Random flights

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Khabarovsk, March 2014

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One-dimensional random motions at finite velocity

1.The symmetric telegraph process

Definition 1.

$$X(t) = V(0)\int_0^t (-1)^{N(s)}ds \tag{1}$$

where $V(0)$ is a symmetric two-valued r.v. taking values $\pm c$. It can be regarded as the initial velocity of motion. $N(t)$ is the number of events of a homogeneous Poisson process independent from $V(0)$.

Known results:

1. Let $\Pr\{X(t) \in dx\}/dx = p(x,t)$. The absolutely continuous part of the probability distribution satisfies the telegraph equation

$$\frac{\partial^2 p}{\partial t^2} + 2\lambda \frac{\partial p}{\partial t} = c^2 \frac{\partial^2 p}{\partial x^2}, \qquad p(x,0) = \delta(x), \qquad p_t(x,0) = 0. \tag{2}$$

The discrete part of the distribution is concentrated at $\pm ct$ with probabilities $\frac{1}{2}e^{-\lambda t}$.

2. The probability density of the a.c. part is

$$p(x,t) = \frac{e^{-\lambda t}}{2c}\left[\lambda I_0\left(\frac{\lambda}{c}\sqrt{c^2t^2 - x^2}\right) + \frac{\partial}{\partial t} I_0\left(\frac{\lambda}{c}\sqrt{c^2t^2 - x^2}\right)\right], \quad |x| < ct, \tag{3}$$

where $I_0(x) = \sum_{k=0}^{\infty} \left(\frac{x}{2}\right)^{2k} \frac{1}{k!}$ and

$$\int_{-ct}^{+ct} p(x,t)dx \;=\; 1 - e^{-\lambda t}, \qquad t \geq 0. \tag{4}$$

3. For a fixed number $N(t)$ of changes of direction we have that

$$\begin{aligned}
\Pr\{X(t) \in dx | N(t) = 2k+1\} \;&=\; dx \frac{(2k+1)!}{(k!)^2} \frac{\left(c^2t^2 - x^2\right)^k}{(2ct)^{2k+1}}, && k \geq 0, \\
\Pr\{X(t) \in dx | N(t) = 2k\} \;&=\; dx \frac{ct(2k)!}{k!(k-1)!} \frac{(c^2t^2 - x^2)^{k-1}}{(2ct)^{2k}}, && k \geq 1.
\end{aligned} \tag{5}$$

For example,

$$\begin{aligned}
\Pr\{X(t) \in dx | N(t) = 1\} &= \Pr\{X(t) \in dx | N(t) = 2\} = \frac{1}{2ct}, \quad x \in (-ct, ct) \\
\Pr\{X(t) \in dx | N(t) = 3\} &= \Pr\{X(t) \in dx | N(t) = 4\} = \frac{3!}{(2ct)^3}\left(c^2t^2 - x^2\right)
\end{aligned} \tag{6}$$

4. The first-passage time $T_a = \inf\{t > 0 : X(t) = a\}$ in case of positive initial velocity reads

$$\begin{aligned}
\Pr\{T_a > t | V(0) = c\} &= \begin{cases} \int_t^\infty \frac{\lambda a}{\sqrt{c^2s^2 - a^2}} I_1\left(\frac{\lambda}{c}\sqrt{c^2s^2 - a^2}\right) ds, & t > \frac{a}{c} \\ 1, & t < \frac{a}{c} \end{cases} \\
\Pr\left\{T_a = \frac{a}{c} | V(0) = c\right\} &= e^{-\lambda \frac{a}{c}}
\end{aligned} \tag{7}$$

5. The characteristic function

$$\mathbb{E}e^{i\beta X(t)} = $$

$$= \frac{e^{-\lambda t}}{2}\left[\left(1+\frac{\lambda}{\sqrt{\lambda^2-c^2\beta^2}}\right)e^{t\sqrt{\lambda^2-c^2\beta^2}}+\left(1-\frac{\lambda}{\sqrt{\lambda^2-c^2\beta^2}}\right)e^{-t\sqrt{\lambda^2-c^2\beta^2}}\right] \tag{8}$$

for $|\beta| < \frac{\lambda}{c}$.

6.

Remark 1. *For $\lambda \to \infty$, $c \to \infty$, $c^2/\lambda \to 1$, we have that*

(a) The telegraph equation becomes the heat-equation
(b) The distribution $p(x, t)$ of the telegraph process converges to the transition function of Brownian motion
(c) The first-passage time distribution converges to the law of the first-passage time of Brownian motion

(d) For the characteristic function we have

$$\mathbb{E}e^{i\beta X(t)} \to e^{-\frac{t\beta^2}{2}} \tag{9}$$

2. Telegraph process with drift

Some generalization is introduced by assuming that the telegraph process has drift.

1. The particle moves rightward with velocity c_1 or leftward with velocity $-c_2$.

2. When moving rightward the particle reverses velocity after an exponentially distributed time interval (with parameter λ_1) and when moving leftward with velocity $-c_2$ reverses the direction of motion after a time with exponential distribution of parameter λ_2.

In this last case the telegraph equation governing the distribution reads

$$\begin{aligned} \frac{\partial^2}{\partial t^2}p =& c_1 c_2 \frac{\partial^2 p}{\partial x^2} + (c_2 - c_1)\frac{\partial^2 p}{\partial x \partial t} - (\lambda_1 + \lambda_2)\frac{\partial p}{\partial t} \\ & + \frac{1}{2}\left[(c_2 - c_1)\left(\lambda_1 + \lambda_2\right) - (\lambda_2 - \lambda_1)(c_1 + c_2)\right]\frac{\partial p}{\partial x} \end{aligned} \tag{10}$$

By means of a relativistic transformation

$$\begin{cases} x' = \alpha x + \beta t \\ t' = \gamma x + \delta t \end{cases} \tag{11}$$

we can pass from the asymmetric telegraph process in (x,t) to a symmetric telegraph process in (x',t') where the probability law $p(x',t')$ is governed by

$$\frac{\partial^2 p}{\partial t'^2} = \frac{4(c_1 + c_2)^2 \lambda_1^2 \lambda_2^2}{\left(\lambda_1 + \lambda_2\right)^4}\frac{\partial^2 p}{\partial x'^2} - \frac{4\lambda_1 \lambda_2}{(\lambda_1 + \lambda_2)}\frac{\partial p}{\partial t'} \tag{12}$$

where the new velocity is $c' = \pm\frac{2(c_1+c_2)\lambda_1\lambda_2}{(\lambda_1+\lambda_2)^2}$ while the rate of velocity reversal is $\lambda' = \frac{2\lambda_1\lambda_2}{\lambda_1+\lambda_2}$. The explicit form of the absolutely continuous part of the distribution reads

$$
\begin{aligned}
& p(x,t) = \\
& \frac{e^{-\frac{(\lambda_1+\lambda_2)t}{c}+\frac{\lambda_2-\lambda_1}{c_1+c_2}x+\frac{(\lambda_2-\lambda_1)(c_2-c_1)}{2(c_1+c_2)}t}}{c_1+c_2}\left[\frac{\lambda_1+\lambda_2}{2}I_0\left(\frac{2\sqrt{\lambda_1\lambda_2}}{c_1+c_2}\sqrt{(x+c_2t)(c_1t-x)}\right)\right. \\
& +\frac{\partial}{\partial t}I_0\left(\frac{2\sqrt{\lambda_1\lambda_2}}{c_1+c_2}\sqrt{(x+c_2t)(c_1t-x)}\right) \\
& \left.-\frac{(c_2-c_1)}{2}\frac{\partial}{\partial x}I_0\left(\frac{2\sqrt{\lambda_1\lambda_2}}{c_1+c_2}\sqrt{(x+c_2t)(c_1t-x)}\right)\right]. \qquad (13)
\end{aligned}
$$

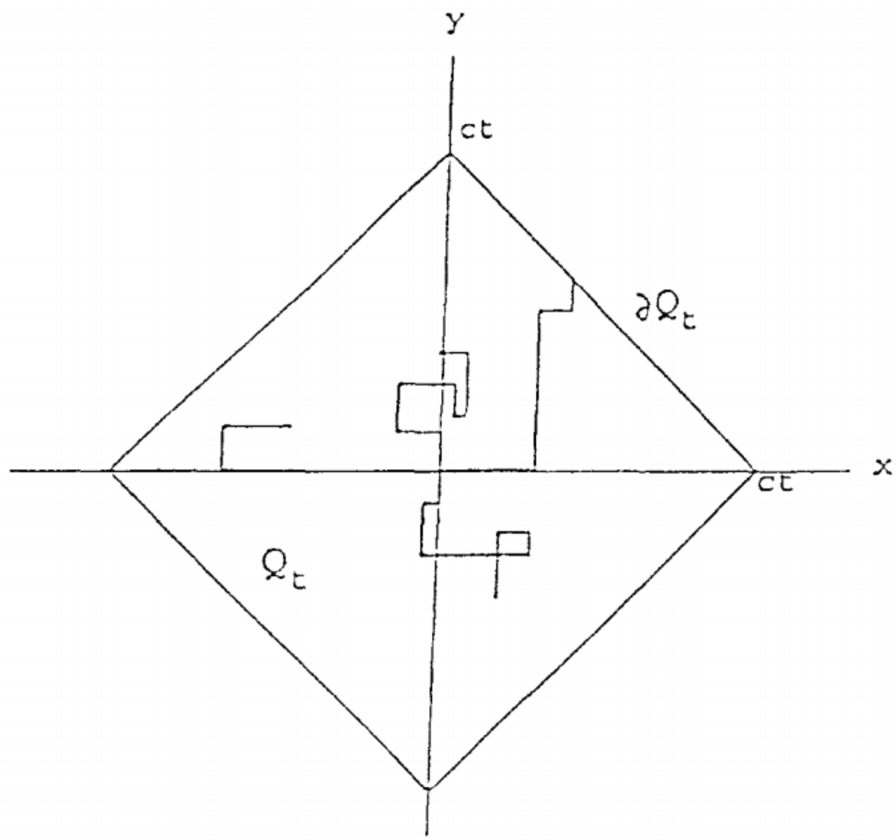
The singular part of the distribution concentrated at C_1t has weight $e^{-\lambda_1 t}/2$ and at $-c_2t$ has weight $e^{-\lambda_2 t}/2$.

Multidimensional extensions

For the sake of simplicity we consider two types of motions in $\mathbb{R}^2$.

1. A planar random motion with a finite number of possible directions

2. A planar motion with an infinite number of possible directions



In the first case the simplest model has four orthogonal directions which are assumed initially with equal probability.

The changes of direction are governed by a homogenous Poisson process.

The velocity is c and changes of direction of motion occur at Poisson times and at each event the particle moves on the line orthogonal with respect to that on which it was moving before the deviation.

Sample paths are formed by segments parallel to the axes.

At each instant the particle is located inside a square Q_t with vertices $(-ct, 0)$, $(ct, 0)$, $(0, -ct)$, $(0, ct)$.

If the particle points outward up to time t it will be located on the boundary ∂Q_t with probability

$$\Pr\left\{X(t), Y(t) \in \partial Q_t\right\} = \Pr\left\{N(t) = 0\right\} + \sum_{k=1}^{\infty} \Pr\left\{N(t) = k\right\} \frac{1}{2^k} = 2e^{-\lambda\frac{t}{2}} - e^{-\lambda t} \tag{14}$$

and clearly with probability $e^{-\lambda t}$ the moving particle is on one of the four vertices.

At each time t a part of the distribution is on the boundary and the absolutely continuous one is inside Q_t.

The inside component grows as time passes and at time $t^\star = -2\log\left(1 - \frac{1}{\sqrt{2}}\right)$ their weight coincides.

The position $X(t), Y(t)$ of this moving particle has the following representation

$$\begin{cases} X(t) = U(t) + V(t) \\ Y(t) = U(t) - V(t) \end{cases} \tag{15}$$

where U and V are independent, symmetric, one-dimensional telegraph processes of parameters $\frac{\lambda}{2}, \frac{c}{2}$. Therefore the absolutely continuous part of the distribution inside Q_{ct}, reads

$$\begin{aligned} g(x,y,t) = & \frac{e^{-\lambda t}}{2c^2} \left[\frac{\lambda^2}{4} I_0 \left(\frac{\lambda}{2c} \sqrt{c^2t^2 - (x+y)^2} \right) I_0 \left(\frac{\lambda}{2c} \sqrt{c^2t^2 - (x-y)^2} \right) \right. \\ & + \frac{\lambda}{2} \frac{\partial}{\partial t} \left(I_0 \left(\frac{\lambda}{2c} \sqrt{c^2t^2 - (x+y)^2} \right) I_0 \left(\frac{\lambda}{2c} \sqrt{c^2t^2 - (x-y)^2} \right) \right) \\ & \left. + \frac{\partial}{\partial t} I_0 \left(\frac{\lambda}{2c} \sqrt{c^2t^2 - (x+y)^2} \right) \frac{\partial}{\partial t} I_0 \left(\frac{\lambda}{2c} \sqrt{c^2t^2 - (x-y)^2} \right) \right] \\ & \text{for } |x-y| < ct, |x+y| < ct \end{aligned} \tag{16}$$

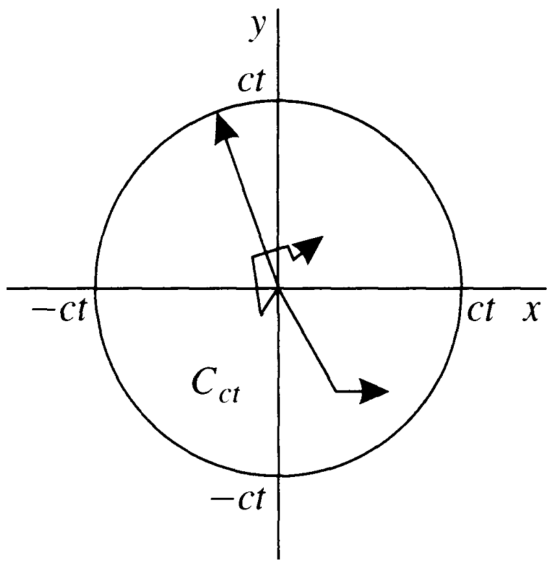
On each of the four segments of the boundary ∂Q_{ct} of Q_t the distribution coincides with that of a telegraph process. The probability function g satisfies the following fourth-order $p.d.e.$

$$\left(\frac{\partial}{\partial t}+\lambda\right)^2\left(\frac{\partial^2}{\partial t^2}+2\lambda-c^2\left\{\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right\}\right)u+c^4\frac{\partial^4 u}{\partial x^2\partial y^2}\;=\;0. \qquad (17)$$

Of course, there are many other models with an arbitrary number of possible directions of motions and with different rules of change among directions (for example, the cyclic one).

Infinite number of directions

The particle moves on $\mathbb{R}^2$ with velocity c and its direction changes at Poisson times.



At each Poisson event it takes a direction with uniformly distributed orientation. Each change of direction is independent from the previous one. The set of possible positions at time t is a circle $C_{ct} = \left\{x, y : x^2 + y^2 \leq c^2t^2\right\}$ and the particle is located on ∂C_{ct} with probability $e^{-\lambda t}$ if no Poisson event disturbs its motion until time t.

A complete probabilistic description of this motion has been given.

The position of the moving particle is given by

$$\begin{cases} X(t) = c\sum_{j=1}^{n+1}(s_j - s_{j-1})\cos\theta_j \\ Y(t) = c\sum_{j=1}^{n+1}(s_j - s_{j-1})\sin\theta_j \end{cases} \tag{18}$$

where $s_0 = 0$, $s_{n+1} = t$, θ_j are independent r.v.'s uniformly distributed in $[0, 2\pi)$. Thanks to some particular properties of Bessel functions it is possible to obtain explicitly the characteristic function

$$\mathbb{E}\left[e^{i\alpha X(t)+i\beta Y(t)}\middle| N(t) = n\right] = \frac{2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}+1\right)}{\left(ct\sqrt{\alpha^2+\beta^2}\right)^{\frac{n}{2}}} J_{\frac{n}{2}}\left(ct\sqrt{\alpha^2+\beta^2}\right), \quad n \geq 1. \tag{19}$$

For $n = 0$

$$\mathbb{E}\left[e^{i\alpha X(t)+i\beta Y(t)}\middle| N(t) = 0\right] = J_0\left(ct\sqrt{\alpha^2+\beta^2}\right). \tag{20}$$

The characteristic function above can be inverted and for $n \geq 1$ we have that

$$\Pr\{X(t) \in dx, Y(t) \in dy | N(t) = n\} = \frac{n}{2\pi (ct)^n} \left(c^2t^2 - (x^2+y^2)\right)^{\frac{n}{2}-1} dxdy \tag{21}$$

Note that for $n = 2$ the distribution obtained is uniform in the circle C_{ct}. For $n = 1$ it coincides with the Green function of the planar waves equation. For $n \geq 3$ it has the form of a bell which is more and more concentrated around the starting point.

The unconditional distribution

$$\Pr\{X(t) \in dx, Y(t) \in dy\} = p(x,y,t)dxdy = \frac{\lambda dxdy}{2\pi c} e^{-\lambda t} \frac{e^{\frac{\lambda}{c}\sqrt{c^2t^2-(x^2+y^2)}}}{\sqrt{c^2t^2-(x^2+y^2)}} \tag{22}$$

for $(x,y) \in C_{ct} - \partial C_{ct}$, can easily be inferred from the previous conditional distribution.

We observe that p is a solution to the planar wave equation

$$\frac{\partial^2 p}{\partial t^2} + 2\lambda \frac{\partial p}{\partial t} \;=\; c^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) p \tag{23}$$

and for $c \to \infty$, $\lambda \to \infty$, $\frac{c^2}{\lambda} \to 1$, converges to the planar heat equation. The density p coinverges to the transition density of the planar Brownian motion.

By means of $p = e^{-\lambda t} w$ equation (23) becomes

$$\frac{\partial^2 w}{\partial t^2} - c^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) w \;=\; \lambda^2 w \tag{24}$$

which is a Klein-Gordon type equation.

Random flights in $\mathbb{R}^d$

The last model can be extended in an Euclidean space of dimension d. The first who thought of random flights with deterministic displacements, with uniformly distributed orientation was K. Pearson. In the case the motion at finite velocity is governed by a homogeneous Poisson process (of rate λ) and at each event a new direction is chosen. The distribution of the orientation is uniform on the hypersphere, that is has density

$$f\left(\theta_1, \ldots, \theta_{d-2}, \phi\right) = \frac{\Gamma\left(\frac{d}{2}\right)}{2n^{\frac{d}{2}}} \sin^{d-2}\theta_1 \sin^{d-3}\theta_2 \cdots \sin\theta_{d-2} \tag{25}$$

where $0 \le \theta_j \le \pi$, $j = 1, \ldots, d-2$, $0 \le \phi \le 2\pi$.

If $N(t) = n$, the total displacement has the form

$$X_d(t) = c\sum_{j=1}^{n+1}(s_j - s_{j-1})\sin\theta_{1,j}\sin\theta_{2,j}\cdots\sin\theta_{d-2,j}\sin\phi_j \tag{26}$$

$$X_{d-1}(t) \; = \; c\sum_{j=1}^{n+1}(s_j - s_{j-1})\sin\theta_{1,j}\cdots\sin\theta_{d-2,j}\cos\phi_j \tag{27}$$

$$\vdots \tag{28}$$

$$X_2(t) \; = \; c\sum_{j=1}^{n+1}(s_j - s_{j-1})\sin\theta_{1,j}\sin\cos\theta_{2,j} \tag{29}$$

$$X_1(t) \; = \; c\sum_{j=1}^{n+1}(s_j - s_{j-1})\cos\theta_{1,j} \tag{30}$$

where $0 \le \theta_j \le \pi$, $0 \le \phi_j \le 2\pi$, s_j are the instants of Poisson events with $s_0 = 0$,

$s_{n+1} = t$.

The general expression of the conditional characteristic function for $n \geq 1$, $d \geq 2$ is

$$\mathbb{E}\left[e^{i \sum_{k=1}^{d} \alpha_k X_k(t)} \middle| N(t) = n \right] =$$

$$= \frac{n!}{t^n} \left(2^{\frac{d}{2}-1} \Gamma\left(\frac{d}{2}\right) \right)^{n+1} \int_0^t ds_1 \cdots \int_{s_{n-1}}^t ds_n \prod_{j=1}^{n+1} \frac{J_{\frac{d}{2}-1}\left(c(s_j - s_{j-1}) \sqrt{\sum_{j=1}^{d} \alpha_j^2} \right)}{\left(c(s_j - s_{j-1}) \sqrt{\sum_{k=1}^{d} \alpha_k^2} \right)^{\frac{d}{2}-1}} \tag{31}$$

for $(\alpha_1, \dots, \alpha_d) \in \mathbb{R}^d$, $0 = s_0 < \cdots < s_{n+1} = t$

The distribution corresponding to this characteristic function reads

$$p_n(x_1, x_2, \dots, x_d, t) = \frac{\left(\Gamma\left(\frac{s}{2}\right) 2^{\frac{d}{2}-1}\right)^n}{(2\pi)^{\frac{d}{2}} t^n \left(\sqrt{\sum x_k^2}\right)^{\frac{d}{2}-1}} \int_0^\infty \rho^{\frac{d}{2}} J_{\frac{d}{2}-1}\left(\rho \sqrt{\sum_{k=1}^d x_k^2}\right) d\rho$$
$$\times \int_0^t ds_1 \cdots \int_{s_{n-1}^t} ds_n \prod_{j=1}^{n+1} \frac{J_{\frac{d}{2}-1}\left(c\rho\left(s_j - s_{j-1}\right)\right)}{\left(c\rho\left(s_j - s_{j-1}\right)\right)^{\frac{d}{2}-1}}, \qquad n \geq 1 \tag{32}$$

This integral can be evaluated explicitly for all $n \geq 1$ only for $d = 2, 4$. For example for $d = 2$

$$\frac{n!}{t^n} \int_0^t ds_1 \cdots \int_{s_{n-1}}^t ds_n \prod_{j=1}^{n+1} \frac{J_0\left(c\rho\left(s_j - s_{j-1}\right)\right)}{(ct\rho)^{\frac{n}{2}}} = \frac{2^{\frac{n}{2}} \Gamma\left(\frac{n}{2}+1\right)}{(ct\rho)^{\frac{n}{1}}} J_{\frac{n}{2}}(ct\rho) \tag{33}$$

and thus

$$
\begin{aligned}
p_n(x_1, x_2, t) &= \frac{2^{\frac{n}{2}} n \Gamma\left(\frac{n}{2}\right)}{4\pi (ct)^{\frac{n}{2}}} \int_0^\infty \rho^{1-\frac{n}{2}} J_0\left(\rho\sqrt{x_1^2 + x_2^2}\right) J_{\frac{n}{2}}\left(ct\rho\right) d\rho \\
&= \frac{n}{2\pi (ct)^{\frac{n}{2}}} \left(c^2t^2 - \left(x_1^2 + x_2^2\right)\right)^{\frac{n}{2}-1}
\end{aligned}
\tag{34}
$$

and a previous result is obtained.

Even the distribution after one single Poisson event is extremely entangled for arbitrary d. For $d = 3$ we have

$$
p_1(x_1, x_2, x_3, t) = \frac{\log\left(\frac{ct+\sqrt{x_1^1+x_2^2+x_3^2}}{ct-\sqrt{x_1^1+x_2^2+x_3^2}}\right)}{\pi (2ct)^2 \sqrt{x_1^1 + x_2^2 + x_3^2}}, \qquad (x_1, x_2, x_3) \in S_{ct}^3 \tag{35}
$$

while for $d = 2$

$$p_1(x_1, x_2; t) = \frac{1}{2\pi ct\sqrt{c^2t^2 - (x_1^2 + x_2^2)}}, \qquad (x_1, x_2) \in S_{ct}^2$$

and $d = 4$ (36)

$$p_1(x_1, x_2, x_4, x_4; t) = \frac{2}{\pi^2(ct)^4}, \qquad (x_1, x_2, x_3, x_4) \in S_{ct}^4 \quad (37)$$

In the last case we have a uniform distribution!

For this reason we are able to produce explicit unconditional distributions in $\mathbb{R}^2$ and $\mathbb{R}^4$, only. For $x_1^2 + x_2^2 < c^2t^2$ we have that

$$\Pr\{X_1(t) \in dx_1, X_2(t) \in dx_2\} = \frac{\lambda}{2\pi c} \frac{e^{-\lambda t + \frac{\lambda}{c}\sqrt{c^2t^2 - (x_1^2 + x_2^2)}}}{\sqrt{c^2t^2 - (x_1^2 + x_2^2)}} dx_1 dx_2, \tag{38}$$

and on ∂C_{ct} the distribution is uniform.

Its projection on the x-axis has an absolutely continuous distribution for $|x| < ct$

$$\Pr\{X_1(t) \in dx_1\} = dx_1 \frac{\lambda e^{-\lambda t}}{2c} \sum_{k=0}^{\infty} \left(\frac{\lambda}{2c} \sqrt{c^2t^2 - x_1^2} \right)^{k-1} \frac{1}{\Gamma^2\left(\frac{k+1}{2}\right)} \tag{39}$$

which differs from that of the classical telegraph process because it behaves as a forward backward motion with random velocities.

In $\mathbb{R}^4$ the explicit unconditional distribution has the form of a cut-off fourdimensional Gaussian

$$p(x_1, x_2, x_3, x_4, t) = \frac{\lambda}{c^4 t^3 \pi^2} e^{-\frac{\lambda}{c^2 t} \sum_{k=1}^{4} x_k^2} \left(2 + \frac{\lambda}{c^2 t} \left(c^2 t^2 - \sum_{k=1}^{4} x_k^2\right)\right) \qquad (40)$$

Dirichlet displacements

A random walker starts from the origin of a frame of reference, moves in the d-dimensional real space ($d \geq 2$) at finite velocity c. In $[0,t]$, n changes of direction are recorded. The instants at which the random walker changes direction are $0 < t_1 < \cdots < t_n < t$, $t_0 = 0$, $t_{n+1} = t$, and by $\tau_j = t_j - t_{j-1}$, $1 \leq j \leq n+1$, we represent the length of time between successive changes of direction. Each displacement has uniformly distributed orientation. In this case we assume that $\tau_1, \ldots, \tau_n$ has distribution either equal to

$$f_1(\tau_1, \ldots, \tau_n) = \frac{\Gamma\left((n-1)(d-1)\right)}{\left(\Gamma(d-1)\right)^{n+1}} \frac{1}{t^{(n+1)(d-1)-1}} \prod_{j=1}^{n+1} \tau_j^{d-2}, \qquad d \geq 2 \tag{41}$$

or

$$f_2(\tau_1, \ldots, \tau_n) = \frac{\Gamma\left((n-1)(\frac{d}{2}-1)\right)}{\left(\Gamma(\frac{d}{2}-1)\right)^{n+1}} \frac{1}{t^{(n+1)(\frac{d}{2}-1)-1}} \prod_{j=1}^{n+1} \tau_j^{\frac{d}{2}-2}, \qquad d \geq 3 \tag{42}$$

where

$$0 < \tau_j < t - \sum_{k=0}^{j-1} \tau_k, \qquad 1 \leq j \leq n, \qquad \tau_{n+1} = t - \sum_{j=1}^{n+1} \tau_j. \tag{43}$$

The two distributions are Dirichlet with parameters $(d-1), \ldots, (d-1)$ and $\left(\frac{d}{2}-1, \ldots, \frac{d}{2}-1\right)$, respectively.

The model consists of a triple $(\boldsymbol{\theta}, \boldsymbol{\tau}, N_d(t))$ where $\boldsymbol{\theta}$ represents the orientations, $\boldsymbol{\tau}$ the displacements and $N_d(t)$ the number of changes of direction up to time t.

By using the first Dirichlet distribution we get

$$p_{\boldsymbol{X}_d}(\boldsymbol{x}_d, t; n) = \frac{\Gamma\left(\frac{n+1}{2}(d-1)+\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}(d-1)\right)} \frac{\left(c^2t^2 - \|\boldsymbol{x}_d\|^2\right)^{\frac{n}{2}(d-1)-1}}{\pi^{\frac{d}{2}}(ct)^{(n+1)(d-1)-1}}, \qquad \|\boldsymbol{x}_d\| < ct, d \geq 2 \tag{44}$$

while by using the second Dirichlet distribution we get

$$p_{\boldsymbol{Y}_d}(\boldsymbol{y}_d, t; n) = \frac{\Gamma\left(\frac{n+1}{2}(d-1)+1\right)}{\Gamma\left(n(d-1)\right)} \frac{\left(c^2t^2 - \|\boldsymbol{y}_d\|^2\right)^{n(\frac{d}{2}-1)-1}}{\pi^{\frac{d}{2}}(ct)^{2(n+1)(\frac{d}{2}-1)}}, \qquad \|\boldsymbol{y}_d\| < ct, d \geq 3. \tag{45}$$

For $d = 2$ the first one yields

$$p_{\boldsymbol{x}_2}(\boldsymbol{x}_2, t; n) = \frac{n}{2\pi(ct)^n}\left(c^2t^2 - \|\boldsymbol{x}_2\|^2\right)^{\frac{n}{2}-1}, \qquad \|\boldsymbol{x}_2\| < ct \tag{46}$$

while for $d = 4$, the second one gives

$$p_{\boldsymbol{Y}_4}(\boldsymbol{y}_4, t; n) \;=\; \frac{n(n+1)}{\pi^2 (ct)^{2n+1}} \left(c^2 t^2 - \|\boldsymbol{y}_4\|^2\right)^{n-1}, \qquad \|\boldsymbol{y}_4\| < ct \tag{47}$$

The unconditional distribution are obtained by randomizing by means of a sort of fractional Poisson process. In the first case

$$\Pr\{N_d(t) = n\} \;=\; \frac{1}{E_{\frac{d-1}{2},\frac{d}{2}}(\lambda t)} \frac{(\lambda t)^n}{\Gamma\left(\left(\frac{d-1}{2}\right)n + \frac{d}{2}\right)}, \qquad \lambda \geq 0, d \geq 2, n \geq 0 \tag{48}$$

while in the second case

$$\Pr\{M_d(t) = n\} \;=\; \frac{1}{E_{\frac{d}{2}-1,\frac{d}{2}}(\lambda t)} \frac{(\lambda t)^n}{\Gamma\left(\left(\frac{d}{2} - 1\right)n + \frac{d}{2}\right)}, \qquad \lambda \geq 0, d \geq 3, n \geq 0. \tag{49}$$

The unconditional distribution obtained in this way has the form (in the case of the first Dirichlet distribution)

$$\Pr\{\boldsymbol{X}_d \in d\boldsymbol{x}_d\} = \frac{\lambda t}{\pi^{\frac{d}{2}}} \frac{\left(c^2t^2 - \|\boldsymbol{x}_d\|\right)^{\frac{d-1}{2}-1}}{(ct)^{2(d-1)-1}} \frac{E_{\frac{d-1}{2},\frac{d-1}{2}}\left(\frac{\lambda t(c^2t^2-\|\boldsymbol{x}_d\|^2)^{\frac{d-1}{2}}}{(ct)^{d-1}}\right)}{E_{\frac{d-1}{2},\frac{d}{2}}(\lambda, t)}. \tag{50}$$

As a special case for $d = 3$ we have that

$$\Pr\{\boldsymbol{X}_3(t) \in d\boldsymbol{x}_3\} = \frac{\lambda}{\pi^{\frac{3}{2}}c^3t^2} \frac{e^{\frac{\lambda\left(c^2t^2-\|\boldsymbol{x}_3\|^2\right)}{c^2t}}}{E_{1,\frac{3}{2}}(\lambda t)} \tag{51}$$

Fractional extension

The telegraph process has been extended in several directions. The simplest one (perhaps) stems from the fractionalization of the telegraph equation

$$\frac{\partial^{2\alpha} u}{\partial t^{2\alpha}} + 2\lambda \frac{\partial^{\alpha} u}{\partial t^{\alpha}} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad \alpha \in (0,1) \tag{52}$$

where

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}} u(x,t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{\partial^m u(x,s)}{\partial s^m} \frac{1}{(t-s)^{\alpha+1-m}} ds, \qquad m-1 < \alpha < m, \tag{53}$$

is the Dzerbayshan-Caputo fractional derivative.

For $0 < \alpha < \frac{1}{2}$ the initial condition assumed is

$$u(x,0) = \delta(x) \tag{54}$$

while for $\frac{1}{2} < \alpha < 1$ a further initial condition is necessary and taken to be equal to

$$u_t(x,0) = 0. \tag{55}$$

For $\lambda \to \infty$, $c \to \infty$, $\frac{c^2}{\lambda} \to 1$, one obtains the so called fractional diffusion equation of which a vast literature exists.

The Fourier transform of the solution of the Cauchy problem for the time-fractional telegraph equation reads

$$\int_{-\infty}^{\infty} e^{i\beta x} u(x,t)dx$$
$$= \frac{1}{2}\left[\left(1+\frac{\lambda}{\sqrt{\lambda^2-c^2\beta^2}}\right)E_{\alpha,1}(\eta_1 t^\alpha)+\left(1-\frac{\lambda}{\sqrt{\lambda^2-c^2\beta^2}}\right)E_{\alpha,1}(\eta_2 t^\alpha)\right] \quad (56)$$

where

$$\eta_1 = -\lambda + \sqrt{\lambda^2-c^2\beta^2}, \qquad \eta_2 = -\lambda - \sqrt{\lambda^2-c^2\beta^2}. \quad (57)$$

Since solutions to the fractional telegraph equations are non-negative for $0 < \alpha < 1$, the expression above can be viewed as the characteristic function of some r.v. $X_\alpha(t)$, $t > 0$. For $\alpha = 1$ (56) coincides with the characteristic function of the telegraph process. Here the exponentials are replaced by the Mittag-Leffler functions $E_{\alpha,1}(x)$.

For $\alpha = \frac{1}{2}$ the characteristic function (56) coincides with that of the composition of a telegraph process T with a reflecting Brownian motion

$$T\left(|B(t)|\right), \qquad t > 0. \tag{58}$$

This means that we can construct a telegraph process moving inside the interval $(-c|B(t)|, +c|B(t)|)$ according to the rules of the usual symmetric process, taken at the random time $|B(t)|$.

Furthermore

$$\begin{aligned} \mathrm{Var} X_\alpha(t) &= 2c^2 t^{2\alpha} E_{\alpha,\alpha+1}(-2\lambda t^\alpha), \qquad 0 < \alpha \leq 2, \\ \mathrm{Var} X_{\frac{1}{2}}(t) &= \int_0^\infty \frac{e^{-\frac{w^2}{4t}}}{\sqrt{\pi t}} \frac{c^2}{\lambda} \left(w + \frac{e^{-2\lambda w} - 1}{2\lambda} \right) dw \overset{t\to\infty}{\sim} \frac{2c^2}{\lambda\sqrt{\pi}} \sqrt{t} \end{aligned} \tag{59}$$

A second type of fractionalization is based on the idea that the telegraph equation

$$\frac{\partial^2 u}{\partial t^2} + 2\lambda \frac{\partial u}{\partial t} \;=\; c^2 \frac{\partial^2 u}{\partial x^2} \tag{60}$$

after the transformation $u = e^{-\lambda t} w$ becomes a sort of Klein-Gordon equation

$$\left(\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2} \right) u \;=\; \lambda^2 u. \tag{61}$$

The fractionalization of the K.G. equation of order $\alpha \in (0,1)$

$$\left(\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2} \right)^{\alpha} u \;=\; \lambda^2 u \tag{62}$$

is based on the idea that by means of the transformation

$$w = \sqrt{c^2 t^2 - x^2} \tag{63}$$

we obtain

$$\left(\frac{d^2}{dw^2} + \frac{1}{w}\frac{d}{dw} \right)^{\alpha} u(w) \; = \; \frac{\lambda^2}{c^{2\alpha}} u(w) \tag{64}$$

where the fractional power of the Bessel operator L_B appears

$$L_B \; = \; \frac{d^2}{dw^2} + \frac{1}{w}\frac{d}{dw} \; = \; \frac{1}{w^2} \left(w \frac{d}{dw} w \frac{d}{dw} \right) . \tag{65}$$

By means of the Mc Bride theory we have that

$$
\begin{aligned}
(L_B)^\alpha f(w) &= \left(\frac{d^2}{dw^2} + \frac{1}{w}\frac{d}{dw}\right)^\alpha f(w) \\
&= 4^\alpha w^{-2\alpha} I_2^{0,-\alpha} I_2^{0,-\alpha} f(w) \qquad (66)
\end{aligned}
$$

where

$$
I_m^{\eta,\alpha} f = \frac{x^{-m\eta-m\alpha}}{\Gamma(\alpha)} \int_0^x (x^m - u^m)^{\alpha-1} u^{m\eta} f(w) du^m, \qquad \alpha > 0, \qquad (67)
$$

are the Erdelyi-Kober integrals. This is a particular case of the hyper-Bessel operator

$$
L = x^{a_1} D x^{a_2} \cdots x^{a_n} D x^{a_{n+1}}, \qquad n \in \mathbb{N}, a_1, \cdots, a_{n+1} \text{ complex numbers} \qquad (68)
$$

The operator L can be written as

$$Lf = m^n x^{a-n} \prod_{k=1}^{n} x^{m-mb_k} D_m x^{mb_k} f \tag{69}$$

where

$$D_m = \frac{d}{dx^m} = \frac{1}{m} x^{1-m} \frac{d}{dx},\ a = \sum_{k=1}^{n+1} a_k,\ m = |a-n|,\ b_k = \frac{1}{m}\left(\sum_{i=k+1}^{n+1} a_i + k - n\right) \tag{70}$$

The integer power of L writes

$$L^r f = m^{nr} x^{-mr} \prod_{k=1}^{n} I_m^{b_k,-r} \tag{71}$$

where, for $\alpha > 0$

$$I_m^{\eta,\alpha} f \; = \; \frac{x^{-m\eta - m\alpha}}{\Gamma(\alpha)} \int_0^x (x^m - u^m)^{\alpha-1} u^{m\eta} f(u) d(u^m) \tag{72}$$

and for $\alpha < 0$

$$I_m^{\eta,\alpha} f \; = \; (\eta + \alpha + 1) \, I_m^{\eta,\alpha+1} f + \frac{1}{m} I_m^{\eta,\alpha+1} \left(x \frac{d}{dx} f \right) \tag{73}$$

the same representation of the fractional power of the operator holds.

If $m = n - \alpha > 0$, α a complex number, $b_k \in A_{p,\mu,m}$, $k = 1, \ldots, n$, for any function $f \in F_{p,\mu}$ we have that

$$L^\alpha f \; = \; m^{n\alpha} x^{-m\alpha} \prod_{k=1}^{n} I_m^{b_k,-\alpha} f \tag{74}$$

where, again, $I_m^{b_k,-\alpha}$ is an Erdely-Kober integral.

The intuitive idea of this representation can be caught by considering the integral operator

$$I_m^{\alpha} f \;=\; \frac{m}{\Gamma(\alpha)} \int_0^x \left(x^m - u^m\right)^{\alpha-1} u^{m-1} f(u) du \tag{75}$$

which includes, for $m = 1$, the Riemann-Liouville fractional integral. The integrals $I_m^{\alpha} f$ and $I_m^{\eta,\alpha} f$ are related by

$$I_m^{\alpha} f \;=\; x^{m\alpha} I_m^{0,\alpha} f. \tag{76}$$

Furthermore

$$
\begin{aligned}
I_m^{\alpha} f &= D_m I_m^{\alpha+1} f = \frac{m D_m}{\Gamma(\alpha+1)} \int_0^x \left(x^m - u^m\right)^{\alpha} u^{m-1} f(u) du \\
&= \left[\text{since } D_m = m^{-1} x^{1-m} \frac{d}{dx}\right] = \frac{m D_m D_m}{\Gamma(\alpha+2)} \int_0^x \left(x^m - u^m\right)^{\alpha+1} u^{m-1} f(u) du \\
&= \cdots = \underbrace{D_m \cdots D_m}_{r \text{ times}} I_m^{\alpha+r} f
\end{aligned} \tag{77}
$$

$$
I_m^{-r} f = D_m \cdots, D_m I_m^0 f = \left(D_m\right)^r f \tag{78}
$$

and for a real number α

$$
I_m^{-\alpha} f = \left(D_m\right)^{\alpha} f. \tag{79}
$$

Since the Erdelyi-Kober integrals satisfy the semigroup property, we have that

$$\begin{aligned}
(D_m)^{\alpha} f &= (D_m)^{n} (D_m)^{\alpha-n} \\
&= I_m^{-n} I_m^{n-\alpha} f \\
&= D_m \frac{m}{\Gamma(n-\alpha)} \int_0^x (x^m - u^m)^{n-\alpha-1} u^{m-1} f(u) du \qquad (80)
\end{aligned}$$

and for $m = 1$ we recover the R-L fractional derivative.

By applying the previous framework we obtain a fractional telegraph-type process $T^\alpha(t)$, $t > 0$, whose distribution is

$$
\begin{aligned}
p^\alpha(x,t) \;=& \frac{1}{E_{\alpha,\alpha}\left(\lambda t^\alpha\right)}\left[ct\sum_{k=1}^{\infty}\left(\frac{\lambda}{2^\alpha c^\alpha}\right)^{2k}\frac{\left(c^2t^2-x^2\right)^{2k-1}}{\Gamma(\alpha k)\Gamma(\alpha k+1)}\right. \\
&\left.+\sum_{k=1}^{\infty}\left(\frac{\lambda}{2^\alpha c^\alpha}\right)^{2k+1}\frac{\left(c^2t^2-x^2\right)^{\alpha k+1}}{\left(\Gamma\left(\alpha k+\frac{1+\alpha}{2}\right)\right)^2}\right] \\
&+\frac{1}{2E_{\alpha,1}(\lambda t^\alpha)}\left(\delta(x+ct)-\delta(x-ct)\right)
\end{aligned} \tag{81}
$$

The function

$$
f(x,t) \;=\; E_{\alpha,1}\left(\lambda t^\alpha\right)\Pr\left\{T^\alpha(t)\in dx\right\}, \qquad x\in(-ct,+ct) \tag{82}
$$

is a solution to

$$
\left(\frac{\partial^2}{\partial t^2}-c^2\frac{\partial^2}{\partial x^2}\right)^\alpha u_\alpha(x,t) \;=\; \lambda^2 u_\alpha(x,t)+\lambda 2^\alpha c^\alpha\frac{\left(\sqrt{c^2t^2-x^2}\right)}{\left(\Gamma\left(\frac{1-\alpha}{2}\right)\right)^2} \tag{83}
$$

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