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# A Minimax Theorem Without Compactness Hypothesis

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**Abstract.** The object of this short note is the proof of a minimax theorem which does not require compactness conditions and is motivated by a problem of optimal investment; the application to the economic problem is illustrated.

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## 1. Introduction

The method of convex duality, applied to the problem of optimal investment in an incomplete market, is based on a *minimax theorem*. Usual minimax theorems require a compactness condition, but the natural space of functions for the problem of optimal investment is the space of all measurable functions, endowed with the topology of convergence in measure.

The characterization of relatively compact subsets of this space is terribly complicated, however Schachermayer has introduced a result which is in some sense a *substitute of compactness* (on the cone of positively valued measurable functions), i.e. the construction of a convergent sequence of functions by taking convex combinations from the original sequence.

The object of this paper is the proof of a minimax theorem (on a suitable space of functions) which does not require compactness conditions and is based on this substitute of compactness.

In Section 2, there is a generalization of the Schachermayer's lemma to the case of a *net* (or generalized sequence) of positively valued measurable functions.

Section 3 contains the proof of the main result: the Minimax theorem. Section 4 illustrates how this result can be applied to the optimal investment problem.

## 2. A substitute of compactness in $L^0_+$

Let $(E, \mathcal{E})$ be a measurable space and $\mu$ a $\sigma$-finite positive measure on $\mathcal{E}$: we denote by $L^0 = L^0(E, \mathcal{E}, \mu)$ the space of all measurable real–valued functions $f$ defined on $E$, endowed with the topology of convergence in measure. Since this topology depends only on the equivalence class of $\mu$, we suppose that $\mu$ is a probability measure i.e. $\mu(E) = 1$.

The topology of convergence in measure is generated by the distance $d(f, g) = \int \operatorname{arctg}(|f-g|)\, \mathrm{d}\mu$: the space $L^0$, equipped with the metric $d$, is complete.

The characterization of relatively compact subsets of $L^0$ is terribly cumbersome and very hard to apply in practice: see for instance [4] pag. 330 for a precise formulation.

Nevertheless, $L^0$ has been considered an interesting object in many problems related to the theory of stochastic processes; in particular, an important lemma due to W. Schachermayer (see [3] Lemma A1.1 for an easy proof) furnishes in some sense a substitute of compactness in $L^0$, or better in the convex cone $L^0_+$ of positively valued measurable functions. This result has been established for a sequence of functions, but for our purposes we have to extend it to a *net* (or *generalized sequence*) of functions.

More precisely, let $I$ be a *directed set*, and $(f_i)_{i\in I}$ a net of functions contained in $L^0_+$. For every $i \in I$, let $\Gamma_i = \operatorname{Conv}\{f_j\,,\, j \geq i\}$ be the convex envelope of all functions $f_j$ with index $j \geq i$ and $\Gamma_0 = \operatorname{Conv}\{f_i\,,\, i \in I\}$. We have the following result, whose proof is a modification of the result proved in [3]; for convenience of the reader, we give a complete proof.

**Lemma 2.1.** *It is possible to determine, for every $i \in I$, a function $g_i \in \Gamma_i$ in such a way that the net $(g_i)_{i\in I}$ converges in measure to a $[0, +\infty]$ valued function $g$.*

*Moreover, there exists an increasing sequence $i_1 \leq i_n \leq \dots$ contained in $I$ such that the sequence $(g_{i_n})_{n\geq 1}$ converges almost everywhere and*

$$g \;=\; \lim_{i\in I} g_i \;=\; \lim_{n\to\infty} g_{i_n}.$$

*Proof.* Let $u : [0, +\infty] \to [0, 1]$ be defined by $u(x) = 1 - e^{-x}$: it is an exercise to prove that, given $\alpha > 0$, there exists $\beta > 0$ such that if $|x - y| > \alpha$ and $\min(x, y) \leq \alpha^{-1}$, we have

$$u\left(\frac{x+y}{2}\right) \geq \frac{u(x)+u(y)}{2} + \beta.$$

Therefore, given $g, h \in L^0_+$, we have

$$\beta\,\mu\left\{|g-h| > \alpha\,,\, \min(g,h) < \alpha^{-1}\right\} \;\leq\; \int u\left(\frac{g+h}{2}\right)\mathrm{d}\mu \;-\; \int \frac{u(g)+u(h)}{2}\,\mathrm{d}\mu. \tag{2.1}$$

We consider the functions as $[0, +\infty]$-valued, and we recall that a net $(x_i)_{i\in I}$ is a Cauchy net in $[0, +\infty]$ if and only if, for every $\alpha > 0$, there exists $i_0$ such that, for $i, j \geq i_0$ we have $|x_i - x_j| \leq \alpha$ or $\min(x_i, x_j) \geq \alpha^{-1}$.

Therefore, in order to prove that $(g_i)_{i\in I}$ is Cauchy in $L^0_+$, we have to prove that, given $\alpha > 0$ and $\epsilon > 0$, there exists $j_0$ such that, for $i, j \geq j_0$, we have

$$\mu\left\{|g_i - g_j| > \alpha\ ,\ \min(g_i, g_j) < \alpha^{-1}\right\} \leq \epsilon. \tag{2.2}$$

Now set, for every $i$,

$$s_i = \sup\{\int u(g)\,\mathrm{d}\mu \,|\, g \in \Gamma_i\}, \quad \text{and} \quad s_0 = \sup\{\int u(g)\,\mathrm{d}\mu \,|\, g \in \Gamma_0\}.$$

We have that $(s_i)_{i\in I}$ is a decreasing net of numbers and let $s_\infty = \inf_{i\in I} s_i = \lim_{i\in I} s_i$ $(0 \leq s_\infty \leq 1)$.

Choose a sequence $(i_n)_{n\geq 1}$ such that $s_\infty = \lim_{n\to\infty} s_{i_n}$: we can suppose that $i_n \leq i_{n+1}$ and $|s_\infty - s_{i_n}| \leq \frac{1}{n}$. For every $n$, choose $g_{i_n} \in \Gamma_{i_n}$ such that $\int u(g_{i_n})\,\mathrm{d}\mu \geq s_{i_n} - \frac{1}{n}$.

Given $i \in I$, there exists $n$ such that $s_{i_{n+1}} \leq s_i \leq s_{i_n}$, and let $g_i \in \Gamma_i$ such that $\int u(g_i)\,\mathrm{d}\mu \geq s_{i_{n+1}} - \frac{1}{n}$.

Suppose $i, j \geq i_n$ : since $\frac{g_i+g_j}{2} \in \Gamma_{i_n}$, we have $\int u(\frac{g_i+g_j}{2})\,\mathrm{d}\mu \leq s_{i_n}$, but $\int u(g_i)\,\mathrm{d}\mu \geq s_{i_n} - \frac{2}{n}$ and the same for $\int u(g_j)\,\mathrm{d}\mu$. Therefore, starting from equation (2.1), we obtain

$$\beta\,\mu\left\{|g_i - g_j| > \alpha\,,\, \min(g_i, g_j) < \alpha^{-1}\right\} \leq \frac{2}{n} \tag{2.3}$$

and the net $(g_i)_{i\in I}$ is Cauchy for the convergence in measure.

Also the sequence $(g_{i_n})_{n\geq 1}$ is Cauchy, and it is evident that $\lim_{n\to\infty} g_{i_n} = \lim_{i\in I} g_i$ : taking a suitable subsequence, we obtain convergence everywhere. □

Note that the limit $g$ is $[0, +\infty]$-valued, but if we suppose that every $f_i$ belongs to a convex bounded subset of $L^0_+$, it is easy to verify that the limit $g$ is real-valued (recall that a subset $C \subset L^0_+$ is bounded if, for every $\epsilon > 0$, there exists $\alpha > 0$ such that, for every $f \in C$ , $\mu\{|f| > \alpha\} < \epsilon$).

**Corollary 2.2.** *Let $X$ a set and, for every $x \in X$, let $C_x$ a convex closed subset of $L^0_+$. Suppose that*

1. *for some $\bar{x} \in X$ , $C_{\bar{x}}$ is bounded in $L^0_+$ ;*
2. *for every finite subset $\{x_1, \ldots, x_n\}$ of $X$, we have that $C_{x_1} \cap \ldots \cap C_{x_n} \neq \emptyset$ .*

*Then we have also that $\bigcap_{x\in X} C_x \neq \emptyset$.*

*Proof.* Let $I$ be the set of finite subsets of $X$, ordered by inclusion; for every $i = \{x_1, \ldots, x_n\}$, choose $f_i \in C_{x_1} \cap \ldots \cap C_{x_n}$. Taking $g_i$ as in Lemma 2.1, it is easy to check that the limit $g$ is a real-valued function (since it is contained in $C_{\bar{x}}$) and belongs to $\bigcap_{x\in X} C_x$. □

**Corollary 2.3.** *Let $C$ be a convex subset of $L^0_+$ and $\Phi : C \to [-\infty, +\infty[$ concave upper semicontinuous (with respect to the convergence in measure).*

*Suppose that, for some real $b$, the set $C_b = \{f : \Phi(f) \geq b\}$ is bounded nonempty: then there exists $\bar{f} \in C$ such that*

$$\sup_{f \in C} \Phi(f) = \Phi(\bar{f}).$$

*Proof.* Let $(f_n)_{n \geq 1}$ be a sequence such that

$$\sup_{f \in C} \Phi(f) = \lim_{n \to \infty} \Phi(f_n) = \alpha$$

and let $g_n \in \Gamma_n = \mathrm{Conv}\,\{f_n, f_{n+1}, \ldots\}$ converging a.e. to a function $\bar{f}$: it is evident that the limit $\bar{f}$ belongs to $C_b$ (which is convex, closed and bounded).

By the concavity of $\Phi$, we have also that $\lim_{n \to \infty} \Phi(g_n) = \alpha$ and, since $\Phi$ is upper semicontinuous, $\Phi(\bar{f}) \geq \alpha$. □

## 3. The minimax theorem

Let $\Phi : C \times M \to \mathbb{R}$ be a real valued function defined on a product set: we have evidently

$$m_1 = \sup_{x \in C} \inf_{y \in M} \Phi(x, y) \leq \inf_{y \in M} \sup_{x \in C} \Phi(x, y) = m_4. \tag{3.1}$$

We call *minimax theorem* a result which states, under suitable conditions, the equality $m_1 = m_4$.

The usual hypotheses for a minimax theorem are that $C$ and $M$ are convex sets, $\Phi$ is *concave* with respect to $x$ (i.e. the functions $x \to \Phi(x, y)$, for $y$ fixed, are concave) and *convex* with respect to $y$ ; moreover a sort of *semicontinuity* and *compactness* is needed.

For instance, we have a minimax theorem if the functions $y \to \Phi(x, y)$ are lower semicontinuous and $M$ is compact (or, equivalently, the functions $x \to \Phi(x, y)$ are upper semicontinuous and $C$ is compact): see for instance [1] Theorem 8.1 pag. 126 or [9] Theorem 45.8 for such results.

The object of this section is to prove a minimax theorem by using the *substitute of compactness* introduced in the previous section (Lemma 2.1).

Before stating the theorem, we introduce two further numbers $m_2$ and $m_3$ as follows: let $\mathcal{F}$ (respectively $\mathcal{G}$) be the collection of finite subsets of $M$ (respectively of $C$) , and define

$$m_2 = \sup_{K \in \mathcal{F}} \inf_{y \in M} \max_{x \in K} \Phi(x, y), \tag{3.2}$$

$$m_3 = \inf_{H \in \mathcal{G}} \sup_{x \in C} \min_{y \in H} \Phi(x, y). \tag{3.3}$$

It is easy to check the inequalities $m_1 \leq m_2 \leq m_4$ and $m_1 \leq m_3 \leq m_4$.

**Lemma 3.1.** *Suppose that $C$ and $M$ are convex sets and that the function $\Phi$ is concave with respect to $x \in C$ and convex with respect to $y \in M$: then we have the equalities $m_1 = m_2$ and $m_3 = m_4$.*

*Proof.* The equality $m_1 = m_2$ is proved in [1] Proposition 8.3 pag. 124 (see also [2] Theorem 2 pag. 316); the second equality is easily obtained by applying the same proposition to the function $\Psi(x, y) = -\Phi(y, x)$. $\square$

We can now state our main result.

**Theorem 3.2.** *Let $C$ be a convex subset of $L^0_+$, $M$ a convex set and $\Phi : C \times M \to \mathbb{R}$. Suppose that*

1. *for every $y \in M$, the function $f \to \Phi(f, y)$ is upper semicontinuous and concave;*
2. *for every $f \in C$, the function $y \to \Phi(f, y)$ is convex;*
3. *for every $y \in M$ and $b \in \mathbb{R}$, the set $C_{y,b} = \{f \in C \,|\Phi(f, y) \geq b\}$ is closed and bounded in $L^0_+$.*

*Then we have the equality $m_1 = m_4$; moreover there exists $\bar{f} \in C$ such that*

$$\inf_{y \in M} \Phi(\bar{f}, y) = \inf_{y \in M} \sup_{f \in C} \Phi(f, y).$$

*Proof.* Since we have the equality $m_3 = m_4$, it is sufficient to prove the equality $m_1 = m_3$; more precisely that we have

$$\sup_{f \in C} \inf_{y \in M} \Phi(f, y) = \inf_{H \in \mathcal{G}} \sup_{f \in C} \min_{y \in K} \Phi(f, y). \tag{3.4}$$

Let $a = m_3 = \inf_{H \in \mathcal{G}} \sup_{f \in C} \min_{y \in K} \Phi(f, y)$: for every $H = \{y_1, \ldots, y_n\}$ the function $\min_{i=1,\ldots,n} \Phi(f, y_i)$ (which is concave upper semicontinuous) attains its *maximum* in $C$ (see Corollary 2.3).

This means that the sets $C_{y,a}$ ( $y \in M$ ) have the finite intersection property, more precisely that $C_{y_1,a} \cap \ldots \cap C_{y_n,a} \neq \emptyset$.

By Corollary 2.2, also $\bigcap_{y \in M} C_{y,a} \neq \emptyset$: a function $\bar{f} \in \bigcap_{y \in M} C_{y,a}$ satisfies the equality

$$\inf_{y \in M} \Phi(\bar{f}, y) = \inf_{y \in M} \sup_{f \in C} \Phi(f, y).$$

$\square$

In a similar way we obtain the following result, which is an analogue of the well known *Von Neumann's Minimax Theorem*.

**Corollary 3.3.** *Let $C, D$ be convex subsets of $L^0_+$, and $\Phi : C \times D \to \mathbb{R}$. Suppose that*

1. *for every $g \in D$ , the function $f \to \Phi(f, g)$ is upper semicontinuous and concave;*
2. *for every $g \in D$ and $b \in \mathbb{R}$ , the set $C_{g,b} = \{f \in C \,|\Phi(f, g) \geq b\}$ is closed and bounded in $L^0_+$;*
3. *for every $f \in C$ , the function $g \to \Phi(f, g)$ is lower semicontinuous and convex;*
4. *for every $f \in C$ and $b \in \mathbb{R}$ , the set $D_{f,b} = \{g \in D \,|\Phi(f, g) \leq b\}$ is closed and bounded in $L^0_+$.*

*Then there exists a saddle point* $(\bar{f}, \bar{g})$ *such that*

$$\Phi(\bar{f}, \bar{g}) = \sup_{f \in C} \inf_{g \in D} \Phi(f, g) = \inf_{g \in D} \sup_{f \in C} \Phi(f, g).$$

## 4. Application to the problem of Optimal Investment

A huge literature has been dedicated in recent years to the problem of Optimal Investment in incomplete markets (also known as the *Utility maximization problem*); we refer to the seminal paper [5] for a general presentation of the problem.

Following [5] and [6], we state it in an abstract setting. Given a measurable space $(E, \mathcal{E})$ endowed with a probability measure $\mu$, we consider two subsets $\mathcal{C}$ and $\mathcal{D}$ of $L^0_+$ with these hypotheses:

1. $\mathcal{C}$ is bounded in $L^0_+$ and contains the constant 1;
2. $\mathcal{C}$ and $\mathcal{D}$ are in a polarity relation, more precisely:

$$f \in \mathcal{C} \Longleftrightarrow \int fg \,\mathrm{d}\mu \le 1 \ , \ \text{ for every } g \in \mathcal{D}$$

$$g \in \mathcal{D} \Longleftrightarrow \int fg \,\mathrm{d}\mu \le 1 \ , \ \text{ for every } f \in \mathcal{C}$$

Therefore, $\mathcal{C}$ and $\mathcal{D}$ are convex, closed and solid subsets of $L^0_+$ ($\mathcal{C}$ is said to be *solid* if, $f \in \mathcal{C}$ and $0 \le g \le f$, imply that $g \in \mathcal{D}$); note that the elements of $\mathcal{D}$ are integrable (since $1 \in \mathcal{C}$). We need the following easy result.

**Lemma 4.1.** *There exists a function* $g \in \mathcal{D}$ *which is strictly positive everywhere.*

*Proof.* For every $A \in \mathcal{E}$ with $\mu(A) > 0$, there exists $g_A \in \mathcal{D}$ with $\int_A g_A \,\mathrm{d}\mu > 0$: otherwise the functions $nI_A$ , $n \ge 1$ would be elements of $\mathcal{C}$ by the polarity relation, but this is impossible since $\mathcal{C}$ is bounded. A suitable infinite convex combination of the form $g = \sum_{n \ge 1} 2^{-n} g_{A_n}$ gives the result (as in the Halmos–Savage Theorem). $\square$

We define, for $x > 0$, $\mathcal{C}(x) = x\mathcal{C} = \{xf \mid f \in \mathcal{C}\}$; and analogously $\mathcal{D}(y) = y\mathcal{D}$. Let now $U$ be a utility function: we assume that $U(x) = -\infty$ for $x < 0$, that $U$ is real valued, concave, increasing and differentiable for $x > 0$, and that satisfies the so called Inada's conditions, more precisely $U'(0) = \lim_{x \searrow 0} U'(x) = +\infty$ , $U'(+\infty) = \lim_{x \nearrow \infty} U'(x) = 0$.

Let $V$ be the convex conjugate function of $U$ (the Legendre transform of $-U(-x)$) defined for $y > 0$ by $V(y) = \sup_{x > 0} [U(x) - xy]$: $V$ is convex decreasing and we have that $U(x) = \inf_{y > 0} [V(y) + xy]$ (see [7] for details).

The goal of the *utility maximization problem* is to maximize the average value of $U(f)$ in the set $\mathcal{C}(x)$: the value function of this problem is denoted by

$$u(x) \ = \ \sup_{f \in \mathcal{C}(x)} \int U(f) \,\mathrm{d}\mu. \tag{4.1}$$

The function $x \to u(x)$ is concave increasing, and we assume that

$$u(x) < +\infty \quad \text{for some (hence for all) } x > 0.$$

The dual problem of (4.1) is to minimize the average value of $V(g)$ in the set $\mathcal{D}(y)$: its value function is defined by

$$v(y) \;=\; \inf_{g \in \mathcal{D}(y)} \int V(g)\, \mathrm{d}\mu. \tag{4.2}$$

The function $y \to v(y)$ is convex decreasing. These problems are been extensively analyzed in the papers [5] and [6]; we are only interested in a proof of the following result.

**Theorem 4.2.** *There exists $y_0$ such that $v(y)$ is finitely valued for $y > y_0$ and the functions $u$ and $v$ are conjugate in the sense that*

$$u(x) = \inf_{y>0} \left[ v(y) + xy \right], \tag{4.3}$$

$$v(y) = \sup_{x>0} \left[ u(x) - xy \right]. \tag{4.4}$$

*Remark* 4.3. The statements of Theorem 4.2 correspond to Theorem 2.1 in [5] and this is the only result in the papers [5] and [6] where a *minimax* theorem is used. The authors adapt a minimax result taken from [9] (see also [8] for an alternative approach based on a minimax theorem from [2]): we feel that the result stated in the previous section gives a more direct and natural proof.

*Proof.* Let us suppose that the utility function $U$ is uniformly bounded from above, i.e. that $U(+\infty) = \lim_{x \nearrow +\infty} U(x) = m < +\infty$.

Recall that a positive function $f \in \mathcal{C}(x)$ if and only if $\int fg \, \mathrm{d}\mu \le x$ for every $g \in \mathcal{D}$: the problem (4.1) can be written in the form

$$u(x) \;=\; \sup_{f \ge 0} \inf_{y>0} \left[ \int U(f)\, \mathrm{d}\mu \;-\; y \left( \sup_{g \in \mathcal{D}} \int fg\, \mathrm{d}\mu - x \right) \right].$$

Let us introduce the convex subset $\mathcal{E}$ of $L^0_+$ defined by

$$\mathcal{H} = \left\{ f \ge 0 \,\middle|\, \int U(f)\, \mathrm{d}\mu > -\infty \text{ and } \sup_{g \in \mathcal{D}} \int fg\, \mathrm{d}\mu < +\infty \right\}.$$

The problem (4.1) can be stated equivalently in the form

$$u(x) \;=\; \sup_{f \in \mathcal{H}} \inf_{y>0} \left[ \int U(f)\, \mathrm{d}\mu \;-\; y \left( \sup_{g \in \mathcal{D}} \int fg\, \mathrm{d}\mu. - x \right) \right].$$

We apply Theorem 3.2 to the function $\Phi_1 : \mathcal{H} \times \mathbb{R}^+ \to \mathbb{R}$ defined by

$$\Phi_1(f, y) = \int U(f)\, \mathrm{d}\mu - y \left( \sup_{g \in \mathcal{D}} \int fg\, \mathrm{d}\mu - x \right).$$

(Note that Theorem 3.2 is stated for a real–valued function $\Phi$ , and $\Phi_1 : \mathcal{H} \times \mathbb{R}^+ \to \mathbb{R}$ is real valued, whilst $\Phi_1 : L^0_+ \times \mathbb{R}$ is $[-\infty, +\infty[$ valued).

The function $\Phi_1$ satisfies the hypotheses (1) and (2) of Theorem 3.2: in particular, concerning the semicontinuity, note that $f \to \sup_{g\in\mathcal{D}} \int fg \,\mathrm{d}\mu$ is lower semicontinuous.

Consider now the convex closed sets $\mathcal{H}_{y,b} = \{f \in \mathcal{E} \mid \Phi_1(f,y) \geq b\}$ : note that if $\int U(f)\,\mathrm{d}\mu - y\left(\sup_{g\in\mathcal{D}} \int fg\,\mathrm{d}\mu - x\right) \geq b$ , then $\sup_{g\in\mathcal{D}} \int fg\,\mathrm{d}\mu \leq \frac{m-b}{y} + x = \bar{x}$, and therefore $f \in \mathcal{C}(\bar{x})$ which is bounded in $L^0_+$. We have therefore

$$\begin{aligned} u(x) &= \inf_{y>0} \sup_{f\in\mathcal{H}} \left[\int U(f)\,\mathrm{d}\mu - y\left(\sup_{g\in\mathcal{D}} \int fg\,\mathrm{d}\mu - x\right)\right] \\ &= \inf_{y>0} \sup_{f\in\mathcal{H}} \inf_{g\in\mathcal{D}} \left[\int \left(U(f) - yfg\right)\,\mathrm{d}\mu \, + \, yx\right]. \end{aligned}$$

We consider the function $\Phi_2 : \mathcal{H}\times\mathcal{D} \to \mathbb{R}$ defined by $\Phi_2(f,g) = \int \left(U(f) - yfg\right)\,\mathrm{d}\mu$: also this function satisfies the hypotheses (1) and (2) of Theorem 3.2.

Again, if $\int \left(U(f) - yfg\right)\,\mathrm{d}\mu \geq b$, we have $\int fg\,\mathrm{d}\mu \leq \frac{m-b}{y} = \bar{m}$. If we consider a strictly positive function $g \in \mathcal{D}$ (see Lemma 4.1 for the existence of such a function), the set $\left\{f \in L^0_+ \mid \int fg\,\mathrm{d}\mu \leq \bar{m}\right\}$ is bounded in $L^0_+$.

We can apply Theorem 3.2 and we have that

$$u(x) \;=\; \inf_{y>0} \inf_{g\in\mathcal{D}} \left[\sup_{f\in\mathcal{H}} \int \left(U(f) + yfg\right)\mathrm{d}\mu + yx\right].$$

Note that, for every integer $n$, the set $\mathcal{L}_n = \left\{f \geq 0 \,\middle|\, \frac{1}{n} \leq f \leq n\right\}$ is contained in $\mathcal{H}$ : it is therefore easy to convince ourselves that we have

$$\sup_{f\in\mathcal{H}} \int \left(U(f) + yfg\right)\mathrm{d}\mu = \sup_{f\geq 0} \int \left(U(f) + yfg\right)\mathrm{d}\mu = \int V(yg)\,\mathrm{d}\mu$$

and therefore

$$u(x) = \inf_{y>0} \left[\inf_{g\in\mathcal{D}} \int V(yg)\,\mathrm{d}\mu \, + yx\right] = \inf_{y>0} \left[v(y) + xy\right].$$

We have proved (4.3) for the case where the utility function $U$ is uniformly bounded from above, and (4.4) is equivalent to (4.3) (see e.g. [7] for details).

For the general case, consider for every $n$ a utility function $U^n$ satisfying the Inada's conditions and such that

$$\begin{cases} U^n(x) = U(x), & \text{if } U(x) \leq n, \\ U^n(x) \leq (n+1), \end{cases}$$

and define the functions $V^n$, $u^n$ and $v^n$ as above. It is immediate to verify that $V^n \nearrow V$ and consequently $v^n \nearrow v$: the last statement, which is not immediate, can be proved with hardly any modifications as in Lemma 3.3 of [5]. From these properties, we have $u(x) \geq \inf_{y>0}[v(y) + xy]$.

Conversely, if $f \in \mathcal{C}(x)$ and $g \in \mathcal{D}(y)$, we have

$$U(f) \leq V(g) + fg,$$

and integrating

$$\int U(f)\,\mathrm{d}\mu \le \int (V(g) + fg)\,\mathrm{d}\mu \le \int V(g)\,\mathrm{d}\mu + xy.$$

Consequently $u(x) \le v(y) + xy$. □

*Remark* 4.4. The proof of Theorem 4.2 shows in particular (as a consequence of Theorem 3.2) that if the utility function $U$ is uniformly bounded from above, the problem (4.1) always has a solution (i.e. the *supremum* is in fact a *maximum*); but for the general case this is not proved. In fact this is not true, and the papers [5] and [6], give necessary and sufficient conditions for the existence of a solution to the problem (4.1).

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