An alternative proof of a theorem of Aldous concerning convergence in distribution for martingales.

Maurizio Pratelli

Dipartimento di Matematica, Università di Pisa.
Via Buonarroti 2. I-56127 Pisa, Italy
e-mail: pratelli@dm.unipi.it

We consider regular right continuous stochastic processes $X = (X_t)_{0\le t\le 1}$ defined on the finite time interval $[0,1]$: let $\mathbf{P}^X$ be the distribution of X on the canonical Skorokhod space $\mathbf{D} = \mathbf{D}([0,1];\mathbf{R})$ of "càdlàg" paths.

We consider on $\mathbf{D}$, besides the usual Skorokhod topology referred as S–topology (Jacod–Shiryaev is perhaps the best reference for our purposes, see [4]), the "pseudo-path" or MZ–topology: we refer to the paper of Meyer–Zheng ([6]) for a complete account of this rather neglected topology (see also Kurtz [5]).

We will use the notation $X^n \Longrightarrow^S X$ (respectively $X^n \Longrightarrow^{MZ} X$) to indicate that the probabilities $\mathbf{P}^{X^n}$ converge strictly to $\mathbf{P}^X$ when the space $\mathbf{D}$ is endowed respectively with the S– or the MZ–topology. We will write also $X^n \Longrightarrow^{f.d.d.} X$ to indicate that all finite dimensional distributions of $(X^n_t)_{0\le t\le 1}$ converge to those of $(X_t)_{0\le t\le 1}$.

The following theorem holds true:

Theorem. *Let (M^n) be a sequence of martingales, and M a continuous martingale, and suppose that the following integrability condition is satisfied:*

(1) all random variables $\left(\sup_{0\le t\le 1}|M^n_t|\right)$, $n = 1,2,\dots$ are uniformly integrable.

Then the following statements are equivalent:

(a) $M^n \Longrightarrow^S M$,

(b) $M^n \Longrightarrow^{f.d.d.} M$,

(c) $M^n \Longrightarrow^{MZ} M$.

The implication $(a)\Longrightarrow(b)$ is quite obvious, since Skorokhod convergence implies convergence of finite dimensional distributions for all continuity points of M (see [4]).

The implications $(b)\Longrightarrow(c)$ is an easy consequence of the results of Meyer–Zheng: in fact the sequence (M^n) is "tight" for the MZ–topology ([6] p. 368) and, if X is a limit process, there exists ([6] p. 365) a subsequence (M^{n_k}) and a set $I \subset [0,1]$ of full Lebesgue measure such that all finite dimensional distributions $(M_t^{n_k})_{t\in I}$ converge to those of $(X_t)_{t\in I}$: necessarily $\mathbf{P}^X = \mathbf{P}^M$.

Aldous (in [2]) gives a proof of the implication $(b)\Longrightarrow(a)$, but (although he does not mention the MZ–topology) the implication $(c)\Longrightarrow(a)$ is more or less implicit in his paper (see [2] p. 591).

The purpose of this paper is to give a proof of the implication $(c)\Longrightarrow(a)$, completely different form the Aldous' original one and strictly in the spirit of the paper of Meyer–Zheng; I hope that this contributes also to a better knowledge of the result of "Stopping times and tightness II" ([2]), which is in my opinion very important and seems to be almost unknown.

The proof will be postponed after some remarks.

Remark 1. I want to point out that Aldous' proof of the implication $(b)\Longrightarrow(a)$ requires the following weaker integrability condition:

(2) all random variables M_1^n , $n=1,2,\ldots$ are uniformly integrable.

Condition (2) implies that all r.v. of the form M_T^n , $n=1,2,\ldots$, with T a natural stopping time for M^n , are uniformly integrable; instead our proof needs a more stringent condition, i.e. that all r.v. of the form M_S^n , $n=1,2,\ldots$, with S a random variable in $[0,1]$, are uniformly integrable.

Remark 2. The extension of the Theorem to processes whose time interval is $[0,+\infty)$ is straightforward: in that case the correct hypothesis is that, for every fixed t , the r.v. $\sup_{0\le s\le t}|M_s^n|$, $n=1,2,\ldots$ are uniformly integrable.

In fact, if the limit function f is continuous, $f_n \to f$ for the S–topology (respectively the MZ–topology) on $\mathbf{D}(\mathbf{R}^+;\mathbf{R})$ if and only if the restrictions of f_n to every finite time interval converge to those of f (for the S– or the MZ–topology).

Remark 3. The Theorem fails to be true if the limit martingale M is not continuous ([2] p. 588), and fails for more general processes, e.g. for supermartingales.

Let indeed T be a Poisson r.v. and put, for every n :

$$X_t^n = \left(I_{\{t\ge T\}} - t\wedge T\right) - n\left((t-T)\,I_{\{t\ge T\}} \wedge 1\right) .$$

The processes X^n are supermartingales whose paths converge in measure (but not uniformly) to the paths of the continuous supermartingale $X_t = -(t\wedge T)$.

Remark 4. Suppose that the processes X^n are supermartingales, and consider their Doob–Meyer decompositions $X^n = M^n - A^n$. If separately $M^n \Longrightarrow^{MZ} M$ and the martingale M is continuous, and if $A^n \Longrightarrow^{MZ} A$ and the increasing process A is continuous, then $X^n \Longrightarrow^{S} X = M - A$ (remark that, for monotone processes, convergence in the MZ–sense to a continuous limit implies convergence for the S–topology).

An application of the latter result can be found in [7], theorem 5.5 .

The proof of the implication $(c) \Longrightarrow (a)$ of the Theorem is rather technical, and will be divided in several steps.

Step 1. Given $\epsilon > 0$, there exists $\delta > 0$ such that, if S is a r.v. with values in $[0,1]$ and $0 \le d \le \delta$:

$$\mathbf{E}\left[|M_{S+d} - M_S|\right] \le \epsilon \; . \tag{3}$$

This is an easy consequence of the path-continuity of the limit process M, and of the integrability of $M^* = \sup_{0\le t\le 1} |M_t|$. Remark that the function $f \to \sup_{0\le t\le 1} |f(t)|$ is lower semi-continuous on $\mathbf{D}$ endowed with the topology of convergence in measure (i.e. the MZ–topology); therefore the integrability of M^* is a consequence of condition (1) of the theorem.

Step 2. Suppose that (a) is false; then the sequence does not verify Aldous' tightness condition ([1] p. 335, see also [4]); therefore there exists $\epsilon > 0$ such that for every $\delta > 0$ it is possible to determine a subsequence n_k and, for every k, a natural stopping time T_k (i.e. a stopping time for the filtration generated by M^{n_k}) and $0 < d_k \le \delta$ such that

$$\mathbf{E}^{n_k}\left[\,|M^{n_k}_{T_k+d_k} - M^{n_k}_{T_k}|\,\right] \ge \epsilon \; . \tag{4}$$

(In the sequel, for the sake of simplicity of notations, we will assume that indices have been renamed so that the whole sequence verifies (4)). We choose δ such that, for any r.v. S whatsoever, we also have (step 1) $\mathbf{E}\big[|M_{S+2\delta} - M_S|\big] \le \frac{\epsilon}{4}$.

Step 3. There exists a random variable T with values in [0,1] such that (M^n, T_n) converge in distribution to (M,T) on the space $\mathbf{D}([0,1],\mathbf{R}^+) \times [0,1]$ equipped with the product topology ($\mathbf{D}$ being equipped with the MZ–topology).

In fact the laws of (M^n, T_n) are evidently tight since the laws of M^n are tight on $\mathbf{D}$ ([6] p. 368); we point out that the limit r.v. T is not a natural stopping time for the stochastic process M (but it can be proved that M is a martingale for the canonical filtration on $\mathbf{D} \times [0,1]$, i.e. the smallest filtration that makes M adapted and T a stopping time).

Step 4. For c and d in $[0,1]$, we have the inequality

$$\mathbf{E}^n\left[\big|M^n_{T_n+\delta+c} \; - \; M^n_{T_n-d}\big|\right] \ge \frac{\varepsilon}{2} \; . \tag{5}$$

(It is technically convenient to regard each process M as extended to $[-1,2]$ by putting $M_t = M_0$ for $t < 0$ and $M_t = M_1$ for $t > 1$: this enables us to write $M_{T+\delta}$ instead of $M_{(T+\delta)\wedge 1}$.)

Concerning the inequality (5), firstly we note that

$$\left(M^n_{T_n+d_n} - M^n_{T_n}\right) = \mathbf{E}^n\left[M^n_{T_n+\delta+c} - M^n_{T_n}\;\big|\mathcal{F}_{T_n+d_n}\right]$$

and therefore

$$\mathbf{E}^n\left[\big|M^n_{T_n+\delta+c} - M^n_{T_n}\big|\right] \ge \mathbf{E}^n\left[\big|M^n_{T_n+d_n} - M^n_{T_n}\big|\right] \ge \epsilon \, .$$

Then we remark that $(T_n - c)$ is not a stopping time, but the r.v. $M^n_{T_n-c}$ is $\mathcal{F}_{T_n}$–measurable: in fact $M^n_{T_n-c}.I\{T_n \le t\} = M^n_{(T_n\wedge t)-c}.I_{\{T_n\le t\}}$ and $(T_n \wedge t - c)$ is $\mathcal{F}_t$–measurable.

Let $X = \left(M^n_{T_n+\delta+c} - M^n_{T_n}\right)$, $Y = \left(M^n_{T_n} - M^n_{T_n-c}\right)$ and $\mathcal{G} = \mathcal{F}_{T_n}$: Y is $\mathcal{G}$-adapted and $\mathbf{E}[X|\mathcal{G}] = 0$.

We remark that $\mathbf{E}\left[X^+|\mathcal{G}\right] = \mathbf{E}\left[X^-|\mathcal{G}\right] = \frac{1}{2}\mathbf{E}\left[|X|\,|\mathcal{G}\right]$, and that $|X+Y| \ge X^+.I_{\{Y\ge 0\}} + X^-.I_{\{Y<0\}}$.

One gets $\mathbf{E}\big[\,|X+Y|\,\big|\mathcal{G}\,\big] \ge \frac{1}{2}\mathbf{E}\big[\,|X|\big|\mathcal{G}\,\big]$; and, taking the expectations, the inequality (5).

Step 5. There exists a subsequence and a set $I \subset [-1,1]$ of full Lebesgue measure such that the finite dimensional distributions of $\left(M^n_{T_n+t}\right)_{t\in I}$ converge to those of $(M_{T+t})_{t\in I}$.

The proof of this step is a slight modification of the argument given in [6] (p. 364): Dudley's extension of the Skorokhod representation theorem implies that one can find on some probability space $(\Omega, \mathcal{F}, \mathbf{P})$ some random variables (X^n, S_n) and (X, S) with values in $\mathbf{D}\times[0,1]$ such that the laws of (X^n, S_n) (resp. (X,S)) are equal to those of (M^n, T_n) (resp. (M,T)) and that, for almost all ω, $(X^n(\omega), S_n(\omega))$ converge to $(X(\omega), S(\omega))$: to be accurate, the "paths" $t \to (X^n_t(\omega))$ converge in measure to the path $t \to (X_t(\omega))$ and $S_n(\omega)$ converge to $S(\omega)$.

We remark that the Skorokhod theorem cannot be applied directly since $\mathbf{D}$ is not a Polish space ([6] p. 372), but Dudley's extension works well (see [3]).

By substituting X^n with $\mathrm{arctg}(X^n)$, we can suppose that X^n and X are uniformly bounded: therefore we have

$$\lim_{n\to\infty}\left(\int_{-1}^{+1} dt \int_\Omega \left|X^n_{T_n(\omega)+t}(\omega) - X_{T(\omega)+t}(\omega)\right| d\mathbf{P}(\omega)\right) = 0\,. \tag{6}$$

By taking a subsequence, we find that for every t in a set $I \subset [-1,1]$ of full Lebesgue measure,

$$\lim_{n\to\infty}\int_\Omega \left|X^n_{T_n(\omega)+t}(\omega) - X_{T(\omega)+t}(\omega)\right| d\mathbf{P}(\omega) = 0\,. \tag{7}$$

Hence one gets easily the convergence of finite dimensional distributions of $\left(M^n_{T+t}\right)_{t\in I}$.

Step 6. We choose $0 \le d, c \le 1$ such that $d + c < \delta$ and that $\left(M^n_{T_n+\delta+c}\,,\, M^n_{T_n-d}\right)$ converge in distribution to $(M_{T+\delta+c}\,,\, M_{T-d})$; since the r.v. involved are uniformly integrable, the inequality (5) gives in the limit

$$\mathbf{E}\big[\,|M_{T+\delta+c} - M_{T-d}|\,\big] \ge \frac{\varepsilon}{2}$$

and finally we have a contradiction.

References

[1] Aldous D.: *Stopping Times and Tightness*. Ann. Prob. **6**, 335–340 (1979)

[2] Aldous D.: *Stopping Times and Tightness II*. Ann. Prob. **17**, 586–595 (1989)

[3] Dudley R.M.: *Distances of Probability measures and Random Variables*. Ann. of Math. Stat. **39**, 1563–1572 (1968)

[4] Jacod J., Shiryaev A.N.: *Limit theorems for stochastic processes*. Berlin, Heidelberg, New York: Springer 1987.

[5] Kurtz T.G.: *Random time changes and convergence in distribution under the Meyer–Zheng conditions* Ann. Prob. **19**, 1010–1034 (1991)

[6] Meyer P.A., Zheng W.A.: *Tigthness criteria for laws of semimartingales*. Ann. Inst. Henri Poincaré Vol. **20** No. 4, 353–372 (1984)

[7] Mulinacci S., Pratelli M.: *Functional convergence of Snell envelopes: Applications to American options approximations*. Finance Stochast. **2**, 311–327 (1998)