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Optimal stopping and American options with discrete dividends and exogenous risk

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Abstract

In this paper we analyze some problems arising in the evaluation of American options when the underlying security pays discrete dividends. To this aim, we study the problem of maximizing the expected gain process over stopping times taking values in the union of disjoint, real compact sets. The results we obtain can be applied to evaluate options with restrictions on exercise periods, but are also useful for the evaluation of American options on assets that pay discrete dividends. In particular, we generalize the evaluation formula for American call options due to Whaley [Journal of Financial Economics 9 (1981) 207], allowing for a stochastic jump of the underlying security at the ex-dividend date and discuss the existence of the optimal stopping time. In the same framework, we analyze American put options, justifying the procedure used in Meyer [Journal of Computational Finance 5 (2) (2002)] to account for the presence of discrete dividends in the free boundary formulation from the perspective of optimal stopping theory.

JEL classification: C60; G12

Keywords: No arbitrage evaluation; American options; Discrete dividends; Restrictions on exercise dates; Optimal stopping

1. Introduction

American options on assets that pay discrete dividends are widely traded on financial markets. The evaluation of such derivatives in continuous-time models (e.g. Black–Scholes) presents some mathematical problem, usually neglected in practice or solved by means of purely financial intuition. The dividend payment at a *known* date produces a decrease in the stock value that is *approximately* equal to the dividend amount (cf. Murray and Jagannathan, 1998; Heath and Jarrow, 1988; Battauz and Beccacece, 2004) and consequently the trajectories of the underlying payoff process have a jump at a fixed instant. Therefore, options that without dividends are exercised optimally only at the maturity date, may have positive early-exercise premium. For example, an American call option may be optimally exercised at the end of the cum-dividend date if a dividend is paid during the life of the option. From a

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financial point of view, this fact seems natural (see Roll, 1977; Whaley, 1981; Geske, 1979), but the mathematical description of the phenomenon presents some technical difficulty. Indeed, if the dividend payment occurs at T_D, meaning that T_D^- is the *cum-dividend* date and T_D is the *ex-dividend* date, it may be optimal to exercise the call option just immediately before the decrease of the stock. But formally such an optimal stopping time does not exist.[1] Moreover, the ex-dividend stock jump can be affected by an *additional* source of risk, as documented in the literature (compare Murray and Jagannathan, 1998; Battauz and Beccacece, 2004 and the references therein). The presence of an additional randomness source makes the market model free of arbitrage but incomplete (see Heath and Jarrow, 1988; Ohashi, 1991; Battauz, 2003). In this framework, to evaluate a contingent claim we *choose* an equivalent risk-neutral martingale measure, among all the equivalent probabilities under which the actualized implied gain process is a martingale. Then, if the contingent claim is of American type, its actualized fair value is given by the Snell envelope of the discounted payoff process under the selected risk-neutral probability. The existence of the smallest right continuous with left limits supermartingale dominating the discounted payoff process is guaranteed, since the trajectories of the payoff process are right continuous with left limits (compare El Karoui, 1979). On the contrary, the existence and the characterization of the optimal stopping time does not follow from standard results in optimal stopping theory, as in the case of the call option. Indeed, the usual assumption of left-continuity in expectation, required, for example, in El Karoui (1979), holds not true in this case. Hence, to deduce in our framework the existence of the optimal stopping time and to characterize it as the arrival time in the set where the Snell envelope coincides with the discounted payoff process, we use the following trick. We *stretch* the time-interval $[0, T]$ that represents the life period of the option introducing a *fictitious* interval $[T_1, T_2]$, where T_1 denotes the end of the cum-dividend date and T_2 the beginning of the ex-dividend date. During $[T_1, T_2[$ the stock price process as well as the information structure remain constant and vary only at T_2. This allows us to glue the *discrete* irregularity due to the dividend payment and the exogenous source of risk, acting at the dividend date, to the *continuos-time usual* random evolution of the market. Looking for optimal exercise policies, we prove that investors can exercise optimally *either before* the end of the cum-dividend date, T_1, *or after* the beginning of the ex-dividend date, T_2. Hence the evaluation problem is reduced to compute the "Snell envelope" on stopping times taking values in $[0, T_1] \cup [T_2, T]$. The results we obtain can also be applied to evaluate American options with restrictions on exercise periods.

The paper is organized as follows. In the next section, after a brief description of the adopted market model (introduced in Battauz, 2003), we introduce the stretch of the filtration and market processes from T_1 to T_2. We prove in Theorem 1 that no optimal exercise policy takes values in $]T_1; T_2[$ and hence the introduction of the fictitious interval $]T_1; T_2[$ does not produce any financial anomaly. On the contrary, this trick allows us to characterize the optimal stopping time as the first instant such that the Snell envelope reaches the underlying payoff process, though the trajectories of the latter are not continuous. As a consequence, Corollary 1 provides the backward *discrete* link from T_2 to T_1, formalizing the financial intuition that if no exercise in $]T_1; T_2[$ is allowed, then the discounted value of any option in T_2 is the maximum between its actualized payoff process in T_1 (whenever the immediate exercise is convenient) and the discounted *continuation value* available from T_2 on. We apply these results to evaluate options with restrictions on exercise periods in Remark 1 and focus in Section 3 on American call options and in Section 4 on American put options.

2. The market model

We adopt the market model introduced in Battauz (2003) and used also in Battauz and Beccacece (2004) that generalizes the approach of Björk (1998) for describing the behavior of assets that pay discrete dividends. We recall here briefly the basics of the model and refer to Battauz (2003) for a complete discussion on it. We consider an asset S that pays a dividend D at a *fixed and known* date $t = T_D$. More precisely, the instant T_D^- is the *cum-dividend*

[1] The exercise is optimal at T_D^- but not at T_D anymore.

date, while the instant T_D is the *ex-dividend* date. Beyond an *usual* probability space $(\Omega^0, \mathbf{P^0}, (\mathcal{F}_t^0)_t)$, with standard assumptions on the filtration $(\mathcal{F}_t^0)_t$ (see Protter, 1990), representing the *usual* uncertainty of the market for $t \neq T_D$, we consider the *additional* probability space $(\Omega^x, \mathbf{P^x}, \mathcal{A}^x)$ that carries the exogenous additional randomness at T_D (see Battauz and Beccacece, 2004 and the references therein). Hence the continuous-time evolution of the market is described on the product space $\Omega = \Omega^0 \times \Omega^x$ by the completed right continuous version of the filtration $\mathcal{F} = \mathcal{F}^0 \otimes \mathcal{F}^x$, where $\mathcal{F}_t^x = \{\emptyset, \Omega^x\}$ for $0 \leq t < T_D$ and $\mathcal{F}_t^x = \mathcal{A}^x$ for $T_D \leq t \leq T$, endowed with the product measure $\mathbf{P} = \mathbf{P^0} \otimes \mathbf{P^x}$. We assume that the stock S has a log-normal behavior everywhere but at T_D and between T_D^- (the cum-dividend date) and T_D (the ex-dividend date) the stock S has a jump affected by a random variable $X : (\Omega^x, \mathbf{P^x}, \mathcal{A}^x) \to [\alpha; \beta] \subset [-1; 1]$ as follows:

$$\Delta S(T_D) = -D + X(S(T_D^-) - D). \tag{1}$$

The market, constituted by the stock $S(t)$ and the riskless bond $B(t) = \mathrm{e}^{rt}$ (where r is the constant riskless interest rate), is incomplete due to the additional randomness source. Hence there are many equivalent martingale measures for the implied discounted gain process, constituted by the actualized stock value and the discounted cumulative dividend process. Theorem 2 and the subsequent Remark in Battauz (2003) provide a characterization for the densities of such equivalent martingale measures, that can be stated as follows:

Every martingale measure $\mathbf{Q} \ll \mathbf{P}$ for the discounted cumulative gain process is of the form:

$$\mathbf{Q} = \mathbf{Q^0} \times \mathbf{Q^x}, \tag{2}$$

where $\mathbf{Q^0}$ is the usual risk-neutral measure captured by[2] $\mathrm{d}\mathbf{Q^0}/\mathrm{d}\mathbf{P^0} = \mathcal{E}(-\int_0^T (\mu - r/\sigma_0)\,\mathrm{d}W_s)$, and $\mathrm{d}\mathbf{Q^x} = U\,\mathrm{d}\mathbf{P^x}$ is only constrained by

$$\int_{\Omega^x} U(\omega, x)\,\mathrm{d}\mathbf{P^x}(x) = 1, \qquad \int_{\Omega^x} U(\omega, x)X(x)\,\mathrm{d}\mathbf{P^x}(x) = 0 \tag{3}$$

for $\mathbf{P^0}$*-a.e.* $\omega \in \Omega^0$. The measure $\mathbf{Q}$ is a probability measure if $U > 0$.

The dynamics of the stock price S is described under $\mathbf{Q} = \mathbf{Q^0} \times \mathbf{Q^x}$ by

$$S(0) = S_0, \qquad S(T_D) = (S(T_D^-) - D)(1 + X), \qquad \mathrm{d}S(t) = S(t)(r\,\mathrm{d}t + \sigma\,\mathrm{d}\tilde{W}_t), \quad t \neq T_D, \tag{4}$$

where $\sigma \in]0; +\infty[$ is the volatility of the stock price, $\tilde{W}$ is a $\mathbf{Q^0}$-Brownian motion with respect to $\mathcal{F}^0$ and the distribution of X is captured by $\mathbf{Q^x}$.

In order to evaluate American derivatives in a complete market, one has to compute the Snell envelope of the discounted payoff process under the risk-neutral measure. In incomplete markets, there are many prices that do not permit arbitrage opportunities. Indeed, for a derivative written on S we find an interval of prices, varying the risk-neutral probability according to Formula (3). Therefore, we first determine the Snell envelope of the discounted payoff process under a selected equivalent martingale measure (for example, the quadratic optimal measure; see Battauz, 2003). Then in Sections 3 and 4 we provide upper and lower bounds to the interval of NoArbitrage prices of derivatives whose payoff process is a monotonic function of the underlying security, like call and put options (see Battauz and Beccacece (2004) for numerical results based on the calibration of this model to the Italian market).

In our framework, the trajectories of the stock price are continuous everywhere but at T_D. The presence of the dividend and the discrepancy X produce in the stock a jump described in Eq. (1) and make the process in $t = T_D$ right continuous with left limits (RCLL henceforth). This is enough to guarantee the existence of the Snell envelope for a discounted payoff process[3] $\tilde{\psi}$ that is continuous with respect to the price of the underlying security (compare El Karoui, 1979). Hence there exists the Snell envelope of the discounted payoff for both the call and the put option, but problems arise in general with respect to the existence of an optimal stopping

[2] $\mathcal{E}(\cdot)$ denotes the stochastic exponential, cf. Protter (1990).

[3] A $\tilde{}$ denotes the values actualized with the riskless bond.

time. However, since trading takes place until the end of the cum-dividend date, it is financially consistent to allow for exercise policies in a set $[0; T_1] \cup [T_2; T]$ (with $T_1 < T_2$), where T_1 plays the role of the end of the cum-dividend date T_D^- and T_2 the beginning of the ex-dividend date T_D. In fact, the same trick is implicitly and informally applied to provide an analytical formula for evaluating American call options with known discrete dividends in Roll (1977), Geske (1979) and Whaley (1981). Therefore, to formalize mathematically the possibility of early exercise at the end of the cum-dividend date, we redefine the discounted payoff process in $[T_1; T_2]$ as follows:

$$\hat{\psi}(t) = \begin{cases} \tilde{\psi}(t) & \text{if } t \le T_1, \\ \tilde{\psi}(T_1) & \text{if } T_1 \le t < T_2, \\ \tilde{\psi}(t) & \text{if } T_2 \le t \le T. \end{cases} \tag{5}$$

In the same way we stretch the filtration on $[T_1; T_2[$, setting

$$\hat{\mathcal{F}}_t = \begin{cases} \mathcal{F}_t & \text{if } t \le T_1, \\ \mathcal{F}_{T_1} & \text{if } T_1 \le t < T_2, \\ \mathcal{F}_t & \text{if } T_2 \le t \le T \end{cases} \tag{6}$$

and the stock price process similarly, so that under a risk-neutral measure **Q** it is driven by

$$\begin{aligned} &\hat{S}(0) = S_0, \qquad \mathrm{d}\hat{S}_t = \hat{S}_t(r\,\mathrm{d}t + \sigma\,\mathrm{d}\tilde{W}_t) \quad \text{if } 0 < t \le T_1, \qquad \hat{S}_t = \hat{S}_{T_1} \quad \text{if } T_1 \le t < T_2, \\ &\hat{S}_{T_2} = (\hat{S}_{T_1} - D)(1 + X), \qquad \mathrm{d}\hat{S}_t = \hat{S}_t(r\,\mathrm{d}t + \sigma\,\mathrm{d}\tilde{W}_t) \quad \text{if } T_2 < t \le T. \end{aligned} \tag{7}$$

We prove henceforth that this formal settlement is what we need to represent mathematically the possibility of optimal early-exercise at the end of the cum-dividend date and that in the *fictitious* time-interval $]T_1; T_2[$ nothing new happens. Indeed, the Snell envelope of $\hat{\psi}(t)$ with respect to the filtration $\hat{\mathcal{F}}$ exists[4] and is in fact the smallest RCLL supermartingale greater than $\hat{\psi}$ (compare Theorem 2.15 in El Karoui (1979)). Since both the process $\hat{\psi}$ as well as the filtration $\hat{\mathcal{F}}$ are constant on $[T_1; T_2[$, we prove in Theorem 1, Part 1, that the Snell envelope J is constant on the same interval. This allows us to prove in Part 2 that the optimal stopping time τ^* is the arrival time in the set $\{J = \hat{\psi}\}$. This characterization constitutes also a technical contribution, since standard results do not apply due to the lack of left-continuity in expectation of the process $\hat{\psi}$ (compare Theorem 2.18 in El Karoui (1979)). Having this characterization at our disposal, it is easy to prove in Theorem 1, Part 3, that every exercise policy taking values in $]T_1; T_2[$ is sub-optimal. Hence the introduction of the fictitious interval $]T_1; T_2[$ gains financial consistency without producing modelization anomalies.

Theorem 1. *Let $\hat{\psi}$ be a process continuous for $t \neq T_2$, constant on $[T_1; T_2[$ and uniformly integrable. Denote with $\mathbb{T}$ the set of the $\hat{\mathcal{F}}$-stopping times and with J the Snell envelope of $\hat{\psi}$ with respect to the filtration $\hat{\mathcal{F}}$ under the probability measure* **Q**. *We have that*

1. *The process J is constant on $[T_1; T_2[$.*
2. *The optimal stopping time $\tau^* \in \mathbb{T}$, i.e. $\mathbb{E}[\hat{\psi}(\tau^*)] = \sup_{U \in \mathbb{T}} \mathbb{E}[\hat{\psi}(U)]$, is the arrival time in the set $\{J = \hat{\psi}\}$, i.e. $\tau^* = \inf\{t \ge 0 | \hat{\psi}(t) = J(t)\} \wedge T$.*
3. *The optimal stopping time $\tau^* \in [0; T_1] \cup [T_2; T]$ a.e.*

Proof. To prove Part 1, we first notice that the Snell envelope of $\hat{\psi}$ is nonincreasing on $[T_1; T_2[$. Indeed, since J is a supermartingale and $\hat{\mathcal{F}}_{t_1} = \hat{\mathcal{F}}_{t_2}$ for all $t_1 < t_2 \in [T_1; T_2[$, we have that $J_{t_1} \ge \mathbb{E}[J_{t_2} | \hat{\mathcal{F}}_{t_1}] = \mathbb{E}[J_{t_2} | \hat{\mathcal{F}}_{t_2}] = J_{t_2}$.

[4] Note that the process $\hat{\psi}(t)$ is RCLL in T_2 and continuous elsewhere.

Moreover, J is constant on $[T_1; T_2[$. To see this, define

$$\hat{J}_t = \begin{cases} J_t & \text{if } t \in [0; T_1[, \\ J_{T_2^-} & \text{if } t \in [T_1; T_2[, \\ J_t & \text{if } t \in [T_2; T] \end{cases}$$

that turn out to be the Snell envelope of $\hat{\psi}$. In fact, $\hat{J} \geq \hat{\psi}$, since $\hat{\psi}$ is constant on $[T_1; T_2[$ and it is RCLL. Moreover, $\hat{J}$ is a supermartingale. Since by construction $\hat{J} \leq J$, it turns out that $\hat{J} = J$.

To prove Part 2, we have to generalize the proof of Theorem 2.18 in El Karoui (1979), since the process $\hat{\psi}$ is not left-continuous in expectation everywhere. To this aim, let $\lambda_n \in [0; 1[$, $\lambda_n \uparrow 1$ and denote by $A_n = \{(\omega, t) | \hat{\psi}_t(\omega) \geq \lambda_n J_t(\omega)\}$ and by τ^{A_n} the arrival time in the set A_n, i.e. $\tau^{A_n} = \inf\{t \geq 0 | \hat{\psi}_t \geq \lambda_n J_t\}$. In the same way, denote by τ^A the arrival time in the set $A = \{(\omega, t) | \hat{\psi}_t(\omega) = J_t(\omega)\}$.

Since λ_n is increasing, the sequence of stopping times τ^{A_n} is also increasing and being bounded by T it admits a limit, that we denote by $\bar{\tau}$. Since both $\hat{\psi}$ and J are constant in $[T_1; T_2[$, by Part 1, $\tau^{A_n} \notin]T_1; T_2[$. Indeed, if there exist $\mathcal{B} \subset \Omega$ such that $\tau^{A_n}(\omega) \in [T_1; T_2[$, for $\omega \in \mathcal{B}$, the constancy of J and $\hat{\psi}$ on $[T_1; T_2[$ implies that on the set $\mathcal{B}$ we have $\hat{\psi}(t) = \hat{\psi}(\tau^{A_n}) \geq \lambda_n J(\tau^{A_n}) = \lambda_n J(t)$ for all $t \in [T_1; T_2[$ and hence $\tau^{A_n} \leq T_1$. Therefore, either $\tau^{A_n} \leq T_1$ or $T_2 \leq \tau^{A_n} \leq T$. Moreover, since the sequence τ^{A_n} is increasing, we have that for $\mathbf{Q}$-a.e. $\omega \in \Omega$, the sequence $\tau^{A_n}(\omega)$ lies definitely[5] either in $[0; T_1]$ or in $[T_2; T]$. Being $\hat{\psi}$ continuous for $t \in [0; T_1]$ and for $t \in [T_2; T]$, we can conclude that $\hat{\psi}_{\tau^{A_n}} \to \hat{\psi}_{\bar{\tau}}$ almost everywhere as $n \to +\infty$. Moreover since the family $(\hat{\psi}_{\tau^{A_n}})_n$ is uniformly integrable, it follows that $\mathbb{E}[\hat{\psi}_{\tau^{A_n}}] \to \mathbb{E}[\hat{\psi}_{\bar{\tau}}]$ as $n \to +\infty$. Therefore, since $(1/\lambda_n)\mathbb{E}[\hat{\psi}_{\tau^{A_n}}] \geq \sup_{U \in \mathbb{T}} \mathbb{E}[\hat{\psi}_U]$ for all λ_n, passing to the limit we have that $\mathbb{E}[\hat{\psi}_{\bar{\tau}}] \geq \sup_{U \in \mathbb{T}} \mathbb{E}[\hat{\psi}_U]$, i.e. $\bar{\tau}$ is optimal. Since for all λ_n we have $A_n \supseteq A$, it follows that $\tau^{A_n} \leq \tau^A$ and therefore $\bar{\tau} = \lim_{n \to \infty} \tau^{A_n} \leq \tau^A$. Moreover, since $\bar{\tau}$ is optimal, by Theorem 2.12 in El Karoui (1979) we have that $J(\bar{\tau}) = \hat{\psi}_{\bar{\tau}}$, hence $\bar{\tau} \geq \tau^A$. Since τ^A coincides with the optimal stopping time $\bar{\tau}$, the arrival time τ^A is optimal.

Part 3 follows immediately since τ^* is the first instant t such that $J(t) = \hat{\psi}(t)$. □

Looking at the optimal stopping described in Theorem 1, Part 3, we observe that the first instant after T_1 when early exercise may be optimal is T_2. Hence it is possible to go backward from T_2 and T_1 as in *discrete* time. The following corollary supplies the recursive link from T_2 to T_1, proving in a rigorous way what is also financially intuitive: if exercise in $]T_1; T_2[$ is forbidden, the actualized value of the option in T_2 is the maximum between the actualized payoff process in T_1 (if immediate exercise is convenient) and the discounted *continuation value* available from T_2 on.

Corollary 1. *Under the assumptions of* Theorem 1, *let*

$$\varphi_t = \begin{cases} \hat{\psi}_t & \textit{if } \in [0; T_1[, \\ \max\{\hat{\psi}_{T_1}, \mathbb{E}[J(T_2) | \hat{\mathcal{F}}_{T_1}]\} & \textit{if } t = T_1. \end{cases}$$

Then, the Snell envelope J of the process $\hat{\psi}$ is

$$J_t = \begin{cases} \operatorname{ess\,sup}_{t \leq \tau \leq T_1} \mathbb{E}[\varphi(\tau) | \hat{\mathcal{F}}_t] & \textit{if } t \in [0; T_1], \\ \varphi(T_1) & \textit{if } t \in [T_1; T_2[, \\ \operatorname{ess\,sup}_{t \leq \tau \leq T} \mathbb{E}[\hat{\psi}(\tau) | \hat{\mathcal{F}}_t] & \textit{if } t \in [T_2; T]. \end{cases} \tag{8}$$

Proof. Denote by K_t the process defined on the right hand side in (8). K is a RCLL supermartingale and by construction $K \geq \hat{\psi}$. Hence, $K \geq J$. To prove the opposite inequality, note that K_t coincides already with J_t for

[5] There exists $n_0 \in \mathbb{N}$ such that either $\tau^{A_n}(\omega) \in [0; T_1]$ for all $n > n_0$ or $\tau^{A_n}(\omega) \in [T_2; T]$ for all $n > n_0$.

$t \in [T_2; T]$. Moreover, since J is a supermartingale, we have $J_{T_1} \geq \mathbb{E}[J(T_2)|\mathcal{F}_{T_1}]$ and by construction $J_{T_1} \geq \hat{\psi}_{T_1}$. This implies that $J(T_1) \geq \varphi(T_1)$ and for $t \leq T_1$ we already have $J(t) \geq \hat{\psi}(t) = \varphi(t)$. Since K is the Snell envelope of the process φ on $[0; T_1]$, we obtain $J \geq K$ and the thesis is proved. □

Remark 1 (American options with restrictions on exercise dates). Corollary 1 supplies the solution to the problem of evaluating American options with restrictions on exercise dates. Indeed, denote with ϕ the continuous actualized payoff process of such an option and suppose that the early exercise is allowed only for $t \in [0; T_1] \cup [T_2; T]$. For $t \in [T_2; T]$, the option coincides with an usual American one and its value is given by $\tilde{V}(t) = \operatorname{ess\,sup}_{t \leq \tau \leq T} \mathbb{E}[\phi(\tau)|\hat{\mathcal{F}}_t]$. The troubles for the holder arise at T_1: he has to decide whether to exercise immediately, gaining $\phi(T_1)$, or to wait for the *continuation value* $\mathbb{E}[\tilde{V}(T_2)|\hat{\mathcal{F}}_{T_1}]$, available from T_2 on. Hence $\tilde{V}(T_1) = \max\{\phi(T_1); \mathbb{E}[\tilde{V}(T_2)|\hat{\mathcal{F}}_{T_1}]\}$ and for $t \in [0; T_1]$ the option can be reduced to an usual American option, noticing that the *terminal* condition at T_1 on $\tilde{V}$ is the previously written one. Hence, replacing in Corollary 1 the payoff process $\hat{\psi}$ with ϕ, we see that the right hand side in Formula (8) defines exactly the discounted value of the option with restrictions on exercise dates, $\tilde{V}$. This means that $\tilde{V}$ is the Snell envelope of an *usual* American option whose payoff $\hat{\psi}$ coincides with ϕ everywhere but on the set $]T_1; T_2[$. On $]T_1; T_2[$ the payoff ϕ is undefined and we set $\hat{\psi}(t) = \phi(T_1)$ for $t \in [T_1; T_2[$, as in Formula (5). Hence an option whose exercise is not allowed on $]T_1; T_2[$ can be evaluated as an usual American one, *filling* the forbidden interval $]T_1; T_2[$ with the definition of the *stretched* $\hat{\psi}$, as already explained.

Remark 2. We notice that Theorem 1 and Corollary 1 do not depend on the hypotheses on the dynamics of S, requiring only the continuity of the actualized payoff process ϕ for $t \neq T_1$ and the constancy on $[T_1; T_2[$ of both the payoff ϕ and the filtration $\hat{\mathcal{F}}$.

3. American call options

This section is devoted to the study of the American call option, whose discounted payoff process is represented in our framework by

$$\hat{\psi}(t) = \begin{cases} e^{-rt}(S(t) - K)^+ & \text{if } t \leq T_1, \\ e^{-rT_1}(S(T_1) - K)^+ & \text{if } T_1 \leq t < T_2, \\ e^{-rt}(S(t) - K)^+ & \text{if } T_2 \leq t \leq T. \end{cases} \tag{9}$$

Since the process $\hat{\psi}$ is continuous for $t \neq T_2$ and RCLL for $t = T_2$, Theorem 1 guarantees the existence of the optimal stopping time $\tau^* \in [0; T_1] \cup [T_2; T]$. More precisely, we prove in Proposition 1 that the American call option can be optimally exercised either at T_1 or at T and describe explicitly the continuation region. Rewriting the optimal exercise policy of Proposition 1 in terms of the cum-dividend price of the underlying security, we extend to our framework the analytic evaluation formula for American call options provided in Whaley (1981) (see Proposition 2), accounting for the additional randomness source X. The formula of Proposition 2 is already used in Battauz and Beccacece (2004) to compute prices for American call options on Italian common stocks that pay discrete dividends and that are affected by an exogenous source of risk.

Proposition 1. *With respect to the previous assumptions, the optimal stopping time $\tau^* \in \mathbb{T}$, i.e. $\mathbb{E}[\hat{\psi}(\tau^*)] = \sup_{U \in \mathbb{T}} \mathbb{E}[\hat{\psi}(U)]$, is given by*

$$\tau^* = \begin{cases} T & \text{if } \hat{\psi}_{T_1} \leq \mathbb{E}[\hat{\psi}_T|\hat{\mathcal{F}}_{T_1}], \\ T_1 & \text{otherwise}. \end{cases} \tag{10}$$

Proof. We verify that τ^* is such that $\mathbb{E}[\hat{\psi}(\tau^*)] \geq \mathbb{E}[\hat{\psi}(\tau)]$ for a generic stopping time τ. To this aim, we decompose $\mathbb{E}[\hat{\psi}(\tau)] = \mathbb{E}[\hat{\psi}(\tau)\mathcal{I}_{\{\tau<T_2\}}] + \mathbb{E}[\hat{\psi}(\tau)\mathcal{I}_{\{\tau\geq T_2\}}]$. Since for $t \in [0; T_2[$ and $t \in [T_2; T]$ the process $\hat{\psi}$ is a submartingale, we have that $\hat{\psi}(\tau)\mathcal{I}_{\{\tau<T_2\}} \leq \mathcal{I}_{\{\tau<T_2\}}\mathbb{E}[\hat{\psi}(T_2^-)|\hat{\mathcal{F}}_\tau] = \mathcal{I}_{\{\tau<T_2\}}\mathbb{E}[\hat{\psi}(T_1)|\hat{\mathcal{F}}_\tau]$ and $\hat{\psi}(\tau)\mathcal{I}_{\{\tau\geq T_2\}} \leq \mathcal{I}_{\{\tau\geq T_2\}}\mathbb{E}[\hat{\psi}(T)|\hat{\mathcal{F}}_\tau]$. Hence $\mathbb{E}[\hat{\psi}(\tau)] \leq \mathbb{E}[\hat{\psi}(T_1)\mathcal{I}_{\{\tau<T_2\}}] + \mathbb{E}[\hat{\psi}(T)\mathcal{I}_{\{\tau\geq T_2\}}]$. Since $\{\tau \geq T_2\} = \{\tau < T_2\}^c$ and $\{\tau < T_2\} \in \hat{\mathcal{F}}_{T_1}$, both the functions $\mathcal{I}_{\{\tau<T_2\}}$ and $\mathcal{I}_{\{\tau\geq T_2\}}$ are $\hat{\mathcal{F}}_{T_1}$-measurable: hence, recalling the construction of τ^* in (10) we have that

$$\mathbb{E}[\hat{\psi}(T_1)\mathcal{I}_{\{\tau<T_2\}}] = \mathbb{E}[\mathcal{I}_{\{\tau<T_2\}}\mathbb{E}[\hat{\psi}(T_1)|\hat{\mathcal{F}}_{T_1}]] \leq \mathbb{E}[\mathcal{I}_{\{\tau<T_2\}}\mathbb{E}[\hat{\psi}(\tau^*)|\hat{\mathcal{F}}_{T_1}]],$$
$$\mathbb{E}[\hat{\psi}(T)\mathcal{I}_{\{\tau\geq T_2\}}] = \mathbb{E}[\mathcal{I}_{\{\tau\geq T_2\}}\mathbb{E}[\hat{\psi}(T)|\hat{\mathcal{F}}_{T_1}]] \leq \mathbb{E}[\mathcal{I}_{\{\tau\geq T_2\}}\mathbb{E}[\hat{\psi}(\tau^*)|\hat{\mathcal{F}}_{T_1}]].$$

Therefore $\mathbb{E}[\hat{\psi}(\tau)] \leq \mathbb{E}[\hat{\psi}(\tau^*)\mathcal{I}_{\{\tau<T_2\}}] + \mathbb{E}[\hat{\psi}(\tau^*)\mathcal{I}_{\{\tau\geq T_2\}}]$ and the thesis is proved. □

Proposition 2. *There exists a cum-dividend "critical price" S^* such that the American call option is optimally exercised at T_1 if and only if $\hat{S}(T_1) > S^*$. Such critical price S^* is the unique solution of the equation*:

$$(S^* - K)^+ = \int_{\Omega^x} \mathrm{bsc}((S^* - D)(1 + X), T - T_2, K)\, \mathrm{d}\mathbf{Q}^{\mathbf{x}}, \tag{11}$$

where $\mathrm{bsc}(s, \tau, K)$ *denotes the value of an European call option on a stock with initial value s, time to maturity τ and strike K, computed with the usual Black–Scholes formula*:

$$\mathrm{bsc}(s, \tau, K) = sN(d_1) - K\,\mathrm{e}^{-r\tau}N(d_2),$$

where $d_1 = (\log(s/K) + (r + (\sigma^2/2))\tau)/(1\sigma\sqrt{\tau})$ and $d_2 = d_1 - \sigma\sqrt{\tau}$.

Hence the actualized value of the American call option is given for $0 \leq t \leq T_1$ by

$$J_t = \frac{1}{\mathrm{e}^{rT_1}}\left(\int_{S>S^*}(S-K)^+\,\mathrm{d}\mathbf{Q}^{\mathbf{0}} + \int_{S\leq S^*}\int_{\Omega^x}\mathrm{bsc}((S-D)(1+X), T-T_2, K)\,\mathrm{d}\mathbf{Q}^{\mathbf{x}}\,\mathrm{d}\mathbf{Q}^{\mathbf{0}}\right), \tag{12}$$

where $S = \hat{S}(T_1)|_{\hat{\mathcal{F}}_t}$, and for $T_2 \leq t \leq T$ it is simply

$$J_t = \mathrm{bsc}(\hat{S}(t), T - t, K).$$

Proof. Applying Proposition 1, it is easy to find the critical S^*, adapting the arguments of Whaley (1981). Indeed, $\mathbb{E}[\hat{\psi}_T|\hat{\mathcal{F}}_{T_1}] = \mathbb{E}[\mathbb{E}[\mathrm{e}^{-rT}(\hat{S}(T) - K)^+|\hat{\mathcal{F}}_{T_2}]|\hat{\mathcal{F}}_{T_1}]$ and since

$$\hat{S}(T)|_{\hat{\mathcal{F}}_{T_2}} = \hat{S}(T_2)\exp((r - \tfrac{1}{2}\sigma^2)(T - T_2) + \sigma\tilde{W}(T - T_2)),$$

the inner conditional expectation is the value of an European call option with maturity T, strike K on an underlying log-normal security that pays no dividends and that is worth $\hat{S}(T_2)$ at the initial time T_2. Hence $\mathbb{E}[\hat{\psi}_T|\hat{\mathcal{F}}_{T_1}] = \mathbb{E}[\mathrm{bsc}(\hat{S}(T_2), T - T_2, K)|\hat{\mathcal{F}}_{T_1}]$ with $\hat{S}(T_2)|_{\hat{\mathcal{F}}_{T_1}} = (\hat{S}(T_1) - D)(1 + X)$ and therefore $\mathbb{E}[\hat{\psi}_T|\hat{\mathcal{F}}_{T_1}] = \int_{\Omega^x}\mathrm{bsc}((\hat{S}(T_1) - D)(1 + X), T - T_2, K)\,\mathrm{d}\mathbf{Q}^{\mathbf{x}}$ since $\hat{\mathcal{F}}_{T_1} = \mathcal{F}^0_{T_1} \otimes \{\emptyset, \Omega^x\}$. Defining $f(s) = (s - K)^+ - \int_{\Omega^x}\mathrm{bsc}((s-D)(1+X), T-T_2, K)\,\mathrm{d}\mathbf{Q}^{\mathbf{x}}$, we have from Eq. (10) that $\{\tau^* = T\} = \{\hat{\psi}_{T_1} \leq \mathbb{E}[\hat{\psi}_T|\hat{\mathcal{F}}_{T_1}]\} = \{f(S) \leq 0\}$. For $s > K$ the derivative $f'(s) = 1 - \int_{\Omega^x}N(d_1((s-D)(1+X), T-T_2, K))(1+X)\,\mathrm{d}\mathbf{Q}^{\mathbf{x}}$, where $N(\cdot)$ denotes the cumulative distribution function of a standard Gaussian random variable and $d_1(s, \tau, K) = (\ln(s/K) + (r+\sigma^2/2)\tau)/\sigma\sqrt{\tau}$. Since $1 + X > 0$ and $0 < N(\cdot) < 1$, we have that $f'(s) > 1 - \int_{\Omega^x}(1 + X)\,\mathrm{d}\mathbf{Q}^{\mathbf{x}} = 0$ since the $\mathbf{Q}^{\mathbf{x}}$-expectation of X is zero. Therefore, being $f(s) < 0$ for $s \leq K$ and $f'(s) > 0$ for $s > K$, there exists an eventually infinite,[6] unique $S^* > K$ such that the set $\{\tau^* = T\} = \{S \leq S^*\}$. Hence $J_t = \mathbb{E}[\hat{\psi}(\tau^*)|\hat{\mathcal{F}}_t] = \mathbb{E}[\hat{\psi}(T_1)\mathcal{I}_{\{S>S^*\}} + \hat{\psi}(T)\mathcal{I}_{\{S\leq S^*\}}|\hat{\mathcal{F}}_t]$ and formulae (11) and (12) follow. □

[6] In case of infinite critical price S^*, the early exercise is suboptimal.

We conclude this section characterizing the interval of NoArbitrage prices for an American call option. To emphasize the dependence on the choice of the risk-neutral probability measure $\mathbf{Q}$ of the computed Snell envelope J, we use in the next proposition the notation $J = \tilde{\mathcal{C}}^{\mathbf{Q}}$.

Proposition 3. *Denote with* $\tilde{\mathcal{C}}^{\alpha}$ (*resp.* $\tilde{\mathcal{C}}^{\beta}$) *the Snell envelope of the actualized payoff of a call option written on the security* $\hat{S}$ *that satisfies* Eq. (7) *under the probability measure*[7] $\mathbf{Q}^{\alpha} = \mathbf{Q^0} \times \delta_{\alpha}$ (*resp.* $\mathbf{Q}^{\beta} = \mathbf{Q^0} \times \delta_{\beta}$). *Hence, for every* $\mathbf{Q}$ *defined in* (2) *we have that* $\tilde{\mathcal{C}}^{\mathbf{Q}}$, *the Snell envelope of the actualized payoff of the call option under* $\mathbf{Q}$, *satisfies*

$$\tilde{\mathcal{C}}^{\alpha} < \tilde{\mathcal{C}}^{\mathbf{Q}} < \tilde{\mathcal{C}}^{\beta}.$$

Proof. Under $\mathbf{Q}^{\alpha} = \mathbf{Q^0} \times \delta_{\alpha}$, Eq. (7) has a unique solution $\hat{S}^{\alpha}$, that in particular verifies $\hat{S}^{\alpha}(T_2) = (\hat{S}^{\alpha}(T_1) - D)(1+\alpha)$ a.e. Similarly, under $\mathbf{Q}^{\beta} = \mathbf{Q^0} \times \delta_{\beta}$, Eq. (7) has a unique solution $\hat{S}^{\beta}$, such that $\hat{S}^{\beta}(T_2) = (\hat{S}^{\beta}(T_1) - D)(1+\beta)$. Since $\alpha \le X \le \beta$, for every solution $\hat{S}$ of Eq. (7) under $\mathbf{Q}$ as in Formula (2) we have that $\hat{S}^{\alpha}(T_2) \le \hat{S}(T_2) \le \hat{S}^{\beta}(T_2)$. Since for $t \in [T_2; T]$ all the processes are driven by the same diffusion equation (7), it follows that $\hat{S}^{\alpha}(t) \le \hat{S}(t) \le \hat{S}^{\beta}(t)$ for all $t \in [T_2; T]$. Moreover, since the processes coincide on $[0; T_2[$, we conclude that $\hat{S}^{\alpha} \le \hat{S} \le \hat{S}^{\beta}$ on the whole interval $[0; T]$. Denote by $\hat{\psi}^{\alpha}$ (resp. $\hat{\psi}^{\beta}$) the actualized payoff process of a call option written on $\hat{S}^{\alpha}$ (resp. $\hat{S}^{\beta}$). The conditional expectation of $\hat{\psi}^{\alpha}$ (resp. $\hat{\psi}^{\beta}$) under $\mathbf{Q}^{\alpha}$ (resp. $\mathbf{Q}^{\beta}$) and under every $\mathbf{Q}$ does not involve the $\mathcal{F}^x$ component, since $\hat{\psi}^{\alpha}$ (resp. $\hat{\psi}^{\beta}$) is independent of X. Hence $\tilde{\mathcal{C}}^{\alpha}$ (resp. $\tilde{\mathcal{C}}^{\beta}$), that is simply the Snell envelope of $\hat{\psi}^{\alpha}$ (resp. $\hat{\psi}^{\beta}$) under $\mathbf{Q^0}$ with respect to the *stretched* $\mathcal{F}^0$, coincides with the Snell envelope of $\hat{\psi}^{\alpha}$ (resp. $\hat{\psi}^{\beta}$) under every $\mathbf{Q}$ with respect to the stretched $\hat{\mathcal{F}}$. From $\hat{S}^{\alpha} \le \hat{S} \le \hat{S}^{\beta}$ on $[0; T]$ it follows that the actualized payoff processes of the derived call options satisfy the inequalities: $\hat{\psi}^{\alpha} \le \hat{\psi} \le \hat{\psi}^{\beta}$ on $[0; T]$ as well as their Snell envelopes[8] under $\mathbf{Q}$, such that $\tilde{\mathcal{C}}^{\alpha} \le \tilde{\mathcal{C}}^{\mathbf{Q}} \le \tilde{\mathcal{C}}^{\beta}$ on $[0; T]$. □

4. American put options

The actualized value (for a given risk-neutral measure $\mathbf{Q}$) of an American put option is given in our framework by the Snell envelope J of

$$\hat{\psi}(t) = \begin{cases} e^{-rt}(K - S(t))^{+} & \text{if } t \le T_1, \\ e^{-rT_1}(K - S(T_1))^{+} & \text{if } T_1 \le t < T_2, \\ e^{-rt}(K - S(t))^{+} & \text{if } T_2 \le t \le T. \end{cases} \tag{13}$$

As for the call option, since the process $\hat{\psi}$ is RCLL, there exists the RCLL supermartingale J that aggregates the Snell envelope (compare El Karoui, 1979). The characterization of the optimal stopping time as the arrival time in the set $\{J = \hat{\psi}\}$ is guaranteed by Theorem 1, but can also be achieved in another way. In fact, since the underlying stock can jump only downwards and the payoff of the put option is nonincreasing with respect to the stock, it follows that $\hat{\psi}$ jumps only upwards. Hence the payoff process of the put option satisfies condition (14) in Proposition 4 and the characterization of the optimal stopping times follows. Condition (14) in Proposition 4 relaxes the assumption of left-continuity in expectation required on the payoff process in Theorem 2.18 of El Karoui (1979) to prove that the arrival time in the set $\{J = \hat{\psi}\}$ is optimal.

As we know from Theorem 1, the optimal exercise policy τ^* takes values in $[0; T_1] \cup [T_2; T]$. However, the holder of an American put option does not exercise it at the end of the cum-dividend date, being sure that the day

[7] δ_{α} is the Dirac measure concentrated on the event $\{X = \alpha\}$.

[8] If $\hat{\psi}_1 \le \hat{\psi}_2$ then their Snell envelopes also satisfy the relation $J_1 \le J_2$.

after the option is going to be worth more. In our framework this means that $\tau^* \in [0; T_1[\cup[T_2; T]$, as we prove in Proposition 5.

Proposition 4. *Suppose that the process $\hat{\psi}$ is RCLL and such that*

$$\lim_{n\to\infty} \mathbb{E}[\hat{\psi}_{U_n}] \leq \mathbb{E}[\hat{\psi}_U] \tag{14}$$

for every sequence of stopping times $U_n \uparrow U$. Then, the arrival time in the set $\{J = \hat{\psi}\}$ is optimal.

Proof. The proof of Theorem 2.18 in El Karoui (1979) can be adapted to our case, since the crucial convergence of the limit $\lim_{n\to\infty} \mathbb{E}[\hat{\psi}_{\tau^{A_n}}] = \mathbb{E}[\hat{\psi}_{\bar{\tau}}]$ for $\tau^{A_n} \uparrow \bar{\tau}$, can be replaced by $\lim_{n\to\infty} \mathbb{E}[\hat{\psi}_{\tau^{A_n}}] \leq \mathbb{E}[\hat{\psi}_{\bar{\tau}}]$, that holds true because of (14) with $U_n = \tau^{A_n}$. Indeed, passing to the limit we have that $\mathbb{E}[\hat{\psi}_{\bar{\tau}}] \geq \sup_{U\in\mathbb{T}} \mathbb{E}[\hat{\psi}_U]$, i.e. $\bar{\tau}$ is optimal. And the rest of the proof follows unchanged. □

Proposition 5. *The optimal exercise policy for an American put option $\tau^* \in [0; T_1[\cup[T_2; T]$.*

Proof. Suppose that $\tau^* = T_1$ on a set $B \subset \Omega$ with $\mathbf{Q}(B) > 0$. We prove that in this case τ^* is not optimal. To this aim, define $\tilde{\tau} = \tau^*\mathcal{I}_{B^c} + T_2\mathcal{I}_B$ and notice that $\tilde{\tau}$ is an admissible stopping time. Indeed for $t < T_1$, the set $\{\tilde{\tau} \leq t\} = \{\tau^* \leq t\}$ is $\hat{\mathcal{F}}_t$-measurable and for $t \geq T_1$, the set $\{\tilde{\tau} \leq t\} = (\{\tau^* \leq t\} \cap B^c) \cup B$ is $\hat{\mathcal{F}}_t$-measurable as well. Moreover, $\mathbb{E}[\hat{\psi}(\tilde{\tau})] = \mathbb{E}[\hat{\psi}(\tau^*)\mathcal{I}_{B^c}] + \mathbb{E}[\hat{\psi}(T_2)\mathcal{I}_B] > \mathbb{E}[\hat{\psi}(\tau^*)]$, that contradicts the optimality of τ^*. □

As for the American call option in Proposition 3, we supply here the bounds to the interval of the NoArbitrage prices of an American put option.

Proposition 6. *Denote with $\tilde{P}^\alpha$ (resp. $\tilde{P}^\beta$) the Snell envelope of the actualized payoff of a put option written on the security $\hat{S}$ that satisfies Eq. (7) under the probability measure $\mathbf{Q}^\alpha = \mathbf{Q^0} \times \delta_\alpha$ (resp. $\mathbf{Q}^\beta = \mathbf{Q^0} \times \delta_\beta$). Hence, for every $\mathbf{Q}$ defined in (2) we have that $\tilde{P}^{\mathbf{Q}}$, the Snell envelope of the actualized payoff of the put option under $\mathbf{Q}$, satisfies*

$$\tilde{P}^\beta < \tilde{P}^{\mathbf{Q}} < \tilde{P}^\alpha.$$

Proof. The proof is the same of Proposition 3. We only notice that from $\hat{S}^\alpha \leq \hat{S} \leq \hat{S}^\beta$ on $[0; T]$ it follows that the actualized payoff processes of the derived put options satisfy the reverse inequalities: $\hat{\psi}^\beta \leq \hat{\psi} \leq \hat{\psi}^\alpha$ on $[0; T]$. □

We conclude the section with some comments on the results we obtained. Proposition 5 describes some feature of the optimal stopping policy for an American put option, but is not exhaustive as Propositions 1 and 2 for American call options. Indeed, the evaluation of American put options presents many problems, even if no dividends are paid during the life of the option, since no closed formula is available neither for the critical stock price (that describes the exercise boundary) nor for the value of the option. In continuous-time models the price is computed solving numerically either the variational inequality system or the free-boundary formulation of the optimal stopping problem (see Myneni, 1992). The variational inequality approach does not determine explicitly the stopping boundary and is particularly useful for the evaluation on multi-assets options (see Jaillet (1990) for a finite-difference and Marcozzi (2001) for a finite-elements treatment of the variational inequality system). The free-boundary formulation consists of the Black–Scholes partial differential equation and its usual boundary conditions plus a Neumann condition to determine the unknown exercise boundary. The method of lines is applied in Meyer (2002) to solve numerically the free-boundary formulation, accounting also for the presence of discrete dividends. In particular, the (actualized) put option value $\tilde{P}(\hat{S}(t), t)$ must satisfy a suitable *interface condition* due to the dividend payment. In our framework, under the assumption $X = 0$, the interface condition of Meyer (2002) can be written as

$$\tilde{P}(\hat{S}(T_1), T_1) = \tilde{P}((\hat{S}(T_1) - D), T_2). \tag{15}$$

Condition (15) can be explained in the light of Corollary 1 and Proposition 5. Indeed, Proposition 5 guarantees that early exercise cannot occur at T_1; applying Corollary 1 this means that the Snell envelope (i.e. the actualized put option value) at T_1 is $J(T_1) = \max\{\hat{\psi}_{T_1}, \mathbb{E}[J(T_2)|\hat{\mathcal{F}}_{T_1}]\} = \mathbb{E}[J(T_2)|\hat{\mathcal{F}}_{T_1}]$ that is exactly the previously written condition (15) if $X = 0$ and becomes

$$\tilde{P}(\hat{S}(T_1), T_1) = \int_{\Omega^x} \tilde{P}((\hat{S}(T_1) - D)(1 + X), T_2)\, \mathrm{d}\mathbf{Q}^{\mathbf{x}}.$$

if $X \neq 0$.

Among all the analytical approximations available for the American put options, we recall here Carr et al. (1992), Bunch and Johnson (2000) and Barone-Adesi and Whaley (1988), which also accounts for the presence of dividends. In Carr et al. (1992), Bunch and Johnson (2000) the price of the American put option is decomposed in various ways to get intuition on the structure of the option value and to deduce analytical approximation of the critical stock price. In particular, Bunch and Johnson (2000) decompose the option value into the European put price plus the early-exercise premium, as in Myneni (1992). They argue financially that on the exercise boundary the value of the option cannot depend on the time to maturity, i.e.:

$$\frac{\partial P}{\partial t_{\mathrm{tm}}}(S^*(t_{\mathrm{tm}}), t_{\mathrm{tm}}) = 0,$$

where t_{tm} denotes the time to maturity of the option. Mathematically, this can be justified noticing that the exercise boundary S^* is defined by the equation:

$$(K - S^*(t_{\mathrm{tm}}))^+ - P(S^*(t_{\mathrm{tm}}), t_{\mathrm{tm}}) = 0.$$

Differentiating with respect to t_{tm} the left hand side of the equation for[9] $K > S^*$ and applying the smooth pasting condition of the free-boundary formulation[10] we obtain what the authors argue in Bunch and Johnson (2000). Using the put decomposition with the early exercise premium, Bunch and Johnson (2000) write an implicit equation for the critical stock price and provide for S^* an analytical approximation. Also Carr et al. (1992) supply, among other contributions, tighter analytic bounds and analytic approximation to the American put value, starting from the early-exercise decomposition. However, as the authors explicitly write in Carr et al. (1992), the extension of their work to account for discrete dividends constitutes a significant avenue for future research.

5. Conclusions

To evaluate American options on assets that pay discrete dividends, we have analyzed an optimal stopping problem with constraints on the stopping times. More precisely, we force the stopping times to take values in the union of disjoint, real compact sets. We characterize the optimal exercise policy as the first instant such that the value of the option equals the underlying payoff, even if the assumption of left-continuity in expectation fails (see El Karoui, 1979). We provide also the (backward) link between different exercise periods and apply these results to evaluate options with restrictions on exercise dates. The same results are also useful to study American options on assets that pay discrete dividends. Indeed, for American call options, it is possible to formalize the existence of the optimal exercise policy, that equals either the (end of the) cum-dividend date or the maturity date. Moreover, we are able to generalize the evaluation formula for American call options due to Whaley (1981), allowing also for a stochastic jump of the underlying security at the ex-dividend date. The generalized formula has been used in Battauz and Beccacece (2004) to analyze the impact of the extra randomness source on the derivatives traded on the Italian market. Finally, we focus on American put options, discussing some recent literature (see Carr, 1992;

[9] If $K \leq S^*(t_{\mathrm{tm}})$, the early exercise is suboptimal at $t = T - t_{\mathrm{tm}}$.

[10] With our notations the smooth pasting condition is $(\partial P/\partial S)(S^*(t_{\mathrm{tm}}), t_{\mathrm{tm}}) = -1$ for all $t_{\mathrm{tm}} \in [0; T]$.

Bunch and Johnson, 2000; Meyer, 2002) and applying the backward link between exercise periods to extend the time-discretization of the free boundary formulation to the case of an extra randomness source at the dividend date (compare to Meyer, 2002).

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