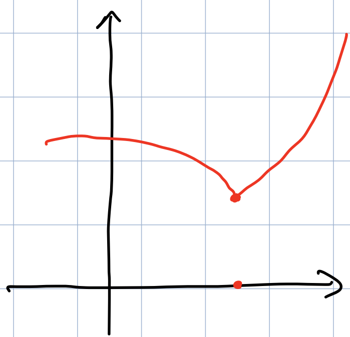
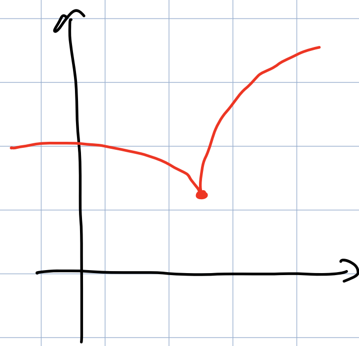
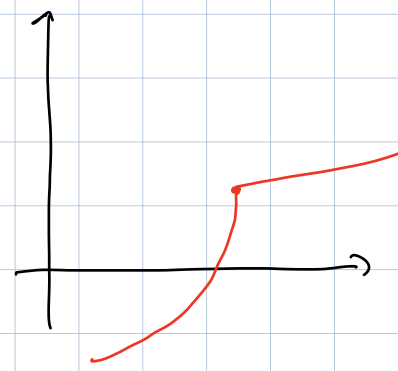




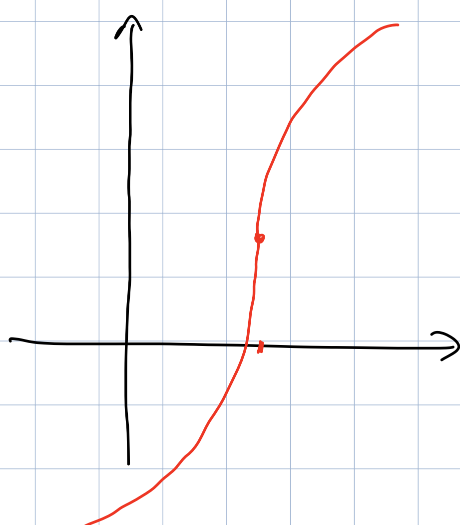
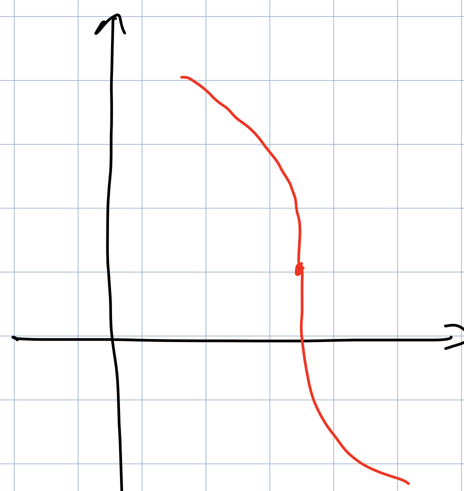
— o —

$$f'_\pm(x) = \lim_{h \to 0^\pm} \frac{f(x+h) - f(x)}{h}$$

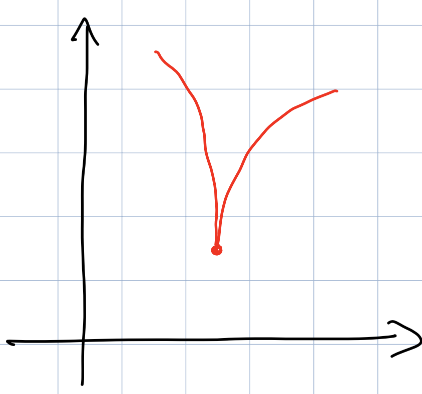
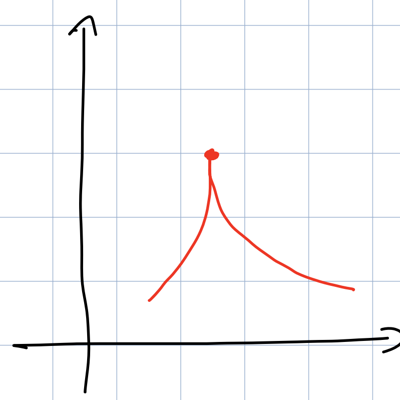
Chiamo x : punto angoloso se
$\exists\, f'_\pm(x)$ diverse e almeno una finita

a tangente verticale se $\exists f'_{\pm}(x)$ uguali tra loro, valore $\pm\infty$

cuspide se $\exists f'_{\pm}(x)$ una $-\infty$ e una $+\infty$

Ese: $D(\arccos(x)) = -\frac{1}{\sqrt{1-x^2}}$

$D(\arctan(x)) = \frac{1}{1+x^2}$

Libro Bramanti ... pag. 135

(14) $\lim_{x \to +\infty} x \cdot \sin\left(\frac{1}{x}\right) = \lim_{y \to 0^+} \frac{\sin(y)}{y} = 1$

(15) $\lim_{x \to 0} \frac{\cos(x)-1}{\sin^2(x)} = \underbrace{\lim_{x \to 0} \left(- \underbrace{\frac{1-\cos(x)}{x^2}}_{1/2} \cdot \underbrace{\left(\frac{x}{\sin(x)}\right)^2}_{1} \right)}_{-1/2}$

(15) $\lim_{x \to \infty} x \cdot \log\left(\frac{x+3}{x+1}\right)$ $\quad \infty \cdot 0$

$= \lim_{x \to \infty} \underbrace{\underbrace{\frac{\log\left(1+\frac{2}{x+1}\right)}{\frac{2}{x+1}}}_{\downarrow\, 1} \cdot \underbrace{\frac{x}{x+1}}_{\downarrow\, 1} \cdot 2}_{2}$

⑰ $\lim_{x \to +\infty} \left(\frac{x^2+3}{x^2+2} \right)^{x}$ $\quad 1^{\infty}$

$$= \underbrace{\lim_{x \to +\infty} \Bigg(\underbrace{\left(1 + \frac{1}{x^2+2}\right)^{x^2+2}}_{\lim_{y\to\infty}\left(1+\frac{1}{y}\right)^y = e} \Bigg)^{\underbrace{\frac{x}{x^2+2}}_{\to 0}}}_{e^0 = 1}$$

⑱ Che andamento hanno in 0

$\tan(x)$	0	$\sim x$
$\cot(x)$	$\pm\infty$	$\sim \frac{1}{x}$
$\arctan(x)$	0	$\sim x$
$\arcsin(x)$	0	$\sim x$

⑲ $\lim_{x \to \pm\infty} \frac{2^{x+\frac{1}{x}} + \log|x|}{x^2+1}$

$+\infty$: $\quad \frac{\overbrace{2^x \cdot \underbrace{2^{1/x}}_{\to 1} + \log(x)}^{2^x}}{\underbrace{x^2+1}_{x^2}} \quad 2^x \longrightarrow +\infty$

$$-\infty \qquad \frac{\underbrace{2^x}_{0} \cdot \underbrace{2^{1/x}}_{1} + \underbrace{\log(-x)}_{\to +\infty}}{\underbrace{x^2+1}_{x^2}} \qquad \frac{\log(-x)}{x^2} \longrightarrow 0$$

(20) $$\lim_{x\to 0} \frac{e^{x^2}-1}{\log(1+2x^2)} \qquad \frac{0}{0}$$

$$= \lim_{x\to 0} \underbrace{\frac{e^{x^2}-1}{x^2}}_{\to 1} \cdot \underbrace{\frac{x^2}{2x^2}}_{=\frac{1}{2}} \cdot \underbrace{\frac{2x^2}{\log(1+2x^2)}}_{\to 1} = \frac{1}{2}$$

(21) $$\lim_{x\to 0} \frac{x\cdot(\sin(2x))^2}{\sin(x^3)} \qquad \frac{0}{0}$$

$$\frac{x\cdot(2x)^2}{x^3} = 4 \qquad \longrightarrow 4$$

(22) $$\lim_{x\to\pm\infty} \left(\sqrt{x^2+3x+2} - \underbrace{|x|}_{\sqrt{x^2}}\right)$$

$$\frac{\cancel{x^2}+3x+2-\cancel{x^2}}{\sqrt{x^2+3x+2}+\sqrt{x^2}} \sim \frac{3x+2}{2|x|} \longrightarrow \pm\frac{3}{2}$$

(23) $$\lim_{x \to +\infty} \frac{\log(\log(x))}{1+\log(x)} \qquad \frac{\infty}{\infty}$$

$y = 1+\log(x)$

$$\lim_{y \to \infty} \frac{\log(y-1)}{y} = 0$$

(24) $$\lim_{x \to 1} \frac{(\log(x))^2}{(2x-2)^2} \qquad x = 1+y$$

$$\lim_{y \to 0} \underbrace{\frac{1}{4} \cdot \Bigg(\underbrace{\frac{\log(1+y)}{y}}_{\to 1}\Bigg)^2}_{1} = \frac{1}{4}$$

(25) $$\lim_{x \to 0} \underbrace{\underbrace{x}_{\to 0} \cdot \underbrace{e^{\sin(1/x)}}_{\nexists \lim}}_{0}$$

$$\left(\frac{1}{e} \le\right) e^{\sin(1/x)} \le e$$

(26) $$\lim_{x \to +\infty} x \cdot \log\left(\frac{x+5}{x-1}\right) \qquad \infty \cdot 0$$

$$\underbrace{\frac{\log\left(1+\frac{6}{x-1}\right)}{\frac{6}{x-1}}}_{1} \cdot \underbrace{\frac{6x}{x-1}}_{6} \longrightarrow 6$$

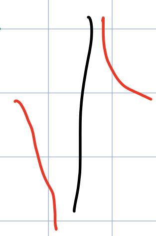
27. $$\lim_{x\to+\infty} \underset{\substack{\downarrow \\ +\infty}}{x} \cdot \underbrace{e^{\sin(x)}}_{\substack{\nexists \lim \\ \dots \geq 1/e}} = +\infty$$

Dire dove è definita e cont. la funzione data e trovare gli asintoti.

28. $$\frac{x^2+3x-1}{x+2}$$ $D = \mathbb{R}\setminus\{-2\}$ cont. su D

$$\lim_{x\to-2^\pm} f(x) = \frac{9}{0^\pm} = \pm\infty$$

$x=-2$ asintoto vert.



$$\lim_{x\to\pm\infty} f(x) = \pm\infty$$

$$\lim_{x\to\pm\infty} \frac{f(x)}{x} = 1$$

$y = 1\cdot x + q$

$$\lim_{x\to\pm\infty} (f(x)-x)$$
$$= \lim \frac{x^2+3x-1-x^2-2x}{x+2}$$
$$= \lim \frac{x-1}{x+2} = 1$$

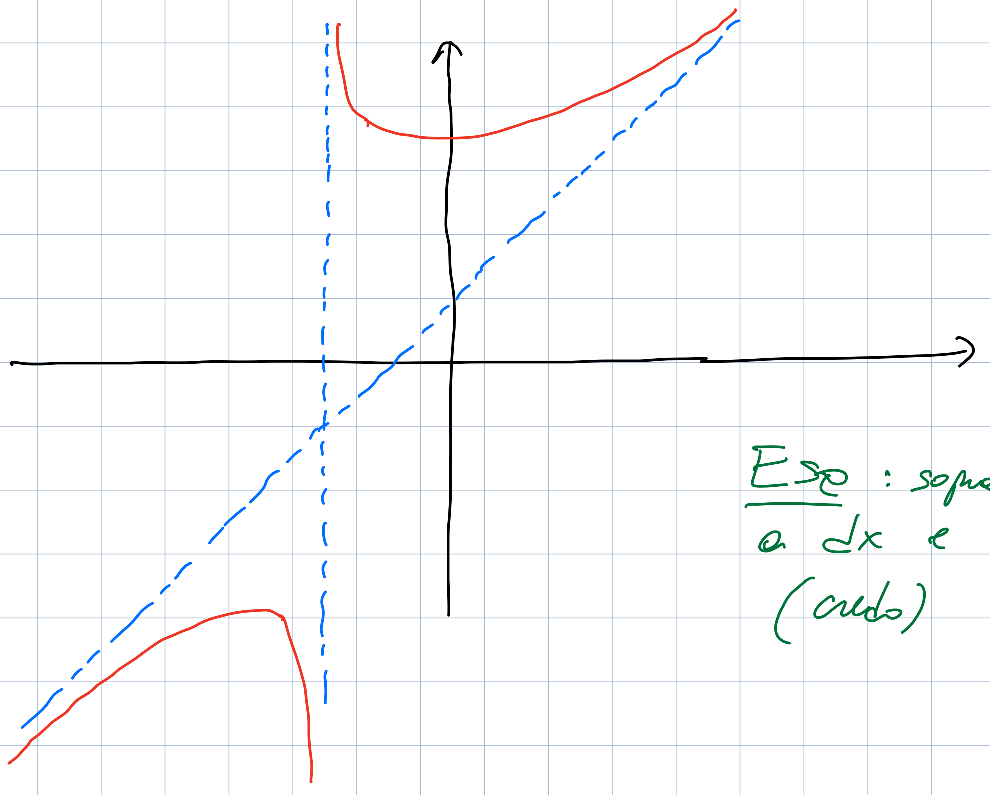
$\exists$ asintoto obl. $y = x+1$

$g(x) = x + \log(x)$ $(0,+\infty)$

$$\lim_{x\to+\infty} \frac{g(x)}{x} = 1$$

$$\lim_{x\to\infty} (g(x) - 1\cdot x)$$
$$= \lim_{x\to\infty} \log(x) = +\infty$$

$\Rightarrow \nexists$ asintoto obliquo



<u>Es.</u> : sopra asintoto a dx e sotto a sx (credo)

(29) $\dfrac{x^3+2x+1}{x+2}$ $D = \mathbb{R} \setminus \{-2\}$ cont.

$$\lim_{x \to -2^{\pm}} f(x) = \frac{-11}{0^{\pm}} = \mp\infty$$

$$\lim_{x \to \pm\infty} f(x) = +\infty$$

$$\lim_{x \to \pm\infty} \frac{f(x)}{x} = \pm\infty$$ $\nexists$ asintoti obliqui

$$\text{(30)} \quad x \cdot e^{\frac{2x+1}{x+3}} \qquad D = \mathbb{R} \setminus \{-3\}$$

$$\lim_{x \to (-3)^{\pm}} f(x) = (-3) \cdot e^{\mp\infty} \begin{cases} \nearrow 0 & \text{in } (-3)^{+} \\ \searrow -\infty & \text{in } (-3)^{-} \end{cases}$$

$$\lim_{x \to \pm\infty} f(x) = \pm\infty \cdot e^{2} = \underline{\pm\infty}$$

$$\lim_{x \to \pm\infty} \frac{f(x)}{x} = e^{2} = m \quad (\text{se } \exists q)$$

$$q = \lim_{x \to \pm\infty} \left(x \cdot e^{\frac{2x+1}{x+3}} - e^{2} \cdot x \right)$$

$$= \lim_{x \to \pm\infty} e^{2} \cdot x \cdot \left(e^{\frac{2x-1}{x+3} - 2} - 1 \right)$$

$$= \lim_{x \to \pm\infty} e^{2} \cdot x \cdot \left(e^{-\frac{7}{x+3}} - 1 \right)$$

$$= \lim_{x \to \infty} e^{2} \cdot \underbrace{\frac{e^{-\frac{7}{x+3}} - 1}{-\frac{7}{x+3}}}_{\uparrow} \cdot \underbrace{-\frac{7}{x+3} \cdot x}_{\to 1}$$

$$= -7e^{2}$$