Ist. Mat. I – CIA
24/4/24

① Risolvere

(a)
$$\begin{array}{l} \text{I} \\ \text{II} \\ \text{III} \end{array}\begin{cases} 2x - y + 3z = 26 \\ -3x + 2y + 7z = 17 \\ 4x + 3y - z = 2 \end{cases} \rightsquigarrow \begin{array}{l} \text{I} \\ \text{II} + \frac{3}{2}\text{I} \\ \text{III} - \frac{1}{2}\text{I} \end{array} \quad \begin{array}{l} \text{I} \\ \text{II} \\ \text{III} \end{array}\begin{cases} 2x + \dots \\ 0 + * y + \dots \\ 0 + *\cdot y + \dots \end{cases}$$

$$\rightsquigarrow \quad \begin{array}{l} \text{I} \\ \text{II} \\ \text{III} + k\cdot\text{II} \end{array} \quad \begin{array}{l} 2x + \dots \\ 0 + * y + \dots \\ 0 + 0 + * z \end{array}$$

$$\begin{array}{l} \text{III} \\ \text{I} \\ \text{II} \end{array}\begin{cases} z = 4x + 3y - 2 \\ \underline{2x} - \underline{y} + \underline{12x} + \underline{9y} - 6 = 26 \\ -\underline{3x} + \underline{2y} + \underline{28x} + \underline{21y} - 14 = 17 \end{cases}$$

$$\begin{array}{l} \text{I} \\ \text{II} \\ \text{III} \end{array}\begin{cases} z = 4x + 3y - 2 \\ \overset{7}{\cancel{14}}x + \overset{4}{\cancel{8}}y = \overset{16}{\cancel{32}} \\ 25x + 23y = 31 \end{cases} \qquad \begin{array}{ll} 23\cdot\text{II} - 4\cdot\text{III} & (23\cdot7 - 4\cdot25)x = (23\cdot16 - 4\cdot31 \\ 25\cdot\text{II} - 7\cdot\text{III} & (25\cdot4 - 7\cdot23)y = 25\cdot16 - 7\cdot31 \end{array}$$

III $\quad y = \dfrac{31 - 25x}{23}$

II $\quad$ solo x → lo trovo → trovo y → trovo z

(c)
$$\begin{cases} 4x - 3y + 7z = 2 \\ -3x + 5y + z = -1 \\ 10x - 13y + 5z = 5 \end{cases} \qquad \begin{cases} z = 3x - 5y - 1 \\ 4x - 3y + 21x - 35y - 7 = 2 \\ 10x - 13y + 15x - 25y - 5 = 5 \end{cases}$$

$$\begin{cases} z = 3x - 5y - 1 \\ 25x - 38y = 9 \\ 25x - 38y = 10 \end{cases}$$

Nessuna soluzione

(e) $$\begin{cases} 7x - y + 2z = 4 \\ 2x + 3y - 5z = -3 \\ x - 10y + 17z = 13 \end{cases}$$

$$\begin{cases} y = 7x + 2z - 4 \\ 2x + 21x + 6z - 12 - 5z = -3 \\ x - 70x - 20z + 40 + 17z = 13 \end{cases}$$

$$\begin{cases} y = 7x + 2z - 4 \\ 23x + z = 9 \end{cases}$$

~~$-69x - 3z = -27$~~

$$\begin{cases} z = -23x + 9 \\ y = 7x - 46x + 18 - 4 \end{cases}$$

$$\begin{cases} z = -23x + 9 \\ y = -39x + 14 \end{cases}$$

OK

$$\begin{cases} x = -t \\ y = 39t + 14 \\ z = 23t + 9 \end{cases}$$

OK

$$\begin{pmatrix} 0 \\ 14 \\ 9 \end{pmatrix} + t \cdot \begin{pmatrix} -1 \\ 39 \\ 23 \end{pmatrix} \qquad t \in \mathbb{R}$$

OK

$$\begin{pmatrix} 0 \\ 14 \\ 9 \end{pmatrix} + \mathrm{Span} \begin{pmatrix} -1 \\ 39 \\ 23 \end{pmatrix}$$

OK

$$\begin{pmatrix} -1 \\ 53 \\ 32 \end{pmatrix} + \mathrm{Span} \begin{pmatrix} \sqrt{2} \\ -39\sqrt{2} \\ -23\sqrt{2} \end{pmatrix}$$

OK

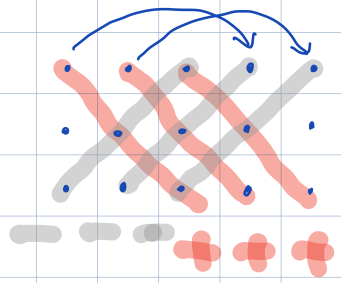
② Quante soluz. ha:

$$\begin{cases} (1-t)x + 2y - 2z = 1 \\ (1+t)x + 3y + z = t \\ 7x + 12y + 2t\,z = -1 \end{cases}$$

$$\left(\underbrace{\begin{matrix} 1-t & 2 & -2 \\ 1+t & 3 & 1 \\ 7 & 12 & 2t \end{matrix}}_{A} \;\middle|\; \begin{matrix} 1 \\ t \\ -1 \end{matrix}\right)$$

So che se $\det(A) \neq 0$ la soluz. è unica.

$$\begin{aligned} \det(A) = \;& 6t - 6t^2 + 14 \\ & -24t \quad -24 \\ & -4t - 4t^2 + 42 \\ & +12t \quad -12 \\ = \;& -10t^2 - 10t + 20 = -10(t^2 + t - 2) \\ & = -10(t+2)(t-1) \end{aligned}$$



Se $t \neq -2$, $t \neq 1$ ho soluz. unica.

$t = -2$
$$\begin{cases} 3x + 2y - 2z = 1 & \checkmark \\ -x + 3y + z = -2 & \checkmark \\ 7x + 12y - 4z = -1 & \checkmark \end{cases}$$

$$\begin{cases} x = 3y + z + 2 \\ 9y + 3z + 6 + 2y - 2z = 1 \\ 21y + 7z + 14 + 12y - 4z = -1 \end{cases}$$

$$\begin{cases} x = \dots \\ 11y + z = -5 \\ \cancel{33y + 3z = -15} \end{cases} \qquad \begin{cases} z = -11y - 5 \\ x = \dots \end{cases}$$

infinite soluzioni (∞^1)

un parametro libero

$t = 1$

$$\begin{cases} 2y - 2z = 1 \\ 2x + 3y + z = 1 \\ 7x + 12y + 2z = -1 \end{cases}$$

$$\begin{cases} z = -2x - 3y + 1 \\ 2y + 4x + 6y - 2 = 1 \\ 7x + 12y - 4x - 6y + 2 = -1 \end{cases}$$

$$\begin{cases} z = \dots \\ 4x + 8y = 3 \\ 3x + 6y = -3 \end{cases} \qquad \begin{cases} z = \\ x + 2y = 3/4 \\ x + 2y = -1 \end{cases}$$

Nessuna soluz.

③ Calcolare det.

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

$$\det\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \rightsquigarrow \text{Sarrus}$$

ⓗ

$$\begin{pmatrix} 6 & 1 & -3 \\ 4 & -7 & 2 \\ -2 & 5 & 8 \end{pmatrix} = 6\cdot(-7)\cdot 8 + 1\cdot 2\cdot(-2) + (-3)\cdot 4\cdot 5$$

$$-(-3)(-7)(-2) - 1\cdot 4\cdot 8 - 6\cdot 2\cdot 5$$

$$= \dots$$

$= $ (sviluppo II riga)

$$-4\cdot\det\begin{pmatrix} 1 & -3 \\ 5 & 8 \end{pmatrix} + (-7)\det\begin{pmatrix} 6 & -3 \\ -2 & 8 \end{pmatrix} - 2\det\begin{pmatrix} 6 & 1 \\ -2 & 5 \end{pmatrix}$$

$$= \dots$$

Uso la I riga per ottenere 0 su II e III riga nella II colonna

$$\begin{array}{c} \text{I} \\ \text{II} \\ \text{III} \end{array}\begin{pmatrix} 6 & 1 & -3 \\ 4 & -7 & 2 \\ -2 & 5 & 8 \end{pmatrix} \rightsquigarrow \begin{array}{c} \text{I} \\ \text{II} + 7\cdot\text{I} \\ \text{III} - 5\cdot\text{I} \end{array}\begin{pmatrix} 6 & 1 & -3 \\ 46 & 0 & -19 \\ -32 & 0 & 23 \end{pmatrix}$$

$$= -1\cdot\det\begin{pmatrix} 46 & -19 \\ -32 & 23 \end{pmatrix} + 0\cdot\dots + 0\dots$$

(j) $\begin{pmatrix} \boxed{3} & \boxed{-5} & \boxed{2} & \textcircled{1} \\ -2 & 4 & 1 & 7 \\ 1 & 6 & -4 & 3 \\ 4 & 2 & -3 & 3 \end{pmatrix}$

NON ESISTE SARRUS 4×4

Uso il coeff nella I riga IV col e faccio operaz. di colonna per ottenere 0 sulle colonne I, II, III nella prima riga

$$(\mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}) \rightsquigarrow (\mathrm{I} - 3\cdot \mathrm{IV}, \mathrm{II} + 5 \cdot \mathrm{IV}, \mathrm{III} - 2 \cdot \mathrm{IV}, \mathrm{IV})$$

$$\rightarrow \begin{pmatrix} 0 & 0 & 0 & 1 \\ -23 & 39 & -13 & 7 \\ -8 & 21 & -10 & 3 \\ -5 & 17 & -9 & 3 \end{pmatrix}$$

$$\pm 0 \cdot \ldots \pm 0 \cdot \ldots \pm 0 \cdot \ldots - 1 \cdot \det \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$

④ Calcolare A^{-1}

Ricordo: nel posto $\overset{\text{riga}}{i}, \overset{\text{colonna}}{j}$ di A^{-1} c'è

$$\frac{(-1)^{i+j}}{\det(A)} \cdot \det \left(\begin{matrix} \text{matrice ottenuta} \\ \text{cancellando} \\ \text{riga } j \text{ e colonna } i \end{matrix} \right)$$

(b) $\begin{pmatrix} -2 & 3 \\ 8 & -5 \end{pmatrix}$ $\qquad A^{-1} = -\frac{1}{14} \begin{pmatrix} -5 & -3 \\ -8 & -2 \end{pmatrix}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

(d) $$A = \begin{pmatrix} 3 & 1 & -1 \\ 1 & -2 & 3 \\ 4 & 1 & -4 \end{pmatrix}$$

$$\det(A) = \begin{pmatrix} 0 & 0 & -1 \\ 10 & 1 & 3 \\ -8 & -3 & -4 \end{pmatrix} = +(-1) \det \begin{pmatrix} 10 & 1 \\ -8 & -3 \end{pmatrix}$$

$$= -(-30+8) = 22$$

$$A = \begin{pmatrix} 3 & 1 & -1 \\ 1 & -2 & 3 \\ 4 & 1 & -4 \end{pmatrix} \qquad A^{-1} = \frac{1}{22} \begin{pmatrix} \boxed{5} & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$

$$A = \begin{pmatrix} 3 & 1 & -1 \\ 1 & -2 & 3 \\ 4 & 1 & -4 \end{pmatrix} \qquad A^{-1} = \frac{1}{22} \begin{pmatrix} \cdot & \boxed{3} & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$

$$A = \begin{pmatrix} 3 & 1 & -1 \\ 1 & -2 & 3 \\ 4 & 1 & -4 \end{pmatrix} \qquad A^{-1} = \frac{1}{22} \begin{pmatrix} \cdot & \cdot & \boxed{1} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$

$$A = \begin{pmatrix} 3 & 1 & -1 \\ 1 & -2 & 3 \\ 4 & 1 & -4 \end{pmatrix} \qquad A^{-1} = \frac{1}{22} \begin{pmatrix} \cdot & \cdot & \cdot \\ \boxed{16} & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$

⑤ Risolvere il sistema usando Cramer se possibile.

Richiamo: $\underset{m\times m}{A}\cdot x = b$

se $\det(A) \neq 0$ la soluz è:

$$x_j = \frac{\det\begin{pmatrix}\text{matrice ottenuta da } A \\ \text{sostituendo colonna } j \text{ con } b\end{pmatrix}}{\det(A)}$$

(a) $\begin{cases} x+5y+3z=3 \\ 3x-4y+2z=6 \\ 5x-y+6z=8 \end{cases}$ $\qquad A = \begin{pmatrix} 1 & 5 & 3 \\ 3 & -4 & 2 \\ 5 & -1 & 6 \end{pmatrix}$

$$\det(A) = -24+50-9+60-90+2 \neq$$

$$x = \frac{\det\begin{pmatrix} 3 & 5 & 3 \\ 6 & -4 & 2 \\ 8 & -1 & 6 \end{pmatrix}}{\det\begin{pmatrix} 1 & 5 & 3 \\ 3 & -4 & 2 \\ 5 & -1 & 6 \end{pmatrix}} \qquad y = \frac{\det\begin{pmatrix} 1 & 3 & 3 \\ 3 & 6 & 2 \\ 5 & 8 & 6 \end{pmatrix}}{\det\begin{pmatrix} 1 & 5 & 3 \\ 3 & -4 & 2 \\ 5 & -1 & 6 \end{pmatrix}}$$

⑧ Calcolare rank(A)

Def: $\underset{m\times n}{\text{rank}(A)} = \dim(\text{Im}(A \text{ pensata } \mathbb{R}^n \to \mathbb{R}^m))$

$= \dim(\text{Span}(\text{colonne di } A))$

= max numero di colonne di A
che sono lin. indip.

Richiamo: $\mathrm{rank}(A) = \max\{k :$ esiste sottomatrice B $k \times k$ di A con $\det(B) \neq 0\}$

Metodo più efficace: estrarre dalle colonne di A una base del generato.

tengo la prima non nulla

tengo la prima non multipla di quella già tenuta

tengo la prima che non è generata dalle due già tenute ....

(c)
$$\begin{pmatrix} 2 & 1 & 5 & 4 \\ 1 & 3 & 5 & 7 \\ 0 & 1 & 1 & 2 \\ 1 & 0 & 2 & 1 \end{pmatrix}$$

✓ ✓

$2 \cdot \mathrm{I} + 1 \cdot \mathrm{II}$ ✗

$\mathrm{I} + 2 \cdot \mathrm{II}$ ✗

$\mathrm{rank}(A) = 2$

$$10 + 35 + 12 - 20 - 30 - 7 = 57 - 57 = 0$$

(Tutte le 3×3 hanno $\det = 0$)